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Null electromagnetic fields in algebraically special Petrov type spaces

par

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ABSTRACT. — An analysis is made of the properties of the null congruence $\mathcal{C}(k)$ tangent to the multiple principal null vector k^α of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ when: *a)* we are given the field of null vectors k^α and not a field of null tetrads associated with k^α; *b)* we postulate the Robinson-Schild type conditions on $C_{\alpha\beta\gamma\delta}$ and not on $\mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + iC^*_{\alpha\beta\gamma\delta}$, *c)* we assume that k^α is at the same time the principal null vector of a null electromagnetic field. It is shown that the shear-free character of $\mathcal{C}(k)$ resulting from the generalized Goldberg-Sachs theorem must then be qualified.

1. In a recent paper [*1*], we established the following result: A null electromagnetic field $\varphi_{\alpha\beta}$ on a riemannian manifold V_4:

$$(1.1a) \qquad \Phi_{\alpha\beta}k^\beta = (\varphi_{\alpha\beta} + i\varphi^*_{\alpha\beta})k^\beta = 0 \qquad k^\sigma k_\sigma = 0$$

$$(1.1b) \qquad \nabla_\beta \Phi^{\alpha\beta} = 0$$

is propagated in its domain of existence, along the null geodesic congruence $\mathcal{C}(k)$ defined by the null vectors $\{ k^\alpha \}$ so that:

$$(1.2a) \qquad \delta\Phi_{\alpha\beta} = k^\sigma \nabla_\sigma \Phi_{\alpha\beta} = \bar{d}\Phi_{\alpha\beta} - \eta\bar{\Phi}_{\alpha\beta}$$

$$(1.2b) \qquad \bar{d} = \bar{m}^\rho m^\sigma \nabla_\rho k_\sigma, \; \eta\bar{\mathcal{A}} = \bar{e}\mathcal{A} \qquad \bar{e} = \bar{m}^\rho \bar{m}^\sigma \nabla_\rho k_\sigma$$

the amplitude $\mathcal{A}$ and the null vector $\overline{m}^\alpha$ being determined by:

(1.3a) $$\Phi_{\alpha\beta} = a(k_\alpha \overline{m}_\beta - k_\beta \overline{m}_\alpha)$$

(1.3b) $$\overline{m}^\alpha = \text{complex conjugate of } m^\alpha \qquad \overline{m}^\sigma \overline{m}_\sigma = 0$$

The shear-free character of the congruence $\mathcal{C}(k)$ results from (1.2) when one postulates the Robinson's conditions of normalized propagation [2]:

(1.4) $$\delta\Phi_{\alpha\beta} = \bar{d}\Phi_{\alpha\beta}$$

However, in the general case the propagation law (1.2) of the null field preserve the orthogonality of k^α and $\overline{m}^\alpha$, but not the null character of $\overline{m}^\alpha$; the geodesic congruence $\mathcal{C}(k)$ is no more shear-free :

(1.5) $$\bar{e} = \overline{m}^\rho \delta \overline{m}_\rho = - k^\lambda \overline{m}^\sigma \nabla_\sigma \overline{m}_\lambda$$

and the law (1.2) determines the variation of the generalized Stokes parameters (1) of hte light described by $\varphi_{\alpha\beta}$. Anyhow, when we give as usual, physical meaning only to the real part of complex quantities, we may continue to use at each point of V_4 the representation of the real $\varphi_{\alpha\beta}$ by the complex antiself-dual bivector $\Phi_{\alpha\beta}$. Note $\delta\varphi^*_{\alpha\beta} \neq (\delta\varphi)^*_{\alpha\beta}$.

The same trend of ideas leads to investigate:

a) the incidence of the non-conservation of the antiself-dual character of a bivector such as $\Phi_{\alpha\beta}$ by differentiation, on the generalized Goldberg-Sachs (g. G. S.) theorem [3],

b) the consequences of weaker field equations than those adopted in the (g. G. S.) theorem,

c) the consequences of a null electromagnetic field energy distribution in V_4, the principal null vector of which is at the same time the multiple principal null vector of the Weyl tensor.

