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SINGULAR PERTURBATIONS FOR A CLASS
OF QUASI-LINEAR HYPERBOLIC EQUATIONS

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Résumé : Nous étudions le comportement pour $\varepsilon \rightarrow 0+$ de la solution d’un problème aux limites relatif à $\varepsilon L_2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + L_1 u_\varepsilon + G(u_\varepsilon) = f$ où $L_j$ ($j = 1, 2$) est un opérateur linéaire hyperbolique d’ordre $j$ et $G$ une fonction lipschitzienne.

Dans le cas "temporel" nous obtenons la convergence de $u_\varepsilon$ vers $u$ et des dérivées de $u_\varepsilon$ dans des espaces de Sobolev locaux où $u$ est la solution d’un problème aux limites relatif à $L_1 u + G(u) = f$.

Summary : We study the behavior for $\varepsilon \rightarrow 0+$ of the solution of a boundary value problem relative to $\varepsilon L_2 \frac{\partial^2 u_\varepsilon}{\partial t^2} + L_1 u_\varepsilon + G(u_\varepsilon) = f$ where $L_j$ ($j = 1, 2$) is a linear hyperbolic operator of order $j$ and $G$ a lipschitzian function.

In the "time like" case, we obtain the convergence of $u_\varepsilon$ to $u$ and of the derivatives of $u_\varepsilon$ in local Sobolev spaces where $u$ is the solution of a boundary value problem relative to $L_1 u + G(u) = f$.

We study a problem of singular perturbations for a class of hyperbolic quasi-linear partial differential equations which are of the type:

$$\varepsilon L_2 u_\varepsilon + L_1 u_\varepsilon + G(u_\varepsilon) = f$$

where $L_2 = \frac{\partial^2}{\partial t^2} - \Delta$, $L_1 = a \frac{\partial}{\partial t} + \sum_{k=1}^{n} b_k \frac{\partial}{\partial x_k}$ and $G: \mathbb{R} \rightarrow \mathbb{R}$ is a lipschitzian function.

In particular, this type of equation includes the Gordon's equation with damping. A similar non
linear problem has been studied by R. Geel [3] with a function \( G(x,t,v) \) whose derivative, with respect to \( v \), satisfies a Hölder condition with exponent \( \alpha > 0 \), the solutions being taken in the classical sense.

We consider the problem in the «time-like» case, that is: when operator \( L_1 \) divides operator \( L_2 \) in the sense of J. Leray [5], L. Garding [2]. The results of convergence are obtained in Sobolev spaces of local type and are analogous, with some supplementary results, to those established in the case when the non-linear term is \( G(v) = |v|^\rho \) [4]. Moreover the theory of non linear interpolation has the interest to give here a theorem of convergence with weakened assumptions.

The following is an outline of this work:

1. Notations hypotheses and two examples
2. Convergence of \( u_\varepsilon \) and \( L_1 u_\varepsilon \)
3. Convergence of the derivatives of \( u_\varepsilon \)
4. Application of the non linear interpolation
5. Some remarks about correctors.

1. NOTATIONS HYPOTHESES AND TWO EXAMPLES

\( \Omega \) is a bounded open set in \( \mathbb{R}^n \) of class \( \mathcal{C}^{1,1} \) (J. Necas [9]) with boundary \( \Gamma = \partial \Omega \).

We set \( Q = \Omega \times ]0,T[ \), \( T \) real > 0, \( \Sigma = \Gamma \times ]0,T[ \) and for every \( t \in [0,T] \), \( Q_t = \Omega \times ]0,t[ \), \( \Sigma_t = \Gamma \times ]0,t[ \).

We represent the norm of the usual Sobolev spaces, by:

\[
\|u\|_{L^p(\Omega)} = \|u\|_p \quad \|u\|_{H^1(\Omega)} = \|u\|_2
\]

\[
\|u\|_{L^p(Q)} = \|u\|_p \quad \|u\|_{L^2(0,T;H^1(\Omega))} = \|u\|_2
\]

and the inner product in \( L^2(\Omega) \) by \( \langle \cdot, \cdot \rangle \). We keep the same notation \( \langle \cdot, \cdot \rangle \) for the duality between \( L^p(\Omega) \), \( L^{p'}(\Omega) (\frac{1}{p} + \frac{1}{p'} = 1) \) and \( H^{-1}(\Omega), H^1_0(\Omega) \).

We note \( u', u'', \ldots \) the derivatives of \( u \) in the sense of vector-value distributions on \( ]0,T[ \) and \( \alpha(u,v) \) the bilinear form \( \int_{\Omega} \nabla u \cdot \nabla v \, dx \).

We consider the following initial boundary value problem:
The condition $H_1$ (iii) implies (see M. Marcus and V.J. Mizel [7]) the:

**LEMME 1.1, G' ((R) and for every $v \in H^1_0(\Omega)$, we have:

$$\left\{ \begin{array}{l}
\varepsilon L_2 u_{\varepsilon} + L_1 u_{\varepsilon} + G(u_{\varepsilon}) = f \\
\varepsilon \left( u'_{\varepsilon}, v \right) + \varepsilon a(u_{\varepsilon}, v) + (L_1 u_{\varepsilon}, v) + (G(u_{\varepsilon}), v) = (f, v) \\
\forall v \in H^1_0(\Omega), a.e \ in \ t \in ]0, T[ \\
\left\{ \begin{array}{l}
u_{\varepsilon} \in L^\infty(0, T ; H^1_0(\Omega)), u'_{\varepsilon} \in L^\infty(0, T ; L^2(\Omega)), \\
u_{\varepsilon}(x, 0) = u_0, \ u'_{\varepsilon}(x, 0) = u_1 \\
u_0, u_1, f \ given \ such \ that:
\end{array} \right.
\end{array} \right.$$ (1.4)

Taking into account hypothesis about $a, b_k$ and $f$, and lemma 1.1, one can show thanks to Galerkin's method (in the case $a, b_k = 0$ see J.C. Saut [10]), the

**THEOREM 1.2.** The problem $\mathcal{P}_e$ has a unique solution, for each $\varepsilon > 0$.

The variable coefficients $a, b_k$ and the function $G$ satisfy the hypothesis

$$H_1 \left\{ \begin{array}{l}
(i) \quad a, b_k \in W^{1, \infty}(\Omega) \cap C^0(\bar{\Omega}) \\
(ii) \quad \inf_{\bar{\Omega}} \ a(x, t) = \delta > 0 \\
(iii) \quad G : \mathbb{R} \rightarrow \mathbb{R} \ is \ a \ lipschitzian \ function \ i.e. : \\
\forall (\lambda, \mu) \in \mathbb{R}^2, \ |G(\lambda) - G(\mu)| \leq \ell \ |\lambda - \mu|, \ \ell \ \text{positive \ constant.}
\end{array} \right.$$ (1.5)

The condition $H_1$ (iii) implies (see M. Marcus and V.J. Mizel [7]) the:

**LEMME 1.1.** $G' \in L^\infty(\Omega)$ and for every $v \in H^1(\Omega)$, we have:

$$\frac{\partial}{\partial t} G(v) = G'(v)v' \in L^2(\Omega) \ and \ |G'(v)v'|_2 \leq \ell v_2$$

$$\frac{\partial}{\partial \lambda_k} G(v) = G'(v) \frac{\partial v}{\partial \lambda_k} \in L^2(\Omega) \ and \ |G'(v) \frac{\partial v}{\partial \lambda_k}|_2 \leq \ell \frac{\partial v}{\partial \lambda_k}_2 \ (k = 1, 2, ..., n)$$

**EXISTENCE AND REGULARITY OF THE SOLUTION $U_{\varepsilon}$ OF $\mathcal{P}_e$:**

Taking into account hypothesis about $a, b_k$ and $f$, and lemma 1.1, one can show thanks to Galerkin's method (in the case $a, b_k = 0$ see J.C. Saut [10]), the

**THEOREM 1.2.** The problem $\mathcal{P}_e$ has a unique solution, for each $\varepsilon > 0$. 
In fact there exists a solution as soon as $G$ is a Hölder function with exponent $\alpha$, $0 < \alpha < 1$.

**THEOREM 1.3.** Under hypothesis

\[ H_2 : H_1 \text{ with } u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \ u_1 \in H^1_0(\Omega), \ f \in L^2(Q) \]

for each $\varepsilon > 0$, there exists a unique solution to the problem $\mathcal{P}_\varepsilon$ such that:

\[ u_\varepsilon \in L^\infty(0; T; H^1_0(\Omega)) \cup H^2(\Omega)), \ u_\varepsilon' \in L^\infty(0; T; H^1_0(\Omega)), \ u^-_\varepsilon \in L^\infty(0; T; L^2(\Omega)). \]

In order to study the convergence, we have to introduce:

1. **The fundamental hypothesis:**

The results of convergence are obtained in the «time-like» case that is with the condition:

\[ (A) \quad \sum_{k=1}^n b_k^2(x, t) < \alpha^2(x, t) \quad \forall (x, t) \in Q \]

One can deduce from (A) the two properties:

If \[ \Phi(\xi_1, \xi_2, \ldots, \xi_n \xi_o) = \xi_o^2 + 2 \sum_{k=1}^n a^{-1} b_k \xi_k \xi_o + \sum_{k=1}^n \xi_k^2 \]

then \[ \Phi(\xi_1, \xi_2, \ldots, \xi_n \xi_o) \geq \frac{\omega}{2} \sum_{k=1}^n \xi_k^2 \text{ where } \omega = \inf \left(1 - \sum_{k=1}^n a^2 b_k^2\right) \]

For every functions $v \in L^2(Q), \ \theta \in C^0(\bar{Q}), \ \theta \geq 0$, such that $\theta \parallel \text{grad} \ v \parallel$ and $\theta v' \in L^2(Q)$, we have:

\[ \int_0^t \left(\parallel \theta \parallel \text{grad} \ v \parallel^2 - \parallel \theta v' \parallel^2 \right) ds \geq -\omega_1 \int_0^t \left(\parallel \theta_L \parallel \text{grad} \ v \parallel^2 \right) ds + \frac{3 \omega_1}{4} \int_0^t \left(\theta \parallel \text{grad} \ v \parallel \right) ds \quad (1.6) \]

where the positive constant $\omega_1$ depends only on the coefficients.

