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## ON THE GREATEST PRIME FACTOR OF $n^2 + 1$

by J.-M. DESHOILLERS and H. IWANIEC

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### 1. Introduction.

In 1967 C. Hooley [2] (see also [3]) showed that if  $D$  is not a perfect square then the greatest prime factor of  $n^2 - D$  exceeds  $n^{11/10}$  infinitely often. In fact Hooley's arguments yield a slightly better result with the exponent  $11/10$  replaced by any  $\theta$  less than  $\theta_0 = 1.100\ 148\ 3 \dots$  the solution of

$$(1) \quad \frac{14}{3} \left( \theta - \frac{12}{11} \right) + \frac{28}{9} \log \left( 1 + \frac{33}{14} \left( \theta - \frac{12}{11} \right) \right) - \frac{41}{33} + \frac{32}{9} \log \frac{11}{8} = 0.$$

Among several innovative ideas in Hooley's proof one finds a very interesting application of A. Weil's estimate for Kloosterman sums

$$(2) \quad S(nm; c) = \sum_{\substack{d \pmod{c} \\ (d, c) = 1}} e \left( n \frac{\bar{d}}{c} + m \frac{d}{c} \right) \ll (n, m, c)^{\frac{1}{2}} c^{\frac{1}{2} + \varepsilon}$$

where the symbol  $\bar{d}$  stands for a solution of  $d\bar{d} \equiv 1 \pmod{c}$ . Recently the authors [1] investigated linear forms in Kloosterman sums  $S(n\bar{Q}, m; c)$  with the variables of the summation  $n, m$  and  $c$  counted with a smooth weight function, showing (see Lemma 3) that there exists a considerable cancellation of terms.

In the paper we inject this result into the Chebyshev-Hooley method to prove the following

**THEOREM.** — *For any  $\varepsilon > 0$  there exist infinitely many integers  $n$  such that  $n^2 + 1$  has a prime factor greater than  $n^{\theta - \varepsilon}$ , where  $\theta$  satisfies*

$$2 - \theta - 2 \operatorname{Log} (2 - \theta) = \frac{5}{4} \quad (\theta = 1.202\ 468 \dots).$$

Our result can be generalized to  $n^2 - D$  by using Hooley's arguments from [3].

The authors express their thanks to Prof. C. Hooley for interesting comments and corrections.

## 2. Chebyshev's method.

Let  $x \geq 2$  and let  $b$  be a non-negative function of  $C^\infty$ -class with support in  $[x, 2x]$  and the derivatives of which satisfy

$$b^{(l)}(\xi) \ll x^{-l}, \quad l = 0, 1, 2, \dots,$$

the implied constant in  $\ll$  depending on  $l$  alone. Denote

$$X = \int b(\xi) d\xi \quad \text{and} \quad |\mathcal{A}_d| = \sum_{n^2+1 \equiv 0 \pmod{d}} b(n).$$

We begin with applying Chebyshev's idea to calculate

$$\begin{aligned} (1) \quad T(x) &= \sum_p |\mathcal{A}_p| \log p = \sum_d |\mathcal{A}_d| \wedge(d) + O(x) \\ &= \sum_n b(n) \sum_{d|n^2+1} \wedge(d) + O(x) = \sum_n b(n) \log(n^2+1) + O(x) \\ &= 2(\log x) \int b(\xi) d\xi + O(x) = 2X \log x + O(x). \end{aligned}$$

The partial sum

$$\begin{aligned} T_0(x) &= \sum_{p \leq x} |\mathcal{A}_p| \log p \\ &= \sum_{p \leq x} \sum_{v^2+1 \equiv 0 \pmod{p}} (\log p) \sum_{n \equiv v \pmod{p}} b(n) \end{aligned}$$

can be evaluated easily by the Poisson summation formula.

LEMMA 1. — For any  $f(\xi)$  of  $C^1$  class with compact support in  $(0, \infty)$  we have

$$\sum_{n \equiv a \pmod{q}} f(n) = \frac{1}{q} \sum_h e\left(-\frac{ah}{q}\right) \hat{f}\left(\frac{h}{q}\right), \quad h \in \mathbf{Z}$$

where  $\hat{f}(t)$  is the Fourier transform of  $f(\xi)$ .

