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1. Introduction.

Let X be a completely regular Hausdorff space; denote by $C(X)$ the ring of real valued continuous functions on X, by $Q(X)$ the maximal ring of quotients of $C(X)$ (for definition and properties of $Q(X)$, see [FGL]). A maximal ideal M of $Q(X)$ is called real when $Q(X)/M$ is isomorphic to the real field.

In a recent work [P], Park proved that if X is realcompact, separable, and perfectly normal, then the absence of isolated points in X implies the absence of real ideals in $Q(X)$.

In the same year 1969 work [H] appeared: Hager proved that $Q(X)$ is isomorphic to some $C(Y)$ if and only if the isolated points of X are dense in X (unless measurable cardinals are considered).

His techniques involve φ-algebras and their uniform completions. By pasting together three propositions from [H] (3.1, 2.2, 1.6), it can be shown that Park's result holds for « practically » all completely regular T_2 spaces. In other words, if X has non measurable cardinality, then every real ideal of $Q(X)$ is the ideal of all « functions » in $Q(X)$ which vanish on some isolated point of X.

In this paper, we shall give a direct proof of this fact. As a side result, we obtain also another proof of Hager's main result.

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2. Preliminaries.

Some basic facts are needed on $Q(X)$. They all can be found in [FGL]. The ring $Q(X)$ is considered here as the direct limit of the rings $C(V)$, V ranging over the dense open subsets of X, with restriction mappings as morphisms. It is well known that $Q(X)$ is a von Neumann regular ring, so that every element of $Q(X)$ is associated to an idempotent, and every ideal of $Q(X)$ is an intersection of maximal ideals. Moreover, in the maximal spectrum $\mathcal{M}(Q(X))$ of $Q(X)$, the sets

$$V(e) = \{M \in \mathcal{M}(Q(X)) \colon e \in M\}$$

are a clopen basis for $\mathcal{M}(Q(X))$ when e runs through the idempotents of $Q(X)$.

3. Results.

Let us start with a simple observation.

PROPOSITION. *Let p be an isolated point of X, and denote by e_p the characteristic function of $X \setminus \{p\}$ on X. Then $e_pQ(X)$ is a maximal ideal of $Q(X)$, and $Q(X)/e_pQ(X)$ is isomorphic to* $\mathbf{R}$.

PROOF. Since p belongs to every dense subset of X, the formula $\omega_p(g) = g(p)$, $g \in Q(X)$ defines a homomorphism $\omega_p \colon Q(X) \to \mathbf{R}$. Clearly ω_p is surjective and its kernel is $e_pQ(X)$, since $g(p) = 0$ is equivalent to $ge_p = g$, for all $g \in Q(X)$.

THEOREM. *Let M be a maximal ideal of $Q(X)$.*
Consider the following propositions:

(i) *M is an isolated point of $\mathcal{M}(Q(X))$;*

(ii) *M is generated by an idempotent of $Q(X)$;*

(iii) *M is generated by e_p, where e_p is the characteristic function (on X) of $X \setminus \{p\}$, p an isolated point of X;*

(iv) *$M \cap C(X) = M_p$, p an isolated point of X;*

(v) *$Q(X)/M$ is isomorphic to* $\mathbf{R}$.

Then: (i) *through* (iv) *are always equivalent and imply* (v). *If* $|X|$ *is non-measurable, then* (v) *implies* (iv) *and all five propositions are equivalent.*

PROOF. (i) is equivalent to (ii). Clearly, M is isolated in $\mathcal{M}(Q(X))$ if and only if $\{M\} = V(e)$, with e idempotent (see Sect. 2). This is equivalent to say that M is the unique maximal ideal of $Q(X)$ containing e; and since the principal ideal $e\,Q(X)$ is an intersection of maximal ideals (Sect. 2), we have that $eQ(X) = M$.

(ii) implies (iii) Assume that M is generated by the idempotent e; observe that $(1-e)Q(X)$ is a field, being isomorphic to $Q(X)/eQ(X)$. Suppose that $1-e \in C(V)$, V dense and open in X. Then $(1-e)Q(X)$ contains a copy of $(1-e)C(V)$, which is isomorphic to $C(e^{\leftarrow}[\{0\}])$, hence $C(e^{\leftarrow}[\{0\}])$ must be an integral domain, so that $e^{\leftarrow}[\{0\}]$ consists of a single point p, clearly isolated in V and hence in X. Obviously, $e = e_p$.

(iii) is equivalent to (iv) Since p is isolated in X, $M_p = e_p C(X) = (e_p Q(X)) \cap C(X)$; and $e_p Q(X)$ is maximal in $Q(X)$, by the preceding Proposition.

Trivially, (iii) implies (ii). From the above proposition, it follows that (iii) implies (v).

The remaining statement will be proved later on. At this point we are already in a position to prove Hager's result [H]:

Assume that $|X|$ *is nonmeasurable. Then* $Q(X)$ *is isomorphic to some* $C(Y)$ *if and only if the isolated points of* X *are dense in* X.