2. THE ARGUMENT

The (g. G. S.) theorem relies mainly on the following facts:

i) Let $(k^\alpha, m^\alpha, \overline{m}^\alpha, l^\alpha)$ be a field of null tetrads given in a domain D of V_4:

(2.1b) $$k^\sigma k_\sigma = 0$$

(2.1b) $$k^\sigma l_\sigma = - m^\sigma \overline{m}_\sigma = 1, \qquad l^\sigma l_\sigma = m^\sigma m_\sigma = k^\sigma m_\sigma = l^\sigma m_\sigma = 0$$

and let the antiself-dual bivectors:

$$\underset{1}{\psi}_{\alpha\beta} = U_{\alpha\beta} = l_\alpha m_\beta - l_\beta m_\alpha$$

(2.2a) $$\underset{2}{\psi}_{\alpha\beta} = V_{\alpha\beta} = k_\alpha \bar{m}_\beta - k_\beta \bar{m}_\alpha$$

$$\underset{3}{\psi}_{\alpha\beta} = M_{\alpha\beta} = k_\alpha l_\beta - k_\beta l_\alpha + m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha$$

(2.2b) $$\underset{a}{\psi}^*_{\alpha\beta} = - i \underset{a}{\psi}_{\alpha\beta} \qquad a = 1, 2, 3$$

and their complex conjugates $\underset{a}{\bar{\psi}}_{\alpha\beta}$ induce a basis for the 2-forms at each point M of D. Then at M, the real Weyl tensor $C_{\alpha\beta\gamma\delta}$ of V_4 is such that:

(2.3a) $$\mathcal{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + iC^*_{\alpha\beta\gamma\delta} = \mathcal{C}^{ab} \underset{a}{\psi}_{\alpha\beta} \underset{b}{\psi}_{\gamma\delta} \qquad \mathcal{C}^{ab} = \mathcal{C}^{ba}$$

or equivalently:

(2.3b) $$\begin{aligned}\mathcal{C}_{\alpha\beta\gamma\delta} = {} & C^5 U_{\alpha\beta} U_{\gamma\delta} + C^4 (U_{\alpha\beta} M_{\gamma\delta} + M_{\alpha\beta} U_{\gamma\delta}) \\ & + C^3 (M_{\alpha\beta} M_{\gamma\delta} - U_{\alpha\beta} V_{\gamma\delta} - V_{\alpha\beta} U_{\gamma\delta}) + C^2 (V_{\alpha\beta} M_{\gamma\delta} + M_{\alpha\beta} V_{\gamma\delta}) \\ & + C^1 V_{\alpha\beta} V_{\gamma\delta}\end{aligned}$$

ii) As the null tetrads are given *a priori*, the bivectors $\underset{a}{\psi}_{\alpha\beta}$ and $\underset{a}{\bar{\psi}}_{\alpha\beta}$ preserve under differentiation their self-dual or antiself-dual character:

(2.4) $$\nabla_\gamma \underset{a}{\psi}_{\alpha\beta} = \Lambda_\gamma{}^b \underset{b}{\psi}_{\alpha\beta} \qquad \nabla_\gamma \underset{a}{\bar{\psi}}_{\alpha\beta} = \bar{\Lambda}_\gamma{}^b \underset{b}{\bar{\psi}}_{\alpha\beta}$$

Now, for these relations to be valid, the propagation of $C_{\alpha\beta\gamma\delta}$ according to the Bianchi identities must preserve the relations (2.1) defining the null tetrad associated with $C_{\alpha\beta\gamma\delta}$ at each point; this invariance is however not obvious as far as (2.1*b*) is concerned.

3. SOME MATHEMATICAL RELATIONS

Let us assume therefore:

a) that the vector field $\{ k^\alpha \}$ is given, as $C_{\alpha\beta\gamma\delta}$ is determined at each point;

b) that the relations (2.1*b*) associating a null tetrad with $C_{\alpha\beta\gamma\delta}$ are not preserved under differentiation.