2. **Weight functions:**

Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n, 0)$ the unit normal outward vector to $\Sigma$ when it exists.

We represent by $\Lambda$ the null-subset of $\Sigma$ where $\nu$ is not defined, and by $\Sigma_-, \Sigma_+, \Sigma_\circ$, the subsets of $\Sigma - \Lambda$ corresponding respectively to:

\[ \sum_{k=1}^n b_k \nu_k < 0, \sum_{k=1}^n b_k \nu_k > 0, \sum_{k=1}^n b_k \nu_k = 0. \]

Under the hypothesis $H_1$ (ii) and the assumption:
H'$_{1}$: \( L_{1} \) is a vector-field of class \( C^{1} \) on an open set of \( \mathbb{R}^{n+1} \) which contains \( \overline{Q} \), we may use functions \( \varphi \) (F. Mignot and J.P. Puel [8]) satisfying the condition:

\[
\varphi \in C^{0}(Q) \cap W^{1,\infty}(Q), \quad 0 \leq \varphi \leq 1
\]

\[\varphi = 0 \text{ in a neighbourhood of } \Sigma_{+} \text{ in } \Sigma \]

\[L_{1} \varphi \leq 0 \text{ on } Q.\]

These functions are such that:

There is a null-set \( Z \subset Q \) such that \( V(x,t) \in (Q-Z) \cup E_{+} \), there exists a function \( \varphi \) satisfying \( \mathcal{A}_{1} \) such that \( \varphi(x,t) \neq 0 \).

For each compact \( K \subset \overline{Q} \) with \( K \cap \mathcal{Y}(\Sigma_{+}) = \varnothing \), where \( \mathcal{Y}(\Sigma_{+}) \) denotes a neighbourhood of \( \Sigma_{+} \) in \( \Sigma \), there exists a function \( \varphi \) satisfying \( \mathcal{A}_{1} \) such that

\[\varphi(x,t) \geq m > 0 \text{ on } K.\]

**GREEN'S FORMULA FOR OPERATOR \( L_{1} \):**

Under the hypothesis \( H'_{1} \) and the condition

(B): \( \partial \Sigma_{-} \) is a finite reunion of \((n-1)\) dimensional \( C^{1} \) submanifolds

\[\forall w \in L^{2}(Q_{t}) \text{ such that } L_{1} w \in L^{2}(Q_{t}), \quad w|_{\Sigma_{-}} = 0, \quad w(x,0) = u_{0}, \quad \text{we have:} \]

\[
\int_{0}^{t} (L_{1}w,w)ds = \frac{1}{2} \int_{\Omega} a \frac{\partial w}{\partial x} dx - \frac{1}{2} \int_{\Omega} a(x,0)u_{0}^{2} dx + \frac{1}{2} \int_{\Sigma_{+}} \left( \sum_{k=1}^{n} b_{k} \frac{\partial w}{\partial x_{k}} \right) w^{2} d\Gamma ds
\]

\[- \frac{1}{2} \int_{Q_{t}} (\alpha' + \sum_{k=1}^{n} \frac{\partial b_{k}}{\partial x_{k}}) w^{2} dx ds \quad (1.7)\]

We will start the study of the general case investigated below with an illustration through two simple examples which are Klein-Gordon equation with \( G(u) = \sin u \).

EXEMPLE 1. We consider the problem \( \mathcal{P}_{\varepsilon} \) where \( \Omega \) is the square \([0,1[ \times \times \times ]0,1[ \) in \( \mathbb{R}^{2} \) (fig. 1)
b constant with $0 < b < 1$.

We note $L_1$ the operator of first order $u \mapsto L_1 u = u' + b \frac{\partial u}{\partial x}$.

We have seen in the general case that if $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $f \in L^2(Q)$ the problem $\mathcal{S}_\epsilon$ has a unique solution $u_\epsilon$ for all $\epsilon > 0$, such that $L_{\infty}(0,T;H_0^1(\Omega))$, $u_\epsilon' \in L_{\infty}(0,T;L^2(\Omega))$.

For the weight functions and the limit problem the subsets of the boundary taken into account are:

$$
\Gamma_- = \{(x,y); x = 0, 0 < y < 1\}, \quad \Gamma_+ = \{(x,y); x = 1, 0 < y < 1\}
$$

$$
\Gamma_o = \{(x,y); 0 < x < 1, y = 0\} \cup \{(x,y); 0 < x < 1, y = 1\}
$$

and $\Sigma_- = \Gamma_- \times [0,T]$, $\Sigma_o = \Gamma_o \times [0,T]$.

(We remark that the subset $\Lambda$ of $\Sigma$ where the outward normal is not defined is composed of the four edges, $00'$, $AA'$, $BB'$ and $CC'$).

The weight functions $\varphi$ satisfy:

$$
\varphi \in C^2(\overline{\Omega}) \cap W^{1,\infty}(\Omega), \quad 0 \leq \varphi \leq 1 \text{ on } \Omega
$$

$$
\mathcal{A}_1 \left\{ \begin{array}{l}
\varphi(1,y) = 0, \quad 0 \leq y \leq 1 \\
\frac{\partial \varphi}{\partial x} \leq 0 \text{ on } \Omega
\end{array} \right.
$$

Let $\varphi(x,y) = 1 - x$.

Obviously $\varphi$ satisfies the condition $\mathcal{A}_1$ and we have here the fact $\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$. Moreover, for each $\gamma$, $0 < \gamma < 1$, $\varphi(x,y) > \gamma$ on $\Omega_\gamma$ where $\Omega_\gamma = ]0.1 - \gamma[ \times ]0,1[.$

The limit problem is here given by:

$$
\mathcal{S} \left\{ \begin{array}{l}
u' + b \frac{\partial u}{\partial x} + \sin u = f \\
u(x,y,0) = u_0 \\
u(x,y,t) = 0 \text{ on } \Sigma
\end{array} \right.
$$

$\mathcal{S}$ has a unique solution such that $u \in L_{\infty}(0,T;L^2(\Omega))$ and $L_1 u \in L^2(Q)$. 

Then, with the use of the function $\varphi$ the results of convergence are

(i) For $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, the solution $u_\varepsilon$ converges to $u$ in $L^\infty(0,T;L^2(\Omega))$ weak-star and in $L^q(\Omega)$, $\forall q < 2$. Moreover $u_\varepsilon$ converges to $u$ in $L^\infty(0,T;L^2(\Omega_\gamma))$ and $L^1 u_\varepsilon$ converges to $L^1 u$ in $L^2(\Omega;L^2(\Omega_\gamma))$, $\forall \gamma \in ]0,1[$.

Besides for $u_\varepsilon'$ we have : $u_\varepsilon'$ converges to $u'$ in $L^\infty(0,T;L^2(\Omega))$ weak star

(ii) If we take $u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega), f \in L^2(0,T;H^1(\Omega))$ such that $f(0,y,t) = f(1,y,t) = 0$ and $f' \in L^2(Q)$ we can state that $u_\varepsilon$ converges to $u$ in $H^1(\Omega_\gamma)$ where $\Omega_\gamma = \Omega \times ]0,1[$, and we have the estimation :

$$|u_\varepsilon - u|_{L^2(\Omega_\gamma)} \leq K_\gamma \varepsilon^{1/2}$$

where the constant $K_\gamma$ can be written $K_\gamma = C \gamma^3$ with $C$ constant independent of $\varepsilon$ and $\gamma$.

Let now, $\varphi(x,y) = (1-x)(1-y)y$.

This new function $\varphi$ satisfies the condition $\partial_1$ and is such that :

$\varphi(x,y) > 0$ for $(x,y) \in \Omega \cup \Gamma_-$ and for each $\gamma$, $0 < \gamma < 1$ ; or each $\gamma$, $0 < \gamma < 1$ ;

$\varphi(x,y) > \gamma^2(1-\gamma)$ on the open subset of $\Omega : ]0,1-\gamma[ \times ]1-\gamma,1-\gamma[$.

Moreover this function has the supplementary property $\varphi(x,y) = 0$ on $\Gamma_\varepsilon$.

Then by the use of this more particular function, we obtain the result of convergence and the estimation of the point (ii), without the condition $f(x,y,t) = 0$ on $\Sigma_\varepsilon$, but with $\Omega_\gamma$ replaced by $]0,1-\gamma[ \times ]1-\gamma,1-\gamma[$, $Q_\gamma$ by $]0,1-\gamma[ \times ]1-\gamma,1-\gamma[ \times ]0,1[$.