By Lemma 1

$$\sum_{n \equiv v \pmod{p}} b(n) = \frac{1}{p} \sum_h e\left(-\frac{vh}{p}\right) \hat{b}\left(\frac{h}{p}\right).$$

For  $h = 0$  we have  $\hat{b}(0) = X$ . If  $h \neq 0$  by partial integration two times we get

$$\begin{aligned} \hat{b}\left(\frac{h}{p}\right) &= \int b(\xi) e\left(\frac{h}{p} \xi\right) d\xi \\ &= \left(\frac{p}{2\pi i h}\right)^2 \int b''(\xi) e\left(\frac{h}{p} \xi\right) d\xi \ll h^{-2} p^2 x^{-1}. \end{aligned}$$

This yields

$$\sum_{n \equiv v \pmod{p}} b(n) = \frac{X}{p} + O\left(\frac{p}{x}\right)$$

whence

$$(2) \quad T_0(x) = X \log x + O(x).$$

Letting  $P_x$  be the greatest prime factor of  $\prod_{x < n < 2x} (n^2 + 1)$  by (1) and (2) it follows that

$$(3) \quad S(x) = \sum_{x < p \leq P_x} |\mathcal{A}_p| \log p = X \log x + O(x).$$

Our aim is to estimate  $S(x)$  from above and deduce from it a lower estimate for  $P_x$ .

### 3. Splitting up of $S(x)$ .

In what follows it will be convenient to have  $p$  counted with a smooth weight function. Therefore we arrange the sum  $S(x)$  as

$$(4) \quad S(x) = \sum_{1 \leq j \leq J} S(x, P_j) + O(x)$$

with  $P_j = 2^j x$ ,  $0 \leq j \leq J \leq 2 \log x$  and

$$(5) \quad S(x, P_j) = \sum_{P_j < p \leq 4P_j} |\mathcal{A}_p| C_j(p) \log p$$

where  $C_j(\xi)$  are non-negative functions of  $C^\infty$ -class which satisfy the three following conditions

$$(6) \quad \text{Supp } C_j \subset [P_j, 4P_j]$$

$$\sum_{0 \leq j \leq J} C_j(\xi) = \begin{cases} 1 & \text{if } 2x < \xi \leq P_x \\ O(1) & \text{if } x < \xi \leq 2x \text{ or } P_x < \xi \leq 2P_x \\ 0 & \text{otherwise} \end{cases}$$

$C_j^{(l)}(\xi) \ll P_j^{-l}$ , with the implied constant depending on  $l$  alone. The error term  $O(x)$  in (4) comes from a trivial estimate for the contribution of primes  $p$  in the interval  $(x, 2x]$  which is not completely covered.

#### 4. Application of the sieve method.

A typical sum to be considered is

$$S(x, P) = \sum_{P < p \leq 4P} |\mathcal{A}_p| C(p) \log p$$

with  $x < P \leq 2P_x$ . Let  $x \geq D \geq 1$  and let  $\{\lambda_d\}_{d \leq D}$  be an upper bound sieve of level  $D$ , i.e. a sequence of real numbers such that

$$\lambda * 1 \geq \mu * 1, \quad \lambda_1 = 1, \quad \lambda_d = 0 \text{ for } d \geq D.$$

We also assume that  $|\lambda_d| \leq 1$  for all  $d$  and that  $\lambda_d = 0$  when  $d$  is not square-free.