Then, we get:

$$
(3.1)\quad
\begin{aligned}
\nabla_\gamma U_{\alpha\beta} &= \lambda_\gamma U_{\alpha\beta} + \mu_\gamma M_{\alpha\beta} + \sigma_\gamma \overline{M}_{\alpha\beta} + \pi_\gamma \overline{V}_{\alpha\beta} + \bar{\rho}_\gamma \overline{U}_{\alpha\beta} \\
\nabla_\gamma V_{\alpha\beta} &= \varepsilon_\gamma V_{\alpha\beta} + \nu_\gamma M_{\alpha\beta} + \tau_\gamma \overline{M}_{\alpha\beta} + \rho_\gamma \overline{V}_{\alpha\beta} \\
\nabla_\gamma M_{\alpha\beta} &= -\, 2\nu_\gamma U_{\alpha\beta} - 2\mu_\gamma V_{\alpha\beta} + \chi_\gamma (M_{\alpha\beta} + \overline{M}_{\alpha\beta}) + 2\bar{\tau}_\gamma \overline{U}_{\alpha\beta} + 2\bar{\sigma}_\gamma \overline{V}_{\alpha\beta}
\end{aligned}
$$

where

$$
(3.2)\quad
\begin{aligned}
\lambda_\gamma &= k^\sigma \nabla_\gamma l_\sigma - \overline{m}^\sigma \nabla_\gamma m_\sigma, & \varepsilon_\gamma &= l^\sigma \nabla_\gamma k_\sigma - m^\sigma \nabla_\gamma \overline{m}_\sigma \\
\mu_\gamma &= \frac{1}{2}(m^\sigma \nabla_\gamma l_\sigma - l^\sigma \nabla_\gamma m_\sigma) & \nu_\gamma &= \frac{1}{2}(k^\sigma \nabla_\gamma \overline{m}_\sigma - \overline{m}^\sigma \nabla_\gamma k_\sigma) \\
\sigma_\gamma &= -\frac{1}{2}(m^\sigma \nabla_\gamma l_\sigma + l^\sigma \nabla_\gamma m_\sigma) & \tau_\gamma &= \frac{1}{2}(k^\sigma \nabla_\gamma \overline{m}_\sigma + \overline{m}^\sigma \nabla_\gamma k_\sigma) \\
\rho_\gamma &= -\, \overline{m}^\sigma \nabla_\gamma \overline{m}_\sigma & \pi_\gamma &= l^\sigma \nabla_\gamma l_\sigma \\
\chi_\gamma &= \frac{1}{2}(k^\sigma \nabla_\gamma l_\sigma + l^\sigma \nabla_\gamma k_\sigma - m^\sigma \nabla_\gamma \overline{m}_\sigma - \overline{m}^\sigma \nabla_\gamma m_\sigma)
\end{aligned}
$$

From (2.3*b*) and (3.1) we get:

$$
(3.3)\quad
\begin{aligned}
\nabla(\mathcal{C}_{\alpha\beta\gamma\delta} k^\beta k^\delta) = {} & A k_\alpha k_\gamma + B(k_\alpha m_\gamma + k_\gamma m_\alpha) + C(k_\alpha \overline{m}_\gamma + k_\gamma m_\alpha) \\
& + D(l_\alpha m_\gamma + l_\gamma m_\alpha) + E(k_\alpha l_\gamma + k_\gamma l_\alpha) + F(m_\alpha \overline{m}_\gamma + m_\gamma \overline{m}_\alpha)
\end{aligned}
$$

where:

$$
(3.4a)\qquad \nabla = \xi^\sigma \nabla_\sigma \qquad \xi^\alpha = \text{an arbitrary direction}
$$

and

$$
(3.4b)\quad
\begin{aligned}
A &= -\, 2C^4 l^\sigma \nabla m_\sigma + \frac{1}{2} C^3 (7 l^\sigma \nabla k_\sigma + 3 k^\sigma \nabla l_\sigma) + \nabla C^3 \\
&\qquad + C^2 (k^\sigma \nabla \overline{m}_\sigma + 3 \overline{m}^\sigma \nabla k_\sigma) \\
B &= C^5 l^\sigma \nabla m_\sigma - C^4 (2 k^\sigma \nabla l_\sigma - 3 l^\sigma \nabla k_\sigma + 2 \overline{m}^\sigma \nabla m_\sigma + m^\sigma \nabla \overline{m}_\sigma) \\
&\qquad - \nabla C^4 + C^3 (2 k^\sigma \nabla \overline{m}_\sigma + \overline{m}^\sigma \nabla k_\sigma) \\
C &= C^4 m^\sigma \nabla m_\sigma - C^3 (k^6 \nabla m_\sigma + 2 m^\sigma \nabla k_\sigma) \\
D &= -\, C^5 m^\sigma \nabla k_\sigma \qquad E = C^4 m^\sigma \nabla k_\sigma \\
F &= -\, C^5 m^\sigma \nabla m_\sigma + C^4 (k^\sigma \nabla m_\sigma + 2 m^\sigma \nabla k_\sigma)
\end{aligned}
$$