EXEMPLE 2. We take the same problems $\partial_\varepsilon$ and $\partial$ but we consider here the open set $\Omega = \{(x,y) \in \mathbb{R}^2 ; (x-1)^2 + y^2 < 1\}$ (fig. 2).

Fig. 2
Then, $\Gamma_- = \{(x,y) ; x = 1 - (1-\gamma^2)^{1/2}, \ -1 < \gamma < 1\}$, $\Sigma_- = \Gamma_- \times [0,T]$ 

$\Gamma_+ = \{(x,y) ; x = 1 + (1-\gamma^2)^{1/2}, \ -1 < \gamma < 1\}$, $\Sigma_+ = \Gamma_+ \times [0,T]$ 

and $\Sigma_0$ is composed of the two generating lines AA' and BB'.

The weight function we will use, is:

$$
\varphi(x,y) = \begin{cases} 
1 - \gamma^2 & \text{if } x \leq 1 \\
1 - (x-1)^2 - \gamma^2 & \text{if } 1 \leq x \leq 2 
\end{cases}
$$

It is such that: $\varphi(x,y) = 0$ on $\Gamma_-$, $\varphi(x,y) > 0$ on $\Omega \cup \Gamma_-$ and for each $\gamma$, $0 < \gamma < 1$, $\varphi(x,y) > \frac{\gamma^2}{4}$ on $\Omega_\gamma$ where $\Omega_\gamma = \Omega \cap \{(x,y) ; (x-1+\gamma)^2 + y^2 < 1\}$ (fig. 3)

![Fig. 3](image-url)

We have the same results as in example 1, point (i). Because of the fact: $\varphi(x,y) = 0$ on $\Gamma_-$, we have

if we take $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $u_1 \in H^1_0(\Omega)$, $f \in L^2(0,T; H^1(\Omega))$ and $f_\epsilon \in L^2(Q)$, $u_\epsilon$ converges to $u$ in $H^1(Q_\gamma)$, $\forall \gamma \in [0,1[$, where $Q_\gamma = \Omega_\gamma \times [0,T[$.

Moreover the use of the interpolation theory can improve the results in the following way:

if $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $f \in L^2(0,T; H^s(\Omega))$, $0 \leq s \leq 1$, we have the estimation
2. CONVERGENCE OF $u_\varepsilon$ AND $L^1 u_\varepsilon$

In this section we obtain under the hypothesis $H_1$ and the condition (A) the convergence of the solution $u_\varepsilon$ of the problem $\mathcal{P}_\varepsilon$ to $u$ solution of the problem.

2.1 - A PRIORI ESTIMATES

THEOREM 2.1.1. We assume condition (A); then $\exists \varepsilon_0 > 0$ such that $\forall \varepsilon < \varepsilon_0$ the solution $u_\varepsilon$ of the problem $\mathcal{P}_\varepsilon$ satisfies:

$$\|u_\varepsilon\|_2 + \sqrt{\varepsilon} \|u'_\varepsilon\|_2 + \sqrt{\varepsilon} \|u_\varepsilon\|_2 \leq C_1 K_1(f; u_0; u_1; \varepsilon)$$

with $K_1^2(f; u_0; u_1; \varepsilon) = \|f\|_2^2 + \|u_0\|_2^2 + \varepsilon^2 \|u_0\|_2^2 + \varepsilon^2 \|u_1\|_2^2 + (G(0))^2$

Preuve. We take off the method used in [4] theorem 2.1.

With assumption $H_2$:

Then we can make $v = u_\varepsilon + 2 \varepsilon a^{-1} u'_\varepsilon$ in (1.4). With the same transformations as in [4] for the linear terms and taking into account that the nonlinear terms are bounded as follows

$$\|G(u_\varepsilon), u_\varepsilon\| \leq (\varepsilon + 1) \|u_\varepsilon\|_2^2 + \|G(0)\|_2^2 \mes(\Omega)$$
we obtain the statement.

With assumption \( H_1 \):

We use a method of approximation. We consider a family satisfying hypothesis \( H_2 \), such that

\[
\left( f_\mu ; u_{0,\mu} ; u_{1,\mu} \right) \to \left( f, u_0 u_1 \right) \text{ in } L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)
\]

Then \( u_{\epsilon,\mu} \) converges to \( v_{\epsilon} \) in \( L^\infty(0,T ; H^1_0(\Omega)) \) weak star, \( u'_{\epsilon,\mu} \) converges weakly to \( v'_{\epsilon} \) in \( L^2(\Omega) \), \( u''_{\epsilon,\mu} \) converges weakly to \( v''_{\epsilon} \) in \( L^2(0,T ; H^{-1}(\Omega)) \).

As \( u_{\epsilon,\mu} \) converges to \( v_{\epsilon} \) in \( L^2(\Omega) \), \( G(u_{\epsilon,\mu}) \) converges to \( G(v_{\epsilon}) \) in \( L^2(\Omega) \).

Hence we can take the limit with respect to \( \mu \) in the equation satisfied by \( u_{\epsilon,\mu} \) and in boundary conditions and initial datas.

We deduce that \( v_{\epsilon} = u_{\epsilon} \) which gives us the estimates of the theorem.

The estimates on the derivatives of \( u_{\epsilon} \) are not sufficient to conclude about the behavior of \( u_{\epsilon} \) as \( \epsilon \to 0^+ \). Under the assumptions of this section, they may be improved by an estimate of \( \sqrt{\varphi} L_1 u_{\epsilon} \) independent of \( \epsilon \), the weight function \( \varphi \) being introduced in order to compensate the behavior of the derivatives of \( u_{\epsilon} \), in a neighborhood of the surface defining the boundary layer.

**THEOREM 2.1.2.** Under assumption \( H'_1 \) and condition \( (A) \), for each function \( \varphi \) satisfying \( \mathcal{A}_1 (i) \), the solution \( u_{\epsilon} \) of problem \( \mathcal{P}_\epsilon \) verifies:

\[
\forall \epsilon \in ]0,\epsilon_0[, \varphi L_1 u_{\epsilon} L^2 + \sqrt{\epsilon} \varphi u_{\epsilon} L^2 \leq C_2 K_1 (f, u_{0,1}, \sqrt{\epsilon})
\]

**Proof:** One can easily check as for theorem 2.1.1 that it is sufficient to show theorem 2.1.2. under hypothesis \( H_2 \).

Then we take the inner product of the two members of (1.1) with \( \varphi L_1 u_{\epsilon} \).

We transform the linear terms as in [4] theorem 2.3, the nonlinear term is bounded by:

\[
\int_0^t (G(u_{\epsilon}), \varphi L_1 u_{\epsilon}) ds \leq \int_0^t \epsilon \varphi L_1 u_{\epsilon} L^2 \sqrt{\epsilon} L_1 u_{\epsilon} L^2 ds + \int_0^t |G(0)| L^2 \varphi L_1 u_{\epsilon} L^2 ds
\]

\[
\leq k_3 K^2_1 (f, u_{0,1}, \epsilon) + \frac{1}{4} \int_0^t \frac{1}{\sqrt{\epsilon}} L_1 u_{\epsilon} L^2 ds
\]
and theorem 2.1.2 follows.

At last, with the additional hypothesis $H_2$, we can obtain an estimate of $u'_\epsilon$ in $L^\infty(0,T; L^2(\Omega))$ which is independent of $\epsilon$, by the method of differential ratios.

**THEOREM 2.1.3.** With assumptions $H_1', H_2$, condition (A) and the coefficients $b_k$ independent of $t$; for each $\epsilon > 0$, $0 < \epsilon < \epsilon_0$, the solution $u_\epsilon$ of $\mathcal{P}_\epsilon$ verifies

$$\| u'_\epsilon \|_2 + \sqrt{\epsilon} \| u'_\epsilon \|_2 + \sqrt{\epsilon} \| u''_\epsilon \|_2 \leq C_3 K_3 (f, u_0, u_1, \epsilon)$$

where $K_3^2 (f, u_0, u_1, \epsilon) = \| f \|_2^2 + \| u_0 \|_2^2 + \| u'_1 \|_2^2 + \| u''_0 \|_2 + \| u'_0 \|_2 + |G(0)|^2 + |f(0)|^2$.

**Proof.** We use a method of differential ratios. We consider equality (1.4) with $v \in H_0^1(\Omega)$, at time $s$ and $s + h$ ($h > 0$).

We set $w_{\epsilon,h}(s) = \frac{1}{h} [u_\epsilon(s+h) - u_\epsilon(s)]$ and throughout the proof the constants $k_j$ are moreover independent of $h$.