PROOF. The sufficiency is obvious: if the set Y of isolated point is dense in X, then Y is the smallest (open) dense subset of X, so that $Q(X) \simeq C(Y)$ canonically. Conversely, assume $Q(X)$ isomorphic to a $C(Y)$. Then, since $Q(X)$ is rationally complete, we have $Q(Y) = C(Y)$. By [FGL, 3.5] Y is an extremally disconnected P-space, which the nonmeasurability condition forces to be discrete. Then $\mathcal{M}(Q(X))$, identified with $\mathcal{M}(C(Y))$, has a dense subset $\{M_y : y \in Y\}$ of isolated points. By (i) and (iv) of the theorem, for every $y \in Y$ there exists an isolated point $p(y) \in X$ such that $M_{p(y)} = M_y \cap C(X)$. Thus, we obtain a subset $\{p(y) : y \in Y\}$ of isolated points of X, which is clearly dense in X, since $\bigcap_{y\in Y} M_y = \{0\}$ implies $\bigcap_{y\in Y} M_{p(y)} = \{0\}$. We return now to the proof of the last part of the theorem. This proof makes use of a suitable modification of an argument due to Isbell [GJ, 12H].

Let M be a real maximal ideal of $Q(X)$. Put $P = M \cap C(X)$. Then P is a prime ideal of $C(X)$, and $Q(X)/M = \mathbb{R}$ contains a copy of $C(X)/P$, hence also a copy of its field of fractions F; and this implies that P is a real maximal ideal of $C(X)$ since, otherwise, F is non-archimedean (see [GJ, 7.16]). Hence $P = M_p$ for a point $p \in X$ (since $Q(X) = Q(vX)$, we may provisionally suppose X realcompact). We want to prove that, if $|X|$ is nonmeasurable, then p is an isolated point of X. We assume that it is not and we get a contradiction. By Zorn's lemma, there exists a family $\mathcal{S}$ of pairwise disjoint open subsets of X, such that $p \in \mathrm{cl}_X S$, for every $S \in \mathcal{S}$, maximal with respect to these properties. Then $V = \cup\, \mathcal{S}$ is (open and) dense in X. For, otherwise, $X \diagdown \mathrm{cl}_X V$ is a nonempty open subset of X, which cannot be $\{p\}$ since, by assumption, p is nonisolated; and hence the family $\mathcal{S}$ can be enlarged, contradicting its maximality.

Let now $(S_\lambda)_{\lambda \in \Lambda}$ be a one-one indexing of $\mathcal{S}$ by a set Λ of cardinality $|\mathcal{S}|$. Consider Λ as a discrete space, and denote by ω the mapping from $C(\Lambda)$ into $Q(X)$ defined by $\omega(f) = \sum_{\lambda \in \Lambda} f(\lambda) e_\lambda$, where e_λ is the characteristic function of S_λ on V. Since $(S_\lambda)_{\lambda \in \Lambda}$ partitions V into disjoint open subsets, and V is open and dense in X, the e_λ's are orthogonal idempotents of $Q(X)$; by consequence ω is a ring homomorphism. Let π be the homomorphism of $Q(X)$ onto $\mathbb{R}$ whose kernel is M. Then $\pi \circ \omega$ is a (nonzero) ring homomorphism of $C(\Lambda)$ onto $\mathbb{R}$; and since $\cup\, |\Lambda| = |\mathcal{S}|$ is nonmeasurable (being not larger than $|X|$), Λ is realcompact, i.e., there exists $\xi \in \Lambda$ such that $\pi \circ \omega(f) = f(\xi)$ for all $f \in C(\Lambda)$. This shows that $1 - e_\xi$, which is the characteristic function of $\{S_\lambda\colon \lambda \in \Lambda \diagdown \{\xi\}\}$ on V, belongs to M. Take $g \in C(X)$ such that g is zero on S_ξ, and $g(p) \neq 0$ (this is possible, since, by construction, $p \notin \mathrm{cl}_X S_\xi$). Clearly, $g(1 - e_\xi) = g$ in $Q(X)$, since $g|V = g(1 - e_\xi)$. Thus $g \in M$, hence $g \in M \cap C(X) = M_p$. But $g(p) \neq 0$, a contradiction.

Corollary. *Let X be a Tychonoff space of nonmeasurable cardinal. Then the real ideals of $Q(X)$ are precisely the ideals $e_p Q(X)$, where e_p is the characteristic function of $X \diagdown \{p\}$, p an isolated point of X.*

REFERENCES

[FGL] N. J. Fine - L. Gillman - J. Lambek, *Rings of Quotients of Rings of Functions*, McGill University Press, Montreal, 1965.

[GJ] L. GILLMAN - M. JERISON, *Rings of Continuous Functions*, Van Nostrand, Princeton, N. J., 1960.

[H] A. W. HAGER, *Isomorphism with a $C(Y)$ of the maximal ring of quotients of $C(X)$*, Fund. Math., **66** (1969), 7-13.

[P] Y. L. PARK, *On quotient-like extensions of $C(X)$*, Canad. Math. Bull., **12**, no. 6 (1969).

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