On the other hand, the Bianchi identities give:

$$
(3.5)\qquad \nabla_\sigma C^{\alpha\beta\gamma\sigma} = \frac{1}{2} \nabla_\sigma (\mathcal{C}^{\alpha\beta\gamma\sigma} + \mathcal{C}^{\alpha\beta\gamma\sigma}) = P^{\alpha\beta\gamma}
$$

(3.6) $$P_{\alpha\beta\gamma} = \frac{1}{2}(\nabla_\alpha R_{\beta\gamma} - \nabla_\beta R_{\alpha\gamma}) - \frac{1}{12}(g_{\beta\gamma}\nabla_\alpha R - g_{\alpha\gamma}\nabla_\beta R)$$

We are now in a position to elucidate the properties of the null congruence $\mathcal{C}(k)$ when k^α is a multiple vector of $C_{\alpha\beta\gamma\delta}$.

4. k^α IS A DOUBLE PRINCIPAL VECTOR

The Weyl tensor is of Petrov type II or D; at each point M, there exist two null vectors l^α and m^α (l^α being another double principal vector in the case D) so that:

(4.1a) $$C_{\alpha\beta\gamma\delta}k^\beta k^\delta = \frac{1}{2}(C^3 + \overline{C}^3)k_\alpha k_\gamma$$

(4.1b) $$C^5 = C^4 = 0 \qquad \nabla C^5 = \nabla C^4 = 0$$

From (3.3), $C_{\alpha\beta\gamma\delta}$ has its type preserved if and only if:

(4.2) $$(2C^3 - \overline{C}^3)k^\sigma \nabla m_\sigma - (2\overline{C}^3 - C^3)\overline{m}^\sigma \nabla k_\sigma = 0$$

for all ξ^α.

On the other hand, if:

(4.3) $$V_{\lambda\gamma}V_{\beta\alpha}P^{\alpha\beta\gamma} = 0$$

We get, using (4.2):

(4.4a) $$(2C^3 - \overline{C}^3)k^\sigma \delta \overline{m}_\sigma - (C^3 + \overline{C}^3)\overline{m}^\sigma \delta k_\sigma = 0 \qquad \delta = k^\sigma \nabla_\sigma$$

(4.4b)
$$(C^3 + \overline{C}^3)\overline{m}^\sigma \delta \overline{m}_\sigma + (2\overline{C}^3 - C^3)\overline{m}^\rho \overline{m}^\sigma \nabla_\rho k_\sigma + C^2 \overline{m}^\sigma(k^\lambda \nabla_\sigma \overline{m}_\lambda + \overline{m}^\lambda \nabla_\sigma k_\lambda) = 0$$

whence:

(4.5) $$\overline{m}^\sigma \delta k_\sigma = 0 \rightarrow \delta k^\alpha = 0$$

The congruence $\mathcal{C}(k)$ is therefore geodesic, but *not shear-free*.

Had we postulated with I. Robinson and A. Schild:

(4.6a) $$V_{\lambda\gamma}V_{\beta\alpha}\mathcal{P}^{\alpha\beta\gamma} = 0$$

(4.6b) $$\mathcal{P}^{\alpha\beta\gamma} = P^{\alpha\beta\gamma} + iP^{*\alpha\beta\gamma}$$

the shear of $\mathcal{C}(k)$ would be vanishing as (4.6) gives:

(4.7) $$V_{\lambda\rho}\nabla_\sigma V^{\rho\sigma} - 2V_\lambda{}^\sigma v_\sigma = 0$$

5. k^α IS A TRIPLE PRINCIPAL VECTOR

The Weyl tensor is of Petrov type III; at each point l^α and m^α can be choosen so that:

(5.1*a*) $$C_{\alpha\beta\gamma\delta}k^\beta k^\delta = 0$$

(5.1*b*) $$C^5 = C^4 = C^3 = 0 \qquad \nabla C^5 = \nabla C^4 = \nabla C^3 = 0$$

$C_{\alpha\beta\gamma\delta}$ has then its type preserved if and only if:

(5.2) $$k^\sigma \nabla \overline{m}_\sigma + 3\overline{m}^\sigma \nabla k_\sigma = 0$$

for all ξ^α.