By subtracting the two equalities, we have:

$$e (w'_{\epsilon,h}, v) + e (\alpha (w_{\epsilon,h}, v) + (a(s+h) w_{\epsilon,h}, v) + \frac{a(s+h) - a(s)}{h} u'_\epsilon(s), v) + \sum_{k=1}^n (b_k \frac{\partial w_{\epsilon,h}}{\partial x_k}, v)$$

$$+ \frac{1}{h} (G(u_\epsilon(s+h)) - G(u_\epsilon(s)), v) = \frac{1}{h} (f(s+h) - f(s), v)$$

By taking $v = w_{\epsilon,h} + 2\epsilon a^{-1}(s+h) w'_{\epsilon,h}$ and integrating from 0 to $t$, we obtain as in the first part of theorem 2.1.1:

$$\frac{\epsilon^2}{4} \delta_o \| w'_{\epsilon,h} \|_2^2 + e^2 \delta_o \| a(w_{\epsilon,h}, w_{\epsilon,h}) \| + \frac{e \omega}{8} \int_0^t (\| w'_{\epsilon,h} \|_2^2 + \alpha (w_{\epsilon,h}, w_{\epsilon,h})) ds + \delta \| w_{\epsilon,h} \|_2^2$$

$$\leq K_e (h) + k_o \int_0^t \| u'_\epsilon(s) \|_2 ds + k_1 \int_0^t \| w_{\epsilon,h} \|_2 ds$$

where $K_e(h) = e (w'_{\epsilon,h}(0), w_{\epsilon,h}(0)) + \frac{1}{2} \| \sqrt{\alpha(x,0)} w_{\epsilon,h}(0) \|_2^2 + e^2 \| \frac{a^{-1}(x,0) w'_{\epsilon,h}(0)}{2} \|_2^2 + \frac{f(s+h) - f(s)}{h} \| v \|_2 ds$.

$\delta_o = \inf_{Q} a^{-1}(x,t)$

and where the nonlinear term has been bounded as follows:

$$\| \frac{1}{h} \int_0^t (G(u_\epsilon(s+h)) - G(u_\epsilon(s)), v) ds \| \leq k_2 \int_0^t \| w_{\epsilon,h}(s) \|_2^2 ds + k_3 \epsilon^2 \int_0^t \| w'_{\epsilon,h}(s) \|_2^2 ds$$
Thanks to (1.1), one can see that $u_\varepsilon'(0)$ is bounded in $L^2(\Omega)$ independently of $\varepsilon$ and so that:

$$K_\varepsilon(h) \leq k_4 \mathcal{K}_3^2(f, u_0, u_1, \varepsilon)$$

for small $h$.

Then (2.1) implies:

$$\frac{\delta}{6} \| u_\varepsilon(h) \|_2^2 \leq k_4 \mathcal{K}_3^2(f, u_0, u_1, \varepsilon) + k_0 \int_0^T \| u_\varepsilon'(s) \|_2^2 \, ds + k_1 \int_0^T \| u_\varepsilon(s) \|_2^2 \, ds$$

from which we deduce, by Gronwall's lemma:

$$\int_0^T \| u_\varepsilon(h) \|_2^2 \, ds \leq k_5 \mathcal{K}_3^2(f, u_0, u_1, \varepsilon) + \int_0^T \| u_\varepsilon'(s) \|_2^2 \, ds. \quad (2.2)$$

It results from (2.1) and (2.2) that a subsequence $w_{\varepsilon,h} \equiv u_\varepsilon'$ converges to $u_\varepsilon$ weakly in $L^2_0(0, T; L^2(\Omega))$, weakly in $L^2(0, T; H^1_0(\Omega))$, and strongly in $L^2(\Omega)$.

$$w_{\varepsilon,h} \rightharpoonup u_\varepsilon \quad \text{weakly in } L^2(\Omega)$$

$$w_{\varepsilon,h} \rightharpoonup u_\varepsilon' \quad \text{weakly in } L^2(\Omega)$$

$$w_{\varepsilon,h} \rightharpoonup u_\varepsilon' \quad \text{weakly in } L^2(\Omega)$$

Consequently, by taking the limit with respect to $h$ in (2.1), we have:

$$\frac{\varepsilon \omega}{8} \int_0^t (\| u_\varepsilon'' \|_2^2 + \| u_\varepsilon' \|_2^2) \, ds + \frac{\delta}{6} \| u_\varepsilon' \|_2^2 \leq k_6 \mathcal{K}_3^2(f, u_0, u_1, \varepsilon) + k_7 \int_0^t \| u_\varepsilon'(s) \|_2^2 \, ds$$

Theorem 2.1.3 follows thanks to Gronwall's lemma.

### 2.2 - CONVERGENCE

#### 2.2.1 - FIRST RESULTS OF CONVERGENCE:

We assume in all this subsection hypotheses $H_1$, $H_2$, conditions (A) and (B). The solution $u_\varepsilon$ of $\mathcal{P}_\varepsilon$ satisfies the estimates of theorems 2.1.1, 2.1.2. Moreover we deduce from (1.1) that for each function $\varphi$ satisfying conditions $\mathcal{O}_1(i)$ and for $\varepsilon < \varepsilon_0$:

$$\varepsilon \| \sqrt{\varphi} \|_{L^2(0, T; L^2(\Omega))} \leq k_0 K_1(f, u_0, u_1, \sqrt{\varepsilon})$$

Then, we can extract a subsequence, still written $u_\varepsilon$, such that:

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^\infty(0, T; L^2(\Omega))$$

$$\sqrt{\varphi} \rightharpoonup \sqrt{\varphi} L_1 u \quad \text{weakly in } L^2(\Omega)$$

$$\varepsilon \sqrt{\varphi} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega)$$

$$G(u_\varepsilon) \rightharpoonup \chi \quad \text{in } L^\infty(0, T; L^2(\Omega))$$

$$\begin{cases}
\varepsilon \rightarrow 0 \\
\sqrt{\varphi} \rightarrow \sqrt{\varphi} L_1 u \\
\varepsilon \sqrt{\varphi} \rightarrow 0 \\
G(u_\varepsilon) \rightarrow \chi
\end{cases} \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{weak-star} \quad \text{for } \varepsilon < \varepsilon_0 \quad (2.3)$$
Where \( u \) verifies ([4], section 3)

\[
\begin{aligned}
L_1 u + \chi &= f \\
u(x,0) &= u_0, \quad u \mid \Sigma = 0.
\end{aligned}
\]

It's remains to prove that \( \chi = G(u) \), which can be established by a monotonicity method ([4], section 3), by noting that we can write \( G(u) = - (\ell + 1) \chi + Mu \) where \( M \) is a strictly monotone and hemicontinuous operator (the monotonicity method is used in [4] when the operator \( L_1 - (\ell + 1) I \) is positive; we are brought back to this case by the change of variable \( U_e = u e^\lambda t \), the constant \( \lambda \) being chosen such that the new first order linear operator is positive. We remark that the new nonlinear function is defined by \( G(U_e) = e^\lambda t G(u_e e^\lambda t) \) and verifies

\[
|G(U) - G(V)| \leq \epsilon |U - V|
\]

We can then apply the monotonicity method to the function \( U_e \) which satisfies the same properties of regularity and the same estimates as \( u_e \), because:

\[
U_e = u_e e^\lambda t
\]

So \( u \) is solution of the problem

\[
P \begin{cases}
L_1 u + G(u) = f \\
u \in L^\infty(0,T;L^2(\Omega)) \\
u(x,0) = u_0, \quad u \mid \Sigma = 0
\end{cases}
\tag{2.4}
\]

Remark. It results from (2.4) that \( L_1 u \in L^\infty(0,T;L^2(\Omega)) \). Moreover it is easy to see that \( u \) is unique, thanks to Green's formula (1.7).

Hence, we have the

**LEMMA 2.2.1.** (weak convergence) With assumptions \( H_1^\ell, H_2^\ell \) conditions (A), (B), the solution \( u_e \) of \( P_e \) converges to the solution of problem \( P \) in \( L^\infty(0,T;L^2(\Omega)) \) weak star.

Moreover \( \sqrt{\varphi} L_1 u_e \) converges to \( \sqrt{\varphi} L_1 u \) weakly in \( L^2(Q) \) and if \( b_k' = 0 \) \( (k=1,2,\ldots,n) \) \( u_e' \) converges to \( u' \) in \( L^\infty(0,T;L^2(\Omega)) \) weak star.

Our aim is now to obtain some results of strong convergence.

**LEMMA 2.2.2.** With the hypotheses of lemma 2.2.1, the solution \( u_e \) of problem \( P_e \) verifies \( \sqrt{\varphi} u_e \) converges to \( \sqrt{\varphi} u \) in \( L^\infty(0,T;L^2(\Omega)) \) for each function \( \varphi \) satisfying \( \mathcal{A}_T \).
**Proof.** We consider $w_e = v_e - u; w_e$ satisfies

\[
\begin{aligned}
& e \int_{\Omega} \left( u''_e \phi u_e \right) + \alpha(u_e, \phi u_e) \right) \, \mathrm{d}s + \frac{1}{2} \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 - \frac{1}{2} \int_{\Omega_t} (L_1 \phi) w_e^2 \, \mathrm{d}x \, \mathrm{d}s \\
= & \int_0^t \left( e \sqrt{\psi} L_2 u_e \phi \right) u_e \, \mathrm{d}s + \frac{1}{2} \int_{\Omega_t} (a' + \sum_{k=1}^n \frac{\partial b_k}{\partial x_k}) \phi w_e^2 \, \mathrm{d}x \, \mathrm{d}s - \int_0^t (G(u_e) - G(u)) \phi w_e \, \mathrm{d}s \\
\end{aligned}
\]

