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**B. J. DAY**

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FURTHER CRITERIA FOR TOTALITY  
 by B.J. DAY

**Résumé.** Cet article est une suite de l'article de Kelly [2] sur l'étude de la totalité pour les catégories enrichies.

**Introduction.** This Note is a sequel to the Kelly survey [2] of totality for enriched categories and some familiarity with the latter is assumed. It is supposed throughout that  $V$  is a symmetric monoidal closed category with  $V_0$  admitting all small limits and arbitrary intersections of monics.

**Generators and totality.**

**THEOREM 1.** Any cocomplete category  $A$  is total if it admits arbitrary cointersections of epics and has a small generating set.

**PROOF.** It suffices to show that the coend  $\int^a fa \otimes a$  can be constructed in  $A$  from the generators  $G$  of  $A$ . First consider the pushout diagram of the canonical map  $1 \otimes \epsilon$  and  $k$  with  $1 \otimes \epsilon$  jointly epic since  $G$  generates  $A$  :

$$\begin{array}{ccc}
 fa \otimes A(g, a) \otimes g & \xrightarrow{k} & \int^a fg \otimes g \\
 1 \otimes \epsilon \downarrow & \text{p.o.} & \downarrow e_a \\
 fa \otimes a & \xrightarrow{\quad} & q_a
 \end{array}$$

This implies that each  $e_a$  is epic; then the pushout of all those epics over  $a$  in  $A$  is easily seen to be precisely  $\int^a fa \otimes a$  in  $A$ , as required.

**REMARK.** In the above result, epics can be replaced by the maps in  $E$  for any  $E$ - $M$ -factorization system on  $A$ ; a general result concerning limits of  $M$ -subfunctors can be found in the Lemma of [1], §3.

### The adjoint-functor Theorem and totality.

**THEOREM 2.** A category  $A$  is total iff it is complete with all intersections of [strong] monics and there exists a functor  $r$  from  $[A^{op}, V]$  to  $A$  and a natural [strong] monic  $\mu: 1 \rightarrow ry$ .

**PROOF.** Necessity is clear. For sufficiency consider the canonical diagram

$$\begin{array}{ccccc}
 fa & \longrightarrow & A(rya, rf) & \xrightarrow{A(\mu, 1)} & A(a, rf) \\
 \alpha_a \downarrow & & \downarrow A(1, r(\alpha)) & & \downarrow A(1, r(\alpha_a)) \\
 & & A(rya, ryb) & \xrightarrow{A(\mu, 1)} & A(a, ryb) \\
 & \nearrow & & \searrow & \\
 yb(a) & \xrightarrow{A(1, \mu)} & & & 
 \end{array}$$

The result now follows from the Adjoint-Functor Theorem [1].

As an application, consider the category  $A$  of coalgebras for a density comonad on a category  $B$ . If such an  $A$  is complete then it is total with no assumption on  $B$ .

### References.

1. B.J. DAY, On adjoint-functor factorization, *Lecture Notes in Math*, 420, Springer (1974), 1-19.
2. G.M. KELLY, A survey of totality for enriched and ordinary categories, *Cahiers Top et Géom, Diff*, XXVII-2 (1986), 109-132 (and references therein).

School of Mathematics and Physics  
 Macquarie University  
 NORTH RYDE, N.S.W., 2113  
 AUSTRALIA