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COMPOSING SPECIAL EPIMORPHISMS AND RETRACTIONS

by Horst HERRLICH and Lutz SCHRÖDER

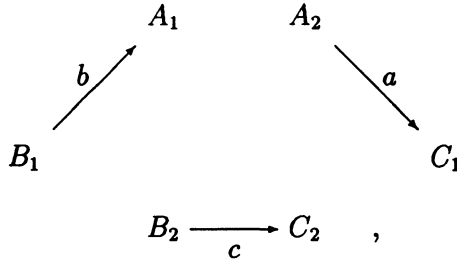
RESUME. Les auteurs démontrent que, dans la catégorie \mathbf{Cat} des catégories petites (qui est localement présentable), le composé d'un épimorphisme régulier et une rétraction n'est pas en général régulier, de même que, dans la catégorie des espaces connexes, le composé d'une rétraction et un épimorphisme régulier. En plus, ils introduisent une catégorie naturelle qui contient \mathbf{Cat} comme sous-catégorie pleine, et dans laquelle le composé d'un épimorphisme extrémal et une rétraction n'est pas en général extrémal.

It is well known that in the locally finitely presentable category \mathbf{Cat} of small categories and functors, regular epimorphisms are not stable under composition (cf. [1], 7.76). A question which naturally arises in this context (and which has a bearing on certain problems concerning the characterization of presentable objects in locally presentable categories, cf. [3]) is whether the composite of a regular epimorphism and a retraction (where "the composite of f and g " means gf) is again regular in \mathbf{Cat} .

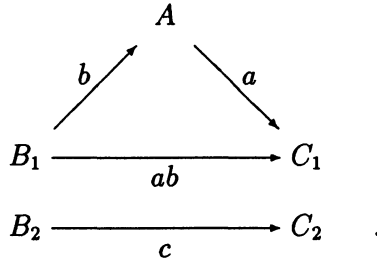
The corresponding general statement can be false even in otherwise quite well behaved categories; in the dual setting, a natural example of topological rather than algebraic nature can be found in [5]: In the category of functionally Hausdorff spaces, which is epireflective in \mathbf{Top} , the composite of two regular monomorphisms need not be regular; it is easily seen that in the given counterexample, the first factor is actually a section.

Now a counterexample of the "simplest conceivable accident" type shows that the same type of irregularity indeed occurs in \mathbf{Cat} :

Example 1 Let \mathbf{A} be the category



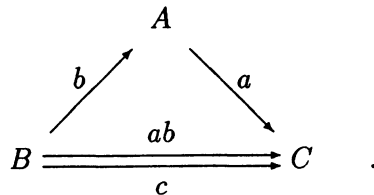
and let $F : \mathbf{A} \rightarrow \mathbf{B}$ be the regular epimorphism obtained by identifying the objects A_1 and A_2 , i.e. \mathbf{B} is the category



Furthermore, let \mathbf{C} be the subcategory of \mathbf{B} generated by a and b and let $R : \mathbf{B} \rightarrow \mathbf{C}$ be the retraction that sends c to ab . Then RF is not a regular epimorphism: The congruence relation \cong on \mathbf{A} induced by RF is generated by

$$A_1 \cong A_2, B_1 \cong B_2 \quad \text{and} \quad C_1 \cong C_2,$$

and in the corresponding quotient of \mathbf{A} , ab and c remain distinct:



Note that \mathbf{C} is even a full subcategory of \mathbf{B} . The example can be modified so that \mathbf{C} becomes instead an object-full subcategory of \mathbf{B} (and R

becomes bijective on objects); to this end, let \mathbf{A}' be the category obtained from \mathbf{A} by identifying B_1 with B_2 and C_1 with C_2 , and take the same two identification steps as above, starting with \mathbf{A}' instead of \mathbf{A} .

Remark 1 Among the more commonly used concepts of special epimorphisms are those of regular, strict, swell, strong, and extremal epimorphisms (where each property implies the following one); see e.g. [1]. In general, only the classes of swell epimorphisms and strong epimorphisms, respectively, are closed under composition; furthermore, it is easy to see that the composite of a strong epimorphism and an extremal epimorphism is always extremal, and that the composite of a retraction and a strict epimorphism is always strict (cf. [1], 7D). In most "reasonable" categories, e.g. in categories with pullbacks, the regular and the strict epimorphisms coincide (cf. [1], 12A), as do the strong and the extremal epimorphisms (cf. [1], 14C); in particular, the above example shows that the composite of a regular epimorphism and a retraction need not be strict.