We can take the inner product of the two members of (2.5) with $\phi w_e \in L^\infty(0, T; L^2(\Omega))$. After integration from 0 to $t$, it comes:

\[
\int_0^t \left\{ (u''_e \phi u_e) + \alpha(u_e, \phi u_e) \right) \right| \, \mathrm{d}s + \frac{1}{2} \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 - \frac{1}{2} \int_{\Omega_t} (L_1 \phi) w_e^2 \, \mathrm{d}x \, \mathrm{d}s
\]

and into account (1.6) with $\theta = \sqrt{\psi}$, $L_1 \phi \leq 0$,

\[
\| G(u_e) - G(u) \| \leq \| w_e \| : \\
\left\{ (u''_e, \phi u_e) + \alpha(u_e, \phi u_e), \phi u_e \right) \right\} \, \mathrm{d}s
\]

and integrating by parts the term

\[
e \int_0^t \left( e \sqrt{\psi} L_2 u_e, \phi u_e \right) u_e \, \mathrm{d}s + \frac{1}{2} \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 - H_e(t) + k_1 \int_0^t \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 \leq H_e(t) + k_1 \int_0^t \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 \\
+ k_2 \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 + \left| \sqrt{\psi} \right| u_e \right| L^2_{\Omega} \right|^2 + e \left( u_1, \phi(x, 0) u_0 \right) + \omega \left| \sqrt{\psi} \right| L_1 u_e \right| L^2_{\Omega} \right|^2 \\
\leq \int_0^t \left( e \sqrt{\psi} L_2 u_e, \phi u_e \right) u_e \, \mathrm{d}s + \frac{1}{2} \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 + k_3 \left| \sqrt{\psi} \right| u_1 \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2
\]

thanks to the theorems 2.1.1, 2.1.2.

From (2.6) we deduce that:

\[
\int_0^t \left| \sqrt{\psi} \right| w_e \right| L^2_{\Omega} \right|^2 \leq k_4 \int_0^t H_e(s) \, \mathrm{d}s.
\]

As (2.3) implies: $H_e(s)$ bounded by a constant independent of $\epsilon$ and $\lim_{\epsilon \to 0} H_e(s) = 0$, lemma 2.2.2 follows thanks to Lebesgue's theorem.

The following lemma gives a result of convergence for $\phi L_1 u_e$.

**Lemma 2.2.3.** We assume hypotheses $H_1, H_2$ and conditions (A), (B). Then, for each function $\phi$ satisfying condition $\phi_1$, the solution $u_e$ of $\mathcal{P}_e$ verifies:

\[
\phi L_1 u_e \to \phi L_1 u \text{ in } L^2(Q)
\]

**Proof.** We consider the inner product of the two members of (2.5) with $\phi L_1 w_e \in L^2(Q)$ and we integrate
from 0 to t. We obtain:

\[ \int_0^t (L_2 u_\varepsilon, \varphi^2 L_1 u_\varepsilon) ds + \int_0^t |\varphi| L_1 w_\varepsilon |^2_2 ds + \int_0^t (G(u_\varepsilon) - G(u), \varphi^2 L_1 w_\varepsilon) ds = \int_0^t (e L_2 u_\varepsilon, \varphi^2 L_1 u) ds \]  

By integrating by parts the term \( e \int_0^t (L_2 u_\varepsilon, \varphi^2 L_1 u_\varepsilon) ds \), then using inequality (1.5), and inequality (1.6) with \( \theta = -\varphi L_1 \varphi \), we show the minoration:

\[ e \int_0^t (L_2 u_\varepsilon, \varphi^2 L_1 u_\varepsilon) ds \geq \frac{\delta \omega}{4} \left\{ |\varphi u_\varepsilon|^2_2 + |\varphi| |\nabla u_\varepsilon|^2_2 \right\} - m_1 \sqrt{\varepsilon} \]

where \( m_1, (i = 1,2) \) is a constant independent of \( \varepsilon \).

Then, it results from (2.7) that:

\[ e \frac{\delta \omega}{4} \left\{ |\varphi u_\varepsilon|^2_2 + |\varphi| |\nabla u_\varepsilon|^2_2 \right\} + \frac{1}{2} \int_0^t |\varphi| L_1 w_\varepsilon |^2_2 ds \leq M_\varepsilon(t) + e m_2 \int_0^t \left\{ |\varphi u_\varepsilon|^2_2 + |\varphi| |\nabla u_\varepsilon|^2_2 \right\} ds \]

where \( M_\varepsilon(t) = \frac{1}{2} \left\{ |\varphi| (G(u_\varepsilon) - G(u)) |^2_2 + \int_0^t (e L_2 u_\varepsilon, \varphi^2 L_1 u) ds \right\} + m_1 \sqrt{\varepsilon} \).

As \( M_\varepsilon(t) \to 0 \) when \( \varepsilon \to 0^+ \), and \( M_\varepsilon(t) \) is bounded independently of \( \varepsilon \) thanks to (2.3), we conclude thanks to Lebesgue's theorem that \( \sqrt{\varepsilon} \varphi u_\varepsilon' \to 0 \) and \( \sqrt{\varepsilon} \varphi |\nabla u_\varepsilon| \to 0 \) in \( L^\infty(0, T; L^2(\Omega)) \).

**Remark.** The proof of the lemma also shows that \( u_\varepsilon' \to 0 \) and \( \sqrt{\varepsilon} \varphi |\nabla u_\varepsilon| \to 0 \) in \( L^\infty(0, T; L^2(\Omega)) \).

### 2.2.2 - CONVERGENCE OF \( u_\varepsilon \) AND \( L_1 u_\varepsilon \)

The results of the subsection 2.2.1 may be improved as follows.

**Theorem 2.2.4.** With hypothesis \( H_1 \), conditions (A) and (B), the solution \( u_\varepsilon \) of problem \( \mathcal{P}_\varepsilon \) verifies:

(i) \( u_\varepsilon \) converges to \( u \) in \( L^\infty(0, T; L^2(\Omega)) \) weak star and in \( L^q(Q), \ \forall q < 2 \) where \( u \) is the solution of the problem \( \mathcal{P} \).

\( u_\varepsilon \) converges to \( u \) and \( L_1 u_\varepsilon \) to \( L_1 u \) in \( L^2(Q') \) where \( Q' \) is an open set of \( Q \) such that \( Q' \cap \gamma' (\Sigma_*) = \emptyset \) where \( \gamma' (\Sigma_*) \) is a neighborhood of \( \Sigma_* \) in \( \Sigma \).
(ii) for each function $\varphi$ satisfying conditions $\mathcal{A}_1$:

\[ \sqrt{\varphi} \ u_\varepsilon \to \sqrt{\varphi} \ u \text{ in } L^\infty(0,T;L^2(\Omega)) \]

\[ \sqrt{\varphi} \ L_1 u_\varepsilon \rightharpoonup \sqrt{\varphi} \ L_1 u \text{ weakly in } L^2(Q) \text{ and } \varphi \ L_1 u_\varepsilon \to \varphi \ L_1 u \text{ strongly in } L^2(Q) \]

(iii) If we also assume hypothesis $H_2$ and that the coefficients $b_k$ are independent of $t$, $u_\varepsilon \to u^*$ in $L^\infty(0,T;L^2(\Omega))$ weak-star.

**Proof.** We remark that existence and uniqueness of $u$ solution of the problem $P$ is insured under the single hypotheses $H_1$ and $H_1'$ (C. Bardos [1], p. 199, by using the transformation $G_u = -(k+1)u + Mu$).

To prove points (i) and (ii), we use a method of regularization as in the proof of theorem 2.1.1. We approximate the triplet $(f,u_0,u_1)$ by a sequence $(f_{\mu},u_{0,\mu},u_{1,\mu})$ satisfying $H_2$ such that:

\[ (f_{\mu},u_{0,\mu},u_{1,\mu}) \to (f,u_0,u_1) \text{ in } L^2(Q) \times H^1_0(\Omega) \times L^2(\Omega) \]

(2.8)

Let $w_\varepsilon, \mu = u_\varepsilon, \mu - u_\varepsilon$ and $w_\mu = u_\mu - u$. We have:

\[ w_\varepsilon, \mu + L_1 w_\varepsilon, \mu + G(u_\varepsilon, \mu) - G(u_\varepsilon) = f_\mu - f \]  

(1)

and it results from theorems 2.1.1 and 2.1.2, since $|G(u_\varepsilon, \mu) - G(u_\varepsilon)| \leq \ell |w_\varepsilon, \mu|$, that

\[ |w_\varepsilon, \mu|^2 + |\sqrt{\varphi} \ L_1 w_\varepsilon, \mu|^2 \leq k_0 (|f_\mu - f|^2 + ||u_{0,\mu} - u_0||^2_2 + |u_{1,\mu} - u_1|^2) \]  

(2.9)

where $k_0$ is a positive constant independent of $\mu$ and $\varepsilon$.

\[ L_1 w_\mu + G(u_\mu) - G(u) = f_\mu - f \]  

(2.10)

from where we deduce by taking the inner product of (2.10) with $w_\mu$, using Green’s formula (1.7) and at last by integrating from 0 to $t$

\[ \frac{\delta}{2} |w_\mu|^2 + \frac{1}{2} \int_0^t \sum_{k=1}^n b_k \nu_k w_\mu^2 \, d\sigma \leq k_1 |u_{0,\mu} - u_0|^2 + \frac{1}{2} |f_\mu - f|^2 + k_2 \int_0^t |w_\mu|^2 \, ds. \]

Then, Gronwall’s lemma implies:

\[ |w_\mu|^2 \leq k_3 (|f_\mu - f|^2 + |u_{0,\mu} - u_0|^2), \ k_3 \text{ positive constant independent of } \mu \]

(11.1)

Now by taking the inner product of (2.10) with $L_1 w_\mu$, we obtain:

\[ |L_1 w_\mu|^2 \leq k_4 (|f_\mu - f|^2 + |u_{0,\mu} - u_0|^2), \ k_4 \text{ positive constant independent of } \mu \]

(11.2)
At last, by using the results of subsection 2.2.1, for each fixed $\mu$

$$
\begin{align*}
  u_{\epsilon, \mu} &\rightarrow u_{\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-star} \\
  \sqrt{\varphi} u_{\epsilon, \mu} &\rightharpoonup \sqrt{\varphi} u_{\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \\
  \sqrt{\varphi} L_1 u_{\epsilon, \mu} &\rightharpoonup \sqrt{\varphi} L_1 u_{\mu} \quad \text{weakly in } L^2(Q) \\
  \varphi L_1 u_{\epsilon, \mu} &\rightarrow \varphi L_1 u_{\mu} \quad \text{in } L^2(Q)
\end{align*}
$$

as $\epsilon \rightarrow 0^+$. 