Thus

$$S(x, P) \leq \sum_{d \leq D} \lambda_d \sum_{m \equiv 0 \pmod{d}} |\mathcal{A}_m| C(m) \log m.$$

By the Poisson summation formula we write

$$\begin{aligned} |\mathcal{A}_m| &= \sum_{v^2 + 1 \equiv 0 \pmod{m}} \sum_{n \equiv v \pmod{m}} b(n) \\ &= \frac{\omega(m)}{m} X + r(\mathcal{A}, m) \end{aligned}$$

where  $\omega(m)$  is the number of incongruent solutions of  $v^2 + 1 \equiv 0 \pmod{m}$  and

$$(7) \quad r(\mathcal{A}, m) = \frac{1}{m} \sum_{h \neq 0} \sum_{v^2 + 1 \equiv 0 \pmod{m}} e\left(-\frac{vh}{m}\right) \hat{b}\left(\frac{h}{m}\right).$$

According to the above we write

$$(8) \quad S(x, P) \leq XV(x, P) + R(x, P)$$

where  $XV(x, P)$  is considered as a main term

$$V(x, P) = \sum_{d < D} \lambda_d \sum_{m \equiv 0 \pmod{d}} \frac{\omega(m)}{m} C(m) \log m$$

and  $R(x, P)$  is the total error term

$$R(x, P) = \sum_{d < D} \lambda_d R(x, d, P)$$

with

$$(9) \quad R(x, d, P) = \sum_{h \neq 0} \sum_{m \equiv 0 \pmod{d}} \frac{C(m)}{m} \log m \sum_{v^2 + 1 \equiv 0 \pmod{m}} \hat{b}\left(\frac{h}{m}\right) e\left(-\frac{vh}{m}\right).$$

### 5. Transformation of $R(x, d, P)$ .

We are searching for  $D$  as large as possible for which the estimate

$$(10) \quad R(x, P) \ll x^{1-\varepsilon}$$

is available. By partial integration  $l = [4\varepsilon^{-1}]$  times we get

$$\hat{b}\left(\frac{h}{m}\right) = \left(-2\pi i \frac{h}{m}\right)^{-l} \int b^{(l)}(\xi) e\left(\frac{h}{m} \xi\right) d\xi \ll x \left(\frac{P}{|h|x}\right)^l \ll h^{-2}$$

for  $|h| \geq Px^{\varepsilon-1} = H$ , say. Hence truncating the series (9) at  $h = H$  we make an error  $O(\tau(d)/d)$  which contributes to  $R(x, P)$  an admissible amount

$$\sum_{d < D} \frac{\tau(d)}{d} \ll (\log D)^2 \ll (\log x)^2.$$

For the remaining terms we need an explicit formula for the solutions of

$$(11) \quad v^2 + 1 \equiv 0 \pmod{m}.$$

LEMMA 2. (Gauss). — Let  $m > 1$ . If (11) is soluble then  $m$  is represented properly as a sum of two squares

$$(12) \quad m = r^2 + s^2, \quad (r, s) = 1, \quad r, s > 0.$$

There is a one to one correspondence between the incongruent solutions  $v \pmod{m}$  of (11) and the solutions  $(r, s)$  of (12) given by

$$\frac{v}{m} = \frac{\bar{r}}{s} - \frac{r}{s(r^2 + s^2)}.$$

*Proof.* — See [5] and [3], p. 34, eq. (68).

By Lemma 2 we get

$$\sum_{v^2 + 1 \equiv 0 \pmod{m}} e\left(-\frac{vh}{m}\right) = \sum_{\substack{r^2 + s^2 = m \\ r, s > 0, (r, s) = 1}} e\left(-h\frac{\bar{r}}{s}\right) \left\{1 + O\left(\frac{r|h|}{sm}\right)\right\}$$

whence letting  $g(m, h) = \frac{C(m)}{m} (\log m) \hat{b}\left(\frac{h}{m}\right)$  we obtain

$$R(x, d, P) = \sum_{0 < |h| \leq H} \sum_{\substack{(r, s) = 1, r, s > 0 \\ r^2 + s^2 \equiv 0 \pmod{d}}} g(r^2 + s^2, h) e\left(-h\frac{\bar{r}}{s}\right) + O(d^{-1} P x^{3\epsilon - 1}).$$

Here the error  $O(d^{-1} P x^{3\epsilon - 1})$  contributes to  $R(x; P)$  less than  $P x^{3\epsilon - 1} \log x \ll x^{1 - \epsilon}$  provided  $P \leq x^{2 - 5\epsilon}$  which we henceforth assume.