The remaining questions of this type are settled by the following two examples, which show that the composite of a retraction and a regular epimorphism need not be regular, and that the composite of an extremal epimorphism and a retraction need not be extremal, respectively; taking into account the facts listed above, it is clear that these examples necessarily involve rather unpleasant, if natural, categories.

Example 2 In the category of nonempty connected topological spaces (which is closed under products and multicoreflective, hence closed under connected colimits, in **Top**), the composite of a retraction and a regular epimorphism need not be regular. To see this, let $r : [0, 3] \rightarrow [0, 2]$ (where the square brackets denote closed intervals on the real line) be the obvious retraction, and let $e : [0, 2] \rightarrow X$ be the regular epimorphism that identifies the points 0 and 1. The composite er is not regular: Assume that er is a coequalizer of a pair (f, g) of maps from a connected space Y into $[0, 3]$. W.l.o.g., there exists $y_0 \in Y$ such that $f(y_0) = 0$ and $g(y_0) = 1$; thus, the set

$$A = \{y \in Y \mid f(y) = 0 \text{ and } g(y) = 1\}$$

is closed, open (Each $y \in A$ has a neighborhood U such that $f(u) < \frac{1}{2}$ and $g(u) > \frac{1}{2}$, and hence $f(u) = 0$ and $g(u) = 1$, for each $u \in U$) and nonempty. By connectedness, $A = Y$, which is a contradiction.

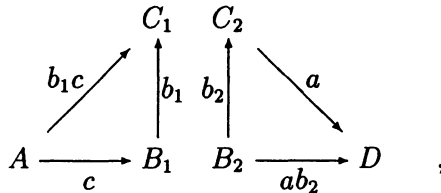
Example 3 Call a directed graph \mathbf{A} (i.e. a unary algebra with two operations c and d , subject to the equations $cc = dc = c$ and $dd = cd = d$) with a partial binary operation $(f, g) \mapsto fg$ such that

- (i) $df = cg$, $d(fg) = dg$, and $c(fg) = cf$ whenever fg is defined;
- (ii) cff and fdf are always defined and equal to f ;
- (iii) whenever fg and gh are defined, then $f(gh)$ and $(fg)h$ are defined and equal, or $fg = f$ and $gh = h$;

a (small) *weak semicategory*; a *functor* is a graph morphism $F : \mathbf{A} \rightarrow \mathbf{B}$ between two weak semicategories such that $FfFg$ is defined and equal to $F(fg)$ whenever fg is defined. The category of weak semicategories and functors is denoted by \mathbf{wSct} . Note that \mathbf{wSct} contains \mathbf{Cat} as a full subcategory; in fact, a semicategory is a category iff it satisfies

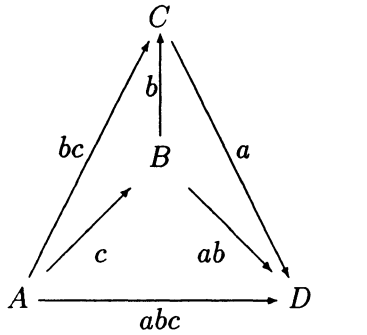
- (iii') $f(gh)$ and $(fg)h$ are defined and equal whenever fg and gh are defined.

In \mathbf{wSct} , the composite of an extremal epimorphism and a retraction need not be extremal: Let \mathbf{A} be the category



and let $F : \mathbf{A} \rightarrow \mathbf{B}$ be the functor that identifies b_1 and b_2 , i.e. \mathbf{B} is the

category



F is an extremal epimorphism in \mathbf{wSct} , since $F[A]$ is not contained in a proper weak sub-semicategory of \mathbf{B} . Note that F is regular in \mathbf{Cat} , but not in \mathbf{wSct} . Now let \mathbf{C} be the subcategory of \mathbf{B} generated by ab and c , and let $R : \mathbf{B} \rightarrow \mathbf{C}$ be the retraction that sends b to B (so that $R(bc) = c$ and $Ra = ab$). Then RF is not extremal in \mathbf{wSct} , since the image of \mathbf{A} in \mathbf{C} is a weak semicategory.

Remark 2 Weak semicategories as defined above can be regarded as special *multiplicative graphs* in the sense of C. EHRESMANN (cf. [2]). In fact, multiplicative graphs are just graphs with a partial operation satisfying axioms (i) and (ii) in the above definition, and the category of multiplicative graphs and functors, which is abstractly equivalent to a quasivariety, contains \mathbf{wSct} as a cone-injectivity class. Multiplicative graphs are used mainly as generating systems for categories; in [4], it is shown that the weak semicategories are characterized by a particularly pleasant property concerning the description of the generated category.

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