As $u_{\epsilon} - u = -w_{\epsilon, \mu} + u_{\epsilon, \mu} - u_{\mu} + w_{\mu}$, one can easily check thanks to (2.8), (2.9), (2.11), (2.12), (2.13) that:

$$
\begin{align*}
  u_{\epsilon} &\rightarrow u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-star} \\
  \sqrt{\varphi} u_{\epsilon} &\rightharpoonup \sqrt{\varphi} u \quad \text{in } L^\infty(0,T;L^2(\Omega)) \\
  \sqrt{\varphi} L_1 u_{\epsilon} &\rightharpoonup \sqrt{\varphi} L_1 u \quad \text{weakly in } L^2(Q) \quad \varphi L_1 u_{\epsilon} \rightarrow \varphi L_1 u \quad \text{in } L^2(Q)
\end{align*}
$$

and the point (ii) follows. To achieve the proof of the point (i) we remark that the convergence of $u_{\epsilon}$ and $L_1 u_{\epsilon}$ in $L^2(Q')$ results from the properties of the functions $\varphi$. These properties also imply that $u_{\epsilon} \to u$ a.e. in $Q$.

As $|u_{\epsilon} - u|^q$ is bounded in $L^{2/q}(Q)$, $\forall q \leq 2$, there is a subsequence of $u_{\epsilon}$ such that $|u_{\epsilon} - u|^q \rightharpoonup 0$ weakly in $L^{2/q}(Q)$ $\forall q < 2$, from where $u_{\epsilon}$ converges to $u$ strongly in $L^q(Q)$, $\forall q < 2$.

The point (iii) results from lemma 2.2.1.

### 3. CONVERGENCE OF THE DERIVATIVES OF $u_{\epsilon}$

In this section, we improve the results of convergence. We aim at obtaining, on the one hand, the strong convergence of the derivatives of $u_{\epsilon}$ in local spaces, on the other hand, the rate of convergence in $\epsilon$ of $\varphi^{3/2}(u_{\epsilon} - u)$ in the space $L^\infty(0,T;L^2(\Omega))$. This kind of results needs hypotheses of regularity on $f$, because of the non-regularity of $u$ under the only assumptions : $f, f' \in L^2(Q)$, the derivatives of the function $u$ generally having poles on the part $\Sigma_0$ of $\Sigma$.

So, we impose on $f$ the hypothesis

$$
H_3 \quad \left\{ \begin{array}{l}
  f \in L^2(0,T;H^1(\Omega)) \\
  f = 0 \text{ on } \mathcal{Y} = \mathcal{Y}(\Sigma_0 \cup \Lambda) \cap \Sigma^- \quad \text{where } \mathcal{Y}(\Sigma_0 \cup \Lambda) \text{ is a neighborhood of } \Sigma_0 \cup \Lambda \text{ in } \Sigma.
\end{array} \right.
$$

Then, there exists $\lambda > 0$, such that $\sum_{k=1}^n b_k \nu_k \leq -\lambda$ on $(\Sigma^-) - \mathcal{Y}$.

With the hypothesis $H_3$, we first establish additional a priori estimates which allow us to obtain by compactness arguments the convergence of $u_{\epsilon}$ to $u$ solution of the problem $P$. 

3.1 - PRIORI ESTIMATES

THEOREM 3.1.1. We suppose hypotheses \( H_1, H_2, H_3 \), conditions (A), (B) and \( G(0) = 0 \). Then for \( \epsilon, 0 < \epsilon < \epsilon_0 \), the solution \( u_\epsilon \) satisfies the estimates of theorems 2.1.1, 2.1.2, 2.1.3 and moreover verifies:

for each function \( \varphi \) satisfying \( \mathcal{A}_1 \)

\[
|\varphi^{3/2}u'_\epsilon|_2 + \|\varphi^{3/2}u_\epsilon\|_2 + \sqrt{\epsilon} \|\varphi^{3/2}\Delta u_\epsilon\|_2 \leq C_4 \left( f, u_\epsilon, u_\epsilon, \epsilon \right).
\]

where

\[
K_4^2(f, u_\epsilon, u_\epsilon, \epsilon) = \frac{1}{\lambda} \left( \int f^2_\Sigma + \|u'_\epsilon\|_2^2 + \|f\|_2^2 + \epsilon^2 \|u_\epsilon\|^2 + \|u'_\epsilon\|^2 + \int_\partial \right)
\]

Proof. The smoothness properties of \( u_\epsilon \), under hypothesis \( H_2 \), allow us to take the inner product of two members of (1.1) with \( -\varphi^3 \Delta u_\epsilon \) . It comes:

\[
-f(u''_\epsilon, \varphi^3 \Delta u_\epsilon) + \epsilon (\varphi^{3/2} \Delta u_\epsilon)^2 - (L_1 u_\epsilon, \varphi^3 \Delta u_\epsilon) - (G(u_\epsilon), \varphi^3 \Delta u_\epsilon) = -(f, \varphi^3 \Delta u_\epsilon) \quad (3.2)
\]

Green's formula gives the following transformations:

\[
-(L_1 u_\epsilon, \varphi^3 \Delta u_\epsilon) = \frac{1}{2} \frac{d}{dt} \left( \int \varphi^3 \right) - \frac{1}{2} \int \left( \sum k \left( b_k \nabla \right) \varphi^3 \left( \left( \nabla \right) \varphi^3 \right) \right) d\Gamma \quad (3.3)
\]

where \( R(u_\epsilon) = 3 \int \varphi^2 L_1 u_\epsilon \left( \nabla \varphi \cdot \nabla u_\epsilon \right) dx + \int \varphi^3 u'_\epsilon \left( \nabla \varphi \cdot \nabla u_\epsilon \right) dx + \sum k \left( b_k \nabla \right) \varphi^3 \left( \left( \nabla \right) \varphi^3 \right) \left( \nabla \varphi^3 \right) \left( \nabla u_\epsilon \right) dx \quad (3.4)
\]

as \( G(u_\epsilon) \in L^2(0,T;H^1_0(\Omega)) \), thanks to lemma 1.1,

\[
-(f, \varphi^3 \Delta u_\epsilon) = \left( \varphi^3, f, u_\epsilon \right) - \int \varphi^3 \frac{\partial u_\epsilon}{\partial \nu} d\Gamma \quad (3.5)
\]

Then we have:

by taking into account theorem 2.1.2:

\[
\int R(u_\epsilon) ds \leq K_4^2(f, u_\epsilon, u_\epsilon, \epsilon) + k_1 \int \varphi^{3/2} u'_\epsilon |_{2}^2 ds + k_2 \int \varphi^{3/2} \left| \nabla u_\epsilon \right|^2 ds \quad (3.6)
\]
thanks to theorem 2.1.1. and lemma 1.1:

\[ \int_0^t (G(u_\varepsilon), \varphi^3 \Delta u_\varepsilon) ds \leq k_3 K_1^2 (f, u_0, u_1, \varepsilon) + k_4 \int_0^t |\varphi^{3/2} \| \text{grad } u_\varepsilon \|^2 \| ds. \]  
(3.7)

and at last:

\[ \int_0^t e(u_{\varepsilon}'', \varphi^3 \Delta u_\varepsilon) ds \leq \frac{\varepsilon}{2} \int_0^t |\varphi^{3/2} u_{\varepsilon}''| \| ds + \frac{\varepsilon}{2} \int_0^t |\varphi^{3/2} \Delta u_\varepsilon| \| ds \]  
(3.8)

\[ \int_{\Sigma_t} f \varphi^3 \frac{\partial u_\varepsilon}{\partial \nu} d \Gamma \leq \frac{k_5}{\lambda} \| f \|_{L^2(\Sigma)}^2 - \frac{1}{4} \int_{\Sigma_t} \left( \sum_{k=1}^n b_k \varphi^3 \right) \| \text{grad } u_\varepsilon \|^2 d \Gamma \]  
(3.9)

So, taking into account results (3.3) to (3.9), \( L_1 \varphi \leq 0 \) on \( Q \), \( \varphi \left( \sum_{k=1}^n b_k \varphi \right) \leq 0 \) on \( \Sigma \) and the properties of the coefficients, equality (3.2) gives:

\[ \frac{\varepsilon}{2} \int_0^t |\varphi^{3/2} \Delta u_\varepsilon| \| ds + \frac{\delta}{2} |\varphi^{3/2} \| \text{grad } u_\varepsilon \|^2 \| \leq k_6 \| f \|_{L^2(\Sigma)}^2 + \frac{k_5}{\lambda} \| f \|_{L^2(\Sigma)}^2 + k_7 \| f(u_0, u_1, \varepsilon) \|_{L^2(\Sigma)}^2 \]  
(3.10)

Now, we consider the method of the differential ratios when the coefficients \( b_k \) depend on \( t \). We have with the same notation as in the proof of the theorem 2.1.3:

\[ e \left( w_{\varepsilon}', h \right) + e \alpha(w_{\varepsilon}, h, \nu) + (a(s+h)w_{\varepsilon}'', h, \nu) + \sum_{k=1}^n \left( \frac{b_k(s+h)}{h} \frac{\partial w_{\varepsilon}}{\partial x_k}, \nu \right) \]  
(3.11)

We first obtain by taking \( \nu = \varepsilon(w_{\varepsilon}, h, 2e a^{-1}(s+h)w_{\varepsilon}, h) \) in (3.11) as in the proof of the theorem 2.1.3:

\[ e^2 \left\| w_{\varepsilon}', h \right\|^2 + e^2 \left\| w_{\varepsilon}, h \right\|^2 \leq k_9 K_3^2 (f, u_0, u_1, \varepsilon) \]  
(3.12)

and then by putting \( \nu = \varphi^3 \left( w_{\varepsilon}, h, 2e a^{-1}(s+h)w_{\varepsilon}, h) \right) \) in (3.11) and taking into account (3.12), it comes:

\[ \frac{\delta}{6} |\varphi^{3/2} u_{\varepsilon}''| \| ds + \int_0^t \left( |\varphi^{3/2} u_{\varepsilon}'| \| + |\varphi^{3/2} \| \text{grad } u_\varepsilon \|^2 \| \right) ds \]  
(3.13)

It results from (3.10) and (3.13):

\[ \frac{\delta}{6} |\varphi^{3/2} u_{\varepsilon}'| \| + \frac{\delta}{2} |\varphi^{3/2} \| \text{grad } u_\varepsilon \|^2 \| \leq k_{11} (K_3^2 (f, u_0, u_1, \varepsilon)) + \int_0^t \left( |\varphi^{3/2} u_{\varepsilon}'| \| + |\varphi^{3/2} \| \text{grad } u_\varepsilon \|^2 \| \right) ds \]  
and theorem 3.1.1 follows.
3.2 - CONVERGENCE

With hypotheses $H_1^*$, $H_2$, $H_3$, conditions (A), (B) and $G(0) = 0$, the following a priori estimates allow us to extract a subsequence still denoted by $u_\varepsilon$ such that:

$$
|u_\varepsilon|_2 \leq C_1 K_1(f,u_0,u_1,\varepsilon), \quad \|\sqrt{\varepsilon} L_1 u_\varepsilon\|_2 \leq C_2 K_1(f,u_0,u_1,\varepsilon)
$$

$$
\|\varphi^{3/2} u_\varepsilon\|_2 + \|\varphi^{3/2} u'_\varepsilon\|_2 \leq C_4 K_4(f,u_0,u_1,\varepsilon)
$$

allow us to extract a subsequence still denoted by $u_\varepsilon$ such that:

$$
u_\varepsilon \rightharpoonup u \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star and } \varphi^{3/2} u_\varepsilon \rightharpoonup \varphi^{3/2} u \text{ in } L^\infty(0,T;H^1_0(\Omega)) \text{ weak star,}
$$

$$
\varphi^{3/2} u'_\varepsilon \rightharpoonup \varphi^{3/2} u' \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star, } \sqrt{\varepsilon} L_1 u_\varepsilon \rightharpoonup \sqrt{\varepsilon} L_1 u \text{ weakly in } L^2(\Omega).
$$

So, in particular, we have: $\varphi^{3/2} u_\varepsilon$ converges to $\varphi^{3/2} u$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$. The properties of functions $\varphi$ imply the existence of a new subsequence such that $u_\varepsilon$ converges to $u$ a.e. on $\Omega$ and so $G(u_\varepsilon)$ converges to $G(u)$ a.e. on $\Omega$. As $|G(u_\varepsilon)|_2 \leq \varepsilon K_1(f,u_0,u_1,\varepsilon)$ we finally have:

$$
G(u_\varepsilon) \rightharpoonup G(u) \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star.}
$$

One can check that $u$ is the solution of problem $P$ (for which we also have shown smoothness properties).

Hence we have the

**THEOREM 3.2.1. (weak-convergence).** Under hypotheses $H_1^*$, $H_2$, $H_3$, conditions (A), (B) and $G(0) = 0$, the solution $u_\varepsilon$ of $P_\varepsilon$ verifies:

1. $u_\varepsilon \rightharpoonup u$ in $L^\infty(0,T;L^2(\Omega))$ weak star.
2. for each function $\varphi$ satisfying $\mathcal{A}_T$:

$$
\sqrt{\varepsilon} L_1 u_\varepsilon \rightharpoonup \sqrt{\varepsilon} L_1 u \text{ weakly in } L^2(\Omega), \quad \varphi^{3/2} u_\varepsilon \rightharpoonup \varphi^{3/2} u \text{ in } L^\infty(0,T;H^1_0(\Omega)) \text{ weak-star,}
$$

$$
\varphi^{3/2} u'_\varepsilon \rightharpoonup \varphi^{3/2} u' \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak-star,}
$$

where $u$ is the solution of the problem $P$.

Of course, the results of strong convergence of theorem 2.2.4 are still valid in the frame of this section. We are now interested by the rate of convergence in $\varepsilon$ of $\varphi^{3/2} (u_\varepsilon - u)$ in $L^\infty(0,T;L^2(\Omega))$. For this, we first improve the estimates satisfied by $u$ in $L^\infty(0,T;H^1_0(\Omega))$ and $u'$ in $L^\infty(0,T;L^2(\Omega))$ which result from the theorem 3.2.1. We obtain the

**LEMMA 3.2.2.** With the same hypotheses as in theorem 3.2.1, we have
Proof. We consider the equality
\[(L_1 u, \varphi^3 (u' - \Delta u)) + (G(u), \varphi^3 (u' - \Delta u)) = (f, \varphi^3 (u' - \Delta u)).\]

Thanks to (3.3), (3.4), (3.5), (3.7), (3.9) where \(u_e\) is replaced by \(u\), inequality
\[
\int_0^t \frac{1}{\varepsilon} \left| \varphi^{3/2} u' \right|^2 \, ds + \frac{\delta}{8} \int_0^t \left| \varphi^{3/2} u' \right|^2 \, ds + k_2 \int_0^t \left| \varphi^{3/2} \nabla u \right|^2 \, ds,
\]
the fact that \(L_1 \varphi \leq 0\) on \(Q\), \(\varphi (\sum_{k=1}^n b_k \phi_k) \leq 0\) on \(\Sigma\), and the properties of the coefficients, it comes:
\[
- \frac{\delta}{2} \left| \varphi^{3/2} \nabla u \right|^2 + \frac{\delta}{2} \int_0^t \left| \varphi^{3/2} u' \right|^2 \, ds \leq k_3 \left( \frac{1}{\lambda} \left| f \right|_{L^2(\Sigma)}^2 + \left| f \right|_{L^2(\Sigma)}^2 + \left| u_0 \right|_{L^2}^2 + \left| u_1 \right|_{L^2}^2 \right)
\]
\[+ k_4 \int_0^t \left| \varphi^{3/2} \nabla u \right|^2 \, ds .
\]

And Gronwall's lemma gives the estimates. (When \(u\) is not smooth enough, the lemma results from the study of the solution of the regularized problem
\[
\begin{cases}
- \eta \Delta v + L_1 v + G(v) = F, & \eta > 0 \\
v(x,0) = u_0
\end{cases}
\]

Now, we may prove the

**THEOREM 3.2.3.** (rate of convergence). With hypotheses of theorem 3.2.1, for each \(0 < \varepsilon < \varepsilon_0\) we have:
\[
\left| \varphi^{3/2} (u_e - u) \right|_{L^2(Q')}^2 \leq K_5 \sqrt{\varepsilon} \quad \text{(for each \( \varphi \) satisfying \(A)\)}
\]
\[
\left| u_e - u \right|_{L^2(Q')}^2 \leq K_5' \sqrt{\varepsilon}
\]

where \(Q'\) is an open set of \(Q\) such that \(\bar{Q}' \cap \gamma(\Sigma_c) = \phi\).

Proof. We set \(w_e = u_e - u\) and we take the inner product of \(\varepsilon L_2 u_e + L_1 w_e + G(u_e) - G(u) = 0\) with \(\varphi^{3} w_e \in L^\infty_{loc}(\Omega)\). It comes:
\[
\varepsilon \frac{3}{4} \omega \int_0^t \left| \varphi^{3/2} \nabla w_e \right|^2 \, ds + \frac{\delta}{2} \left| \varphi^{3/2} w_e \right|^2 \, ds \leq \varepsilon A_e(t) + k_1 \int_0^t \left| \varphi^{3/2} w_e \right|^2 \, ds \quad (3.14)
\]
where \(A_e(t) = -(\varphi^{3/2} u'_e, \varphi^{3/2} w_e) + \omega_1 \int_0^t \left| \varphi^{3/2} L_1 w_e \right|^2 \, ds + \int_0^t \left( \varphi^{3/2} u', \varphi^{3/2} w'_e \right) \, ds \)
\[\quad - \int_{Q_t} \varphi^3 \nabla u \cdot \nabla w_e \, dx \, ds + 3 \int_{Q_t} \varphi^3 w_e \left( \varphi \cdot u'_e + \nabla \varphi \cdot \nabla u_e \right) \, dx \, ds.
\]
As \(\left| A_e(t) \right| \leq k_2 K_5^2 + \frac{\delta}{6} \left| \varphi^{3/2} w_e \right|^2 + \frac{3}{8} \omega \int_0^t \left| \varphi^{3/2} \nabla w_e \right|^2 \, ds + k_3 \int_0^t \left| \varphi^{3/2} w_e \right|^2 \, ds\)

thanks to the theorems 2.1.1, 2.1.2 and lemma 3.2.2, the statement follows by application of Gronwall's lemma.
Remark. We also have shown that \( \| \varphi^{3/2} \| \var{u_\varepsilon} \|_2 \leq K_5 \) and \( \| \varphi^{3/2} u_\varepsilon' \|_2 \leq K_5 \).

The following theorem gives results of strong convergence for the derivatives of \( u_\varepsilon \).

**THEOREM 3.2.4.** (Strong convergence of the derivatives). With hypotheses of theorem 3.2.1, we have:

1. \( \varphi^{3/2} u_\varepsilon \to \varphi^{3/2} u \) in \( L^2(0,T;\mathcal{H}_0^1(\Omega)) \)
2. \( \varphi^{3/2} u_\varepsilon' \to \varphi^{3/2} u' \) in \( L^2(Q) \) for each function \( \varphi \) satisfying \( \mathcal{A}_1 \).

**Proof.** We consider again (3.14).

As \( \lim_{\varepsilon \to 0} A_\varepsilon(t) = 0 \) because \( \sqrt{\varphi} w_\varepsilon \to 0 \) in \( L^\infty(0,T;L^2(\Omega)) \) (theorem 2.2.4)

\[
\varphi^{3/2} u_\varepsilon \to \varphi^{3/2} u \text{ in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak star (theorem 3.2.1)}
\]

\[
\varphi L^1 \varphi w_\varepsilon \to 0 \text{ in } L^2(Q) \quad \text{(theorem 2.2.4)}
\]

\[
\varphi^{3/2} \var{\tilde{\nabla} u_\varepsilon} \to \varphi^{3/2} \var{\tilde{\nabla} u} \text{ in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak star (theorem 3.2.1)}
\]

Gronwall’s lemma and Lebesgue’s theorem allow us to conclude because \( \| A_\varepsilon(t) \| \) is bounded. (ii) results from properties of functions \( \varphi \).

**Remark 3.2.5.** With hypothesis \( H_1 \), conditions (A),(B) and \( G(0) = 0 \), the results of theorem 3.2.3 are still valid if \( f \in L^2(0,T;\mathcal{H}_0^1(\Omega)) \). It is once more enough to approximate the triplet \( (f;u_0;u_1) \) in \( L^2(0,T;\mathcal{H}_0^1(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega) \) by a sequence \( (f_{\mu};u_{0,\mu};u_{1,\mu}) \) satisfying hypothesis \( H_2 \).

---

### 4. APPLICATION OF NON LINEAR INTERPOLATION

The application of non linear interpolation theory (L. Tartar [11]) allows us to explicit \( \| \varphi^{3/2}(u_\varepsilon - u) \|_2 \) in \( \varepsilon \), with less assumptions than in section 3, in particular without condition on \( f \) over \( \mathcal{Y}(\Sigma_0 \cup \Lambda) \).

We first recall the theorem of non linear interpolation of [11] which will be then applied to our problem. The useful result is the following:

Let \( A_0 \subset A_1 \), \( B_0 \subset B_1 \) Banach spaces and \( T \) a map such that \( T(A_1) \subset B_1 \), \( T(A_0) \subset B_0 \) and:

\[
\exists \alpha, \beta : 0 < \alpha < 1, 0 < \beta \text{ such that}
\]

\[
\| T_a - T_b \|_{B_1} \leq f(\| a \|_{A_1}, \| b \|_{A_1}) \| a - b \|_{A_1}^{\alpha}, \forall a, b \in A_1
\]

\[
\| T_a \|_{B_0} \leq g(\| a \|_{A_1}) \| a \|_{A_0}^{\beta}, \forall a \in A_0
\]
Then, if \( 0 < \theta < 1, 1 \leq p < \infty \) we have:

\[
\| T a \|_{(B_0^0, B_1^0)}_{1,q} \leq C h(\| a \|_{A_1^0}) \| a \|_{(A_0^0, A_1^0)}^{(1-\eta)/\beta + \eta \alpha}
\]

where \( \frac{1-\eta}{\eta} = \frac{1-\theta}{\beta} \frac{\alpha}{\beta} \)

\[
q = \max(1, (\frac{1-\theta}{\beta} + \frac{\theta}{\alpha})p)
\]

\[
h(r) = g(2r)^{1-\eta} f(r,2r)^{q}
\]

the space \((A_0^0, A_1^0)_{\theta,p}\) being defined by:

\[
(A_0^0, A_1^0)_{\theta,p} = \left\{ a \in A_0^0 + A_1^0 \mid t^{-\theta} K(t,a) \in L^p(0, \infty; \frac{dt}{t}) \right\}
\]

with the norm

\[
\| a \|_{(A_0^0, A_1^0)_{\theta,p}} = \| t^{-\theta} K(t,a) \|_{L^p(0, \infty; \frac{dt}{t})}
\]

This result applied to our problem gives the:

**THEOREM 4.1.** We suppose hypothesis \( H'_j \), conditions \( (A), (B) \) and \( G(0) = 0 \),

1. Let \( 0 < \theta < 1 \), if \( f \in L^2(0, T; A_0^0(\Omega), L^2(\Omega)) \), for each \( \epsilon < \epsilon_0 \), we have:

\[
\| \varphi^{3/2}(u_\epsilon - u) \|_2 \leq K_6 \epsilon^{2} \quad \text{where} \quad K_6^2 = C_6 \| f \|_{L^2(0, T; A_0^0(\Omega), L^2(\Omega))}^2 + \| u_0 \|^2 + \| u_1 \|_2^2
\]

for each function \( \varphi \) satisfying \( \frac{1-\theta}{1-\theta} \), and:

\[
\| u_\epsilon - u \|_{L^2(Q')} \leq K'_6 \epsilon^{2} \quad \text{where} \quad Q' \text{ is an open set of} \ \Omega \text{ such that} \quad Q' \cap \gamma(\Sigma_{\epsilon}) = \phi.
\]

2. In particular, if \( f \in L^2(0, T; \mathcal{H}^s(\Omega)) \), \( 0 \leq s < \frac{1}{2} \), \( \Omega \) regular, we have:

\[
\| \varphi^{3/2}(u_\epsilon - u) \|_2 \leq K_6 \epsilon^{s/2}
\]

\[
\| u_\epsilon - u \|_{L^2(Q')} \leq K'_6 \epsilon^{s/2}
\]

**Proof.** We consider \( A_0 = L^2(0, T; A_0^0(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega), A_1 = L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \)

\[
B_0 = B_1 = L^\infty(0, T; L^2(\Omega))
\]

\[
T = T_\epsilon : (u_\epsilon, u_1) \rightarrow \varphi^{3/2}(u_\epsilon - u)
\]
It results from theorems 1.2 and 2.2.4 that $T_{\varepsilon}$ maps $A_1$ into $B_1$ and also $A_0$ into $B_0$.

(a) We first consider $T_{\varepsilon} : A_1 \to B_1$. Let $(f,u_0,u_1) \in A_1$ and $(g,v_0,v_1) \in A_1$, $T_{\varepsilon}(f,u_0,u_1) = \varphi^{3/2}(u_0-u)$ and $T_{\varepsilon}(g,v_0,v_1) = \varphi^{3/2}(v_0-v)$.

If we put $w_\varepsilon = u_\varepsilon - v_\varepsilon$ and $w = u - v$, then:

$$
\varepsilon L_2 w_\varepsilon + L_1 w_\varepsilon + G(u_\varepsilon) - G(v_\varepsilon) = f - g
$$

from where by recalling the proof of theorem 2.1.1 and taking into account $|G(u_\varepsilon) - G(v_\varepsilon)| \leq \varepsilon |w_\varepsilon|$, we deduce:

$$
|w_\varepsilon|^2 \leq k_1 \left( |f-g|^2 + \|u_0 - v_0\|^2 + |u_1 - v_1| \right)
$$

(4.1)

$$
L_1 w + G(u) - G(v) = f - g
$$

(4.2)

We take the inner product of two members of (4.2) with $w$ and we integrate from 0 to $t$. Green’s formