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A NOTE ON REARRANGEMENTS OF FOURIER COEFFICIENTS

by **Hugh L. MONTGOMERY**

Let $\{\varphi_k\}$ be a sequence of functions on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, with the property that they are uniformly bounded,

$$(1) \quad \|\varphi_k\|_\infty \leq M,$$

and satisfy a Bessels inequality

$$(2) \quad \sum_k \left| \int_0^1 f \varphi_k \right|^2 \leq M^2 \int_0^1 |f|^2.$$

For the sake of simplicity we suppose that M has the same value in (1) and (2); this does not occasion any loss of generality. Suppose that $\sum_k |a_k|^2 < \infty$. Then

$$(3) \quad f(x) = \sum_k a_k \varphi_k(x)$$

is a member of $L^2(\mathbf{T})$, since the dual of (2) asserts that

$$(4) \quad \int_0^1 \left| \sum_k a_k \varphi_k \right|^2 \leq M^2 \sum_k |a_k|^2.$$

In this note we obtain bounds for $\int_E |f|^2$ in terms of the measure of the set E and the numbers $|a_k|$. Following Hardy and Littlewood, we let the numbers a_0^*, a_1^*, \dots be the numbers $|a_k|$, permuted so that $a_n^* \searrow$. Then we set

$$(5) \quad f^*(x) = \sum_{n=0}^{\infty} a_n^* \cos 2\pi nx.$$

THEOREM 1. — *Let $\{\varphi_k\}$ be a sequence of functions satisfying (1) and (2), let f and f^* be defined by (3) and (5). Then*

for any measurable set $E \subseteq \mathbf{T}$, with measure $|E| = 2\theta$, we have

$$(6) \quad \int_E |f|^2 \leq 20M^2 \int_{-0}^0 |f^*|^2.$$

If $C \in L^2(\mathbf{T})$, $C(x) \sim \sum_{n=0}^{\infty} C_n \cos 2\pi nx$, and if $C_n \searrow$, then $C = C^*$, so (6) implies that

$$\int_E |C|^2 \leq 20 \int_{-0}^0 |C|^2,$$

where $E \subseteq \mathbf{T}$, $|E| = 2\theta$. Thus, although it is not necessarily true that $C(x)$ is decreasing on $\left[0, \frac{1}{2}\right)$, in a certain sense it is still the case that C is largest near 0.

Using a simple inequality of A. Baernstein [2], we shall derive from Theorem 1 the following.

THEOREM 2. — *Let ψ be a convex increasing function from $[0, \infty)$ to \mathbf{R} . Then, in the above notation,*

$$\int_E \psi(|f|^2) \leq \int_{-0}^0 \psi(20 M^2 |f^*|^2).$$

Taking $\psi(t) = t^{q/2}$, we see from the above that

$$(7) \quad \|f\|_q \leq 5M \|f^*\|_q \quad (q \geq 2).$$

Inequalities of this type have a long history. Hardy and Littlewood [3, 4] proved that

$$(8) \quad \|f\|_q \leq c_q \|f^*\|_q \quad (q \geq 2)$$

in the case $\varphi_k(x) = e^{2\pi i k x}$, $-\infty < k < +\infty$. Littlewood [6] has shown that c_q is bounded in this case, and F. R. Keogh [5] has shown that $c_p \rightarrow 1$ as $q \rightarrow \infty$. In the opposite direction, Littlewood [7] showed that $c_q > 1$ except when q is an even integer. Consequently, the constant 20 in Theorems 1 and 2 can not be replaced by 1. R. E. A. C. Paley [9] extended (8) to the case of arbitrary uniformly bounded orthonormal φ_k (see Zygmund [11, XII § 5] for a simple proof). Theorem 2 does not seem to follow from the special case (7), since in general a convex increasing function $\psi(t)$ is not comparable to a sum $\sum_r c_r t^{a_r}$, $c_r \geq 0$, $a_r \geq 1$.

If one were to consider, in place of f^* , a function

$$f^-(x) = \sum_{n=0}^{\infty} a_n * \varphi_n(x),$$

then one does not in general expect the inequality

$$\|f\|_q \leq c_q \|f^-\|_q \quad (q \geq 2)$$

to be valid, even when the φ_n are given in some natural order. (See G. A. Bachelis [1], and H. S. Shapiro [10]). However, in the special case of ordinary Dirichlet series, there are good reasons to believe that something positive may be said. For example, we can formulate a

Conjecture. — Let $\varepsilon > 0$, and $2 \leq q \leq 4$. Then for $T \geq 2$, $N > N_0(\varepsilon, q)$, we have

$$\int_{-T}^T \left| \sum_{n=1}^N a_n n^{-it} \right|^q dt \leq (T + N^{q/2}) N^{q/2+\varepsilon},$$

for arbitrary coefficients a_n satisfying $|a_n| \leq 1$.

The above is known to be true when $q = 2, q = 4$; thus by Hölder's inequality it suffices to consider the case $T = N^{q/2}$. The Conjecture is of special interest in multiplicative number theory, since from it one can deduce (see Montgomery [8, Theorem 12.6]) that the interval $(x, x + x^{\frac{1}{2}+\varepsilon})$ contains a prime number, for all $x > x_0(\varepsilon)$.