For sum over  $r$  we apply Poisson's summation formula giving

$$\begin{aligned} & \sum_{\substack{(r, s) = 1 \\ r^2 + s^2 \equiv 0 \pmod{d}}} g(r^2 + s^2, h) e\left(-h\frac{\bar{r}}{s}\right) \\ & \quad \sum_{\substack{u \pmod{ds} \\ (u, s) = 1 \\ u^2 + s^2 \equiv 0 \pmod{d}}} e\left(-h\frac{\bar{u}}{s}\right) \sum_{r \equiv u \pmod{ds}} g(r^2 + s^2, h) \\ & = \frac{1}{ds} \sum_k \sum_{\substack{u \pmod{ds} \\ (u, s) = 1 \\ u^2 + s^2 \equiv 0 \pmod{d}}} e\left(-h\frac{\bar{u}}{s} - k\frac{u}{ds}\right) G(h, k; s) \end{aligned}$$

where  $G(h, k; s) = \int g(\xi^2 + s^2, h) e(k\xi/ds) d\xi$ . Writing  $u = \alpha s + \beta d$  with  $\alpha^2 + 1 \equiv 0 \pmod{d}$  it becomes

$$\frac{1}{ds} \sum_{\alpha^2 + 1 \equiv 0 \pmod{d}} \sum_k e\left(-\frac{\alpha k}{d}\right) S(-h\bar{\alpha}, -k; s) G(h, k; s).$$

For  $k = 0$  the Kloosterman sum  $S(-h\bar{d}, -k; s)$  reduces to a Ramanujan sum for which we have

$$|S(-h\bar{d}, 0; s)| \leq (h, s).$$

Therefore the terms with  $k = 0$  contribute less than

$$\frac{\tau(d)}{d} \sum_{0 < |h| \leq H} \sum_{s \leq 2\sqrt{P}} \frac{(h, s)}{s} \frac{x \text{Log } P}{P} \ll \frac{\sqrt{P}}{d} x^\epsilon \ll \frac{x^{1-\epsilon}}{d}.$$

Finally

$$(13) \quad R(x, d, P) = \frac{1}{d} \sum_{\alpha^2 + 1 \equiv 0 \pmod{d}} \sum_{0 < |h| \leq H} \sum_{k \neq 0} e\left(-\frac{\alpha k}{d}\right) \sum_{\substack{s > 0 \\ (s, d) = 1}} \frac{1}{s} S(-h\bar{d}, -k; s) G(h, k; s) + O\left(\frac{x^{1-\epsilon}}{d}\right).$$

### 6. Linear forms in Kloosterman sums.

Let  $N, M, C \geq 1$  and  $f(n, m, c)$  be a function of  $C^6$  class with compact support in  $[C, 2C]$  with respect to  $c$  and satisfying

$$(14) \quad \left| \frac{\partial^{l_1+l_2+l_3}}{\partial n^{l_1} \partial m^{l_2} \partial c^{l_3}} f(n, m, c) \right| \leq N^{-l_1} M^{-l_2} C^{-l_3}, \quad 0 \leq l_1, l_2, l_3 \leq 2.$$

In this section we borrow from [1] an estimate for the average of trilinear forms (cf. *Theorem 11*)

$$B_d^\pm(N, M, C) = \sum_{0 < n \leq N} \sum_{0 < m \leq M} \sum_{(c, d) = 1} b_{m, d} S(n\bar{d}, \pm m; c) f(n, m, c)$$

where  $b_{m, d}$  are arbitrary complex numbers.

LEMMA 3. — If  $f(n, m, c)$  satisfies (14) then for any  $\epsilon > 0$  we have

$$\left( \sum_{D < d \leq 2D} |B_d^+(N, M, C)| \right)^2 \ll (CDMN)^\epsilon N \left( \sum_{\substack{0 \leq m \leq M \\ D < d \leq 2D}} |b_{m, d}|^2 \right) \times \left\{ \frac{D(DC^2 + MN + NC^2)(DC^2 + MN + MC^2)}{DC^2 + MN} + \leq \sqrt{D(D+M)} \cdot C^3 \right\}$$

and the same upper bound holds for  $(\sum |B_d^-|)^2$ , the constant implied in  $\ll$  depending on  $\epsilon$  at most.