Now if:

(5.3) $$V_{\beta\alpha}\, P^{\alpha\beta\gamma} = 0$$

We get:

(5.4) $$C^2 V^{\gamma\sigma} v_\sigma - \overline{C}^2 \overline{V}^{\gamma\sigma} \tau_\sigma = 0$$

whence:

(5.5*a*) $$k^\sigma v_\sigma = 0 \qquad k^\sigma \tau_\sigma = 0 \rightarrow \overline{m}^\sigma \delta k_\sigma = k^\sigma \delta \overline{m}_\sigma = 0$$

(5.5*b*) $$2C^2 \overline{e} - \overline{C}^2 d = 0$$

therefore we get in this case also the geodesic character of $\mathcal{C}(k)$ and not the vanishing of its shear.

On the other hand, with Robinson-Schild weak equations:

(5.6) $$V_{\beta\alpha} \mathcal{T}^{\alpha\beta\gamma} = 0$$

the Bianchi identities give:

(5.7) $$V^{\gamma\sigma} v_\sigma = 0$$

i. e. the geodesic and shear-free character of $\mathcal{C}(k)$.

6. k^α IS A QUADRUPLE PRINCIPAL VECTOR

$C_{\alpha\beta\gamma\delta}$ is of Petrov type N, l^α and m^α being chosen so that:

(6.1*a*) $$C_{\alpha\beta\gamma\delta}k^\delta = 0$$

(6.1*b*) $$C^5 = C^4 = C^3 = C^2 = 0 \qquad \nabla C^5 = \nabla C^4 = \nabla C^3 = \nabla C^2 = 0$$

$C_{\alpha\beta\gamma\delta}$ has its type preserved if and only if:

(6.2a) $$C'V_{\alpha\beta}k_\gamma\tau_\sigma + \bar{C}'\bar{V}_{\alpha\beta}k_\gamma\bar{\tau}_\sigma = 0$$

or equivalently:

(6.2b) $$2\tau_\sigma = k^\lambda\nabla_\sigma\bar{m}_\lambda + \bar{m}^\lambda\nabla_\sigma k_\lambda = 0$$

i. e. k^α and $\bar{m}^\alpha$ remain orthogonal.

On the other hand, the Bianchi identities give taking into account (6.2):

(6.3) $$k_\alpha P^{\alpha\beta\gamma} = -(C'V^{\gamma\sigma}v_\sigma + \bar{C}'\bar{V}^{\gamma\sigma}\bar{v}_\sigma)k^\beta$$

(6.4) $$\bar{m}_\alpha P^{\alpha\beta\gamma} = -(C'V^{\gamma\sigma}v_\sigma - \bar{C}'\bar{V}^{\gamma\sigma}\bar{v}_\sigma)\bar{m}^\beta + [C'V^{\gamma\sigma}\rho_\sigma + \bar{C}'(\bar{V}^{\gamma\sigma}\bar{\varepsilon}_\sigma + \nabla_\sigma\bar{V}^{\gamma\sigma}) + \bar{V}^{\gamma\sigma}\nabla_\sigma\bar{C}']k^\beta$$

whence, if:

(6.5) $$V_{\lambda\alpha}P^{\alpha\beta\gamma} = 0$$

(6.6) $$k^\sigma v_\sigma = 0 \qquad \bar{m}^\sigma v_\sigma = 0$$

From (6.2) and (6.6) we derive easily the geodesic and shear-free character of $\mathcal{C}(k)$; so, in the case of Petrov type N Weyl tensor, the (g. G. S.) type theorem results with weaker field equations than those assumed by Robinson and Schild *i. e.*

(6.7) $$V_{\lambda\alpha}\mathcal{J}^{\alpha\beta\gamma} = 0$$

7. NULL ELECTROMAGNETIC FIELDS IN AN ALGEBRAICALLY SPECIAL PETROV TYPE SPACE

In the light of the above considerations, it appears useful to investigate the properties of the null geodesic congruence $\mathcal{C}(k)$ of a null electromagnetic field $\varphi_{\alpha\beta}$, the principal null vector of which is at the same time a multiple principal vector of the Weyl tensor. These properties follow from the structure of the energy-momentum tensor of this field:

(7.1) $$T_{\alpha\beta} = tk_\alpha k_\beta \qquad T = 0$$

this tensor giving, by virtue of the Einstein's field equations:

(7.2a) $$P_{\alpha\beta\gamma} = t\omega_{\alpha\beta}k_\gamma - \Omega_{\alpha\beta}(tk_\gamma)$$

(7.2b) $$\omega_{\alpha\beta} = \nabla_\alpha k_\beta - \nabla_\beta k_\alpha, \qquad \Omega_{\alpha\beta} = k_\alpha\nabla_\beta - k_\beta\nabla_\alpha$$

whence we derive, using $\delta k^\alpha = 0$:

$$k_\alpha P^{\alpha\beta\gamma} = \delta t k^\beta k^\gamma \tag{7.3a}$$

$$\bar{m}_\alpha P^{\alpha\beta\gamma} = t\omega^{\alpha\beta}\bar{m}_\alpha k^\gamma + k^\beta \bar{m}^\sigma \nabla_\sigma (t k^\gamma) \tag{7.3b}$$

and

$$V_{\beta\alpha} P^{\alpha\beta\gamma} = 0 \tag{7.4}$$

i) *If* $C_{\alpha\beta\gamma\delta}$ *is of Petrov type II or D,* (1.5) and (4.4*b*) give:

$$e = 0 \tag{7.5}$$

$\mathcal{C}(k)$ is geodesic and shear-free.

ii) *If* $C_{\alpha\beta\gamma\delta}$ *is of Petrov type III,* (1.5) and (5.2) give again (7.5), *i. e.* the shear-free character of $\mathcal{C}(k)$.

iii) *If* $C_{\alpha\beta\gamma\delta}$ *is of Petrov type N,* (6.5) is no more true in general [*4*]. From (6.3), (6.4) and (7.3) we get:

$$C'\bar{e} = -t\bar{d} \tag{7.6}$$

and:

$$\frac{\delta t}{t} = -(d + \bar{d}) \tag{7.7}$$

The geodesic character of $\mathcal{C}(k)$ is compatible with (6.3), whereas its shear vanishes if $\varphi_{\alpha\beta}$ is covariantly constant.

8. DISCUSSION

The results of the present investigation may be summarized as follows: Let the null vector k^α be at each point of V_4, the multiple principal vector of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ and $\mathcal{C}(k)$ the congruence of null curves tangent to k^α.

i) When $C_{\alpha\beta\gamma\delta}$ is of Petrov typ II or D, $\mathcal{C}(k)$ is *a)* geodesic if $V_{\lambda\gamma}V_{\beta\alpha}\nabla_\sigma C^{\alpha\beta\gamma\sigma} = 0$, *b)* shear-free if either $V_{\lambda\gamma}V_{\beta\alpha}\nabla_\sigma C^{\alpha\beta\gamma\sigma} = 0$ or k^α is at the same time the principal vector of a null electromagnetic field $\varphi_{\alpha\beta}$ present in V_4:

ii) When $C_{\alpha\beta\gamma\delta}$ is of Petrov type III, (k) is *a)* geodesic if $V_{\beta\alpha}\nabla_\sigma C^{\alpha\beta\gamma\sigma} = 0$, *b)* shear-free if either $V_{\beta\alpha}\nabla_\sigma \mathcal{C}^{\alpha\beta\gamma\sigma} = 0$ or k^α is the common multiple principal vector of $\varphi_{\alpha\beta}$ and $\overset{*}{\varphi}_{\alpha\beta}$.

iii) When $C_{\alpha\beta\gamma\delta}$ is of Petrov type N, $\mathcal{C}(k)$ is geodesic and shear-free if $V_{\lambda\alpha}\nabla_\sigma C^{\alpha\beta\gamma\sigma} = 0$ However, when k^α is at the same time the common multiple principal vector of $\varphi_{\alpha\beta}$ and $\overset{*}{\varphi}_{\alpha\beta}$ $\mathcal{C}(k)$ is geodesic and no more shear-free; the interaction of the free gravitational field with the electromagnetic field becomes manifest as for instance in $C'\bar{e} = -\bar{t}d$

Conversely, a reciprocal theorem concerning these results can be easily established.

The above results are based on the view which seems to be a natural one, that in differential considerations the Weyl tensor alone is fundamental and not its dual (important as it is in algebraic considerations); this view appears to us as fundamental for studying the interaction of the algebraically different parts of the curvature tensor, and for investigating the properties of any quantity along a submanifold of V_4.

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