We now prove Theorem 1. We have only countably many functions φ_k , so without loss of generality we may suppose that $0 \leq k < \infty$. Let π be the permutation such that $a_n^* = |a_{\pi(n)}|$. Put $N = [(2\theta)^{-1}]$, and set

$$\mathcal{N} = \{\pi(n) : 0 \leq n \leq N\}.$$

Thus \mathcal{N} is the set of indices of the $N + 1$ coefficients of largest absolute value. Break the sum (3) into two parts,

$$f = \sum_{n \in \mathcal{N}} + \sum_{n \notin \mathcal{N}} = f_1 + f_2,$$

say. On one hand,

$$\int_{\mathbb{R}} |f_1|^2 \leq \|f_1\|_{\infty}^2 \int_{\mathbb{R}} 1 \leq 2\theta \left(M \sum_{n=0}^N a_n^* \right)^2,$$

in view of (1). On the other hand, from (4) we see that

$$\int_{\mathbf{E}} |f_2|^2 \leq \int_0^1 |f_2|^2 \leq M^2 \sum_{n>N} a_n^{*2}.$$

For each x , $|f|^2 \leq 2|f_1|^2 + 2|f_2|^2$, so on combining the above we find that

$$(9) \quad \int_{\mathbf{E}} |f|^2 \leq 4\theta \left(M \sum_{n=0}^N a_n^* \right)^2 + 2M^2 \sum_{n>N} a_n^{*2}.$$

It now remains to relate the right hand side above to $\int_{-1}^1 |f^*|^2$.

Let $K(x) = \max(0, 1 - |x|^{\theta-1})$ for $|x| \leq \frac{1}{2}$. Then

$$\int_{-1}^1 |f^*|^2 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} K|f^*|^2 = \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* (\hat{K}(m+n) + \hat{K}(m-n)).$$

Now $\hat{K}(m) = \theta \left(\frac{\sin \pi m \theta}{\pi m \theta} \right)^2 \geq 0$, so

$$(10) \quad \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* \hat{K}(m-n) \leq \int_{-1}^1 |f^*|^2.$$

If $|m-n| \leq N$ then

$$(11) \quad \hat{K}(m-n) \geq \theta \left(\frac{\sin \pi N \theta}{\pi N \theta} \right)^2 \geq \theta \left(\frac{\sin \frac{1}{2} \pi}{\frac{1}{2} \pi} \right)^2 = 4\pi^{-2}\theta,$$

since $N \leq (2\theta)^{-1}$. But $a_n^* \geq 0$, so

$$(12) \quad \theta \left(\sum_{0 \leq n \leq N} a_n^* \right)^2 \leq \frac{1}{4} \pi^2 \sum_{0 \leq m, n \leq N} \hat{K}(m-n) a_m^* a_n^*.$$

If $0 < n - N \leq m \leq n$ then $a_m^* \geq a_n^*$, so from (11) we find that

$$\sum_{n-N \leq m \leq n} a_m^* \hat{K}(m-n) \geq 4\pi^{-2}\theta(N+1)a_n^* \geq 2\pi^{-2}a_n^*,$$

since $N+1 > (2\theta)^{-1}$. Hence

$$(13) \quad \sum_{n>N} a_n^{*2} \leq \frac{1}{2} \pi^2 \sum_{\substack{n>N \\ n-N \leq m \leq n}} \hat{K}(m-n) a_m^* a_n^*.$$

Combining (9) with (12), (13), we find that

$$\int_E |f|^2 \leq \pi^2 \theta M^2 \sum_{m, n=0}^{\infty} a_m^* a_n^* \hat{K}(m - n).$$

But $2\pi^2 < 20$, so by (10) our proof is complete.

We note that once (9) is established, the remainder of the proof can be effected in several ways. In proving (8), Hardy and Littlewood [3] established that

$$\int_0^1 |f^*|^q \approx_q \sum_{n=0}^{\infty} a_n^{*q} (n + 1)^{q-2}.$$

One can modify their proof of this (see also Keogh [5]) to show that

$$\sum_{n > \theta^{-1}} n^{-2} \left(\sum_{0 \leq m \leq n} a_m^* \right)^2 < c \int_{-\theta}^{\theta} |f^*|^2.$$

Theorem 1 follows easily from the above and (9), apart from the values of constants.

To prove Theorem 2 we require the following result of A. Baernstein [2].

LEMMA. — For $f \in L^1(\mathbf{T})$, $0 \leq \theta \leq \frac{1}{2}$, let $f^+(\theta) = \sup_E \int_E |f|$, where the supremum is taken over all measurable sets $E \subseteq [0, 1)$ such that $|E| = 2\theta$. For two functions $r, s \in L^1(\mathbf{T})$, the following are equivalent:

(a) For all $\theta \in \left[0, \frac{1}{2}\right)$, $r^+(\theta) \leq s^+(\theta)$;

(b) For any $\psi(t)$, convex and increasing on $[0, \infty)$, we have

$$\int_0^1 \psi(|r|) \leq \int_0^1 \psi(|s|).$$

In the language of this lemma, we find from Theorem 1 that $(|f|^2)^+(\theta) \leq (20M|f^*|^2)^+(\theta)$. Hence

$$\|\psi(|f|^2)\|_1 \leq \|\psi(20M|f^*|^2)\|_1.$$

However, with a little more care we obtain the full strength of Theorem 2. Let $E \subseteq [0, 1)$ be a set with $|E| = 2\theta$. Put $r = |f|^2 \chi_E$, $s = 20M^2 |f^*|^2 \chi_{(-\theta, \theta)}$. Then by Theorem 1, $r^+ \leq s^+$,

so by the Lemma above, $\|\psi(r)\|_1 \leq \|\psi(s)\|_1$. If $\psi(0) = 0$, then this asserts that

$$\int_E \psi(|f|^2) \leq \int_{-0}^0 \psi(20M^2|f^*|^2).$$

To obtain this for general ψ we have only to add a constant to both sides of the inequality. This completes the proof of Theorem 2.

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