### 7. Estimation of the error.

In order to make Lemma 3 applicable we first split up the sum over  $s$  in  $R(x,d,P)$  into  $\ll \log P$  sums of the type

$$(14) \quad \sum_{(s,d)=1} \frac{a(s)}{s} S(-hd, \pm k; s) G(h, \mp k; s)$$

where  $a(s)$  is a function of  $C^2$  class with support  $[S, 2S]$ ,  $S \leq 2\sqrt{P}$  and satisfying  $a^{(l)}(s) \ll S^{-l}$  for  $l = 0, 1, 2$ . The terms with  $|k| \geq DSP^{-1/2}x^{3\epsilon} = K$ , say, can be eliminated trivially: integrate by parts  $l = [4\epsilon^{-1}]$  times with respect to  $\xi$  in  $G(h, \pm k, s)$ , getting

$$G(h, \pm k, s) = \left( \frac{-ds}{2\pi ik} \right)^l \int \frac{\partial^l}{\partial \xi^l} g(\xi^2 + s^2, h) e\left( \frac{\xi k}{ds} \right) d\xi \\ \ll \left( \frac{ds}{|k|\sqrt{P}} x^{2\epsilon} \right)^l \sqrt{P} \ll k^{-2} x^{-1}.$$

Therefore such terms contribute to  $R(x,d,P)$  less than

$$\frac{\tau(d)}{d} \sum_{0 < |h| \leq H} \sum_{k \geq 1} \frac{1}{k^2 x} \sum_{0 < s \leq 2\sqrt{P}} 1 \ll \frac{P^{3/2}}{dx^2} x^\epsilon \ll \frac{x^{1-\epsilon}}{d}.$$

For  $0 \leq |h| \leq H$ ,  $0 < |k| \leq K$  and  $S < s \leq 2S$  we trivially have

$$(15) \quad \frac{\partial^{l_1+l_2+l_3}}{\partial h^{l_1} \partial k^{l_2} \partial s^{l_3}} G(h, \mp k, s) \frac{a(s)}{s} \ll |h|^{-l_1} |k|^{-l_2} |s|^{-l_3} \frac{x^{1+12\epsilon}}{S\sqrt{P}}$$

for  $0 \leq l_1, l_2, l_3 \leq 2$ . This shows that Lemma 3 is applicable with

$$f(h, k, s) = \frac{S\sqrt{P}}{x^{1+13\epsilon}} \frac{a(s)}{s} G(h, \mp k, s) \text{ giving}$$

$$\left( \sum_{D < d \leq 2D} |R(x, d, P)| \right)^2 \ll x^{2-2\epsilon} + \frac{x^{2+40\epsilon} HK}{D^2 P} \\ \times \sup_{1 \leq S \leq 2 \leq \sqrt{P}} \left\{ \frac{D^2 (DS^2 + HK + HS^2)(DS^2 + HK + KS^2)}{S^2 (DS^2 + HK)} + DS \sqrt{D(D+K)} \right\} \\ \ll x^{2-2\epsilon} + (D^2 x + DP + DxP^2) x^{48\epsilon}.$$

Therefore (10) holds if

(16)

$$D \leq x^{\frac{1}{2} - 25\epsilon}, \quad D \leq x^{2-10\epsilon} P^{-1} \quad \text{and} \quad D \leq x^{1-10\epsilon} P^{-\frac{1}{2}}.$$

This result can be compared with Hooley's  $D = x^{1-\epsilon} P^{-3/4} \dots$

### 8. Evaluation of the main term.

For  $d$  square-free with  $\omega(d) \neq 0$  consider

$$L(s, d) = \sum_{m=1}^{\infty} \frac{\omega(dm)}{\omega(d)} m^{-s}.$$

LEMMA 4. — *We have*

$$(17) \quad L(s, d) = \frac{\zeta(s)L(s, \chi_4)}{\zeta(2s)} \prod_{p|d} \left(1 + \frac{1}{p^s}\right)^{-1}.$$

*Proof.* — Follow the arguments of [3] on pp. 31-32 and equation (6.1).  
Writing

$$C(m) \frac{\log m}{m} = \frac{1}{2\pi i} \int_{(\sigma)} R(s) m^{-s} ds, \quad \sigma > 0$$

by Mellin's inversion formula and partial integration two times

$$R(s) = \int C(\xi) \frac{\log \xi}{\xi} \xi^{s-1} d\xi \ll (|s| + 1)^{-2} P^{\sigma-1} \log P.$$

Therefore

$$\begin{aligned} \sum_{m \equiv 0 \pmod{d}} \frac{\omega(m)}{m} c(m) \log m &= \frac{1}{2\pi i} \int_{(\sigma)} R(s) \frac{\omega(d)}{d^s} L(s, d) ds \\ &= R(1) \frac{\omega(d)}{d} \frac{L(1, \chi_4)}{\zeta(2)} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} + \frac{1}{2\pi i} \int_{(\frac{1}{2})} R(s) \frac{\omega(d)}{d^s} L(s, d) ds \\ &= \frac{\omega(d)}{d} \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1} \frac{L(1, \chi_4)}{\zeta(2)} \int C(\xi) \frac{\log \xi}{\xi} d\xi + O\left(\frac{\tau^2(d)}{\sqrt{dP}} \log P\right). \end{aligned}$$

This yields

$$V(x, P) = \left( \sum_{d < D} \frac{\lambda_d}{d} \rho(d) \right) \frac{L(1, \chi_4)}{\zeta(2)} \int C(\xi) \frac{\log \xi}{\xi} d\xi + O\left(\sqrt{\frac{D}{P}} (\log x)^4\right)$$

where  $\rho(d) = \omega(d) \prod_{p|d} \left(1 + \frac{1}{p}\right)^{-1}$ . Now we specify  $\lambda_d$  to be those of the Rosser sieve giving (see [4])

$$\begin{aligned} \sum_{d < D} \lambda_d \frac{\rho(d)}{d} &= \prod_{p < D} \left(1 - \frac{\rho(p)}{p}\right) \left(2e^\gamma + O\left(\frac{1}{\log D}\right)\right) \\ &= \prod_{p < D} \left(1 - \frac{1}{p}\right) \frac{\zeta(2)}{L(1, \chi_4)} \left(2e^\gamma + O\left(\frac{1}{\log D}\right)\right) \\ &= \frac{2\zeta(2)}{L(1, \chi_4) \log D} \left(1 + O\left(\frac{1}{\log D}\right)\right) \end{aligned}$$

by the Mertens prime number theorem. Hence we conclude that

$$V(x, P) = \frac{2}{\log D} \int C(\xi) \frac{\log \xi}{\xi} d\xi \left(1 + O\left(\frac{1}{\log D}\right)\right).$$

We choose  $D$  equal to  $x^{1-10\varepsilon} P^{-\frac{1}{2}}$  thus by (6) the total main term is equal to

$$\begin{aligned} X \sum_{0 \leq j \leq J} V(x, P_j) &= 2(1 + O(\varepsilon)) X \int_x^{P_x} \frac{\text{Log } \xi}{\xi \text{Log}(x/\sqrt{\xi})} d\xi \\ &= 2(1 + O(\varepsilon)) X \int_1^\theta \frac{t dt}{1 - t/2} \text{Log } x \\ &= (1 + O(\varepsilon)) f(\theta) X \text{Log } x \end{aligned}$$

where  $f(\theta) = 4(1 - \theta - 2 \text{Log}(2 - \theta))$  and is less than 1 for  $\theta = 1.202\,468\,87$ . The proof the Theorem follows from this and (3).

One may note that the truth of Selberg's eigenvalue conjecture leads to the lower bound  $x^{\sqrt{3/2} - \varepsilon}$  for  $P_x$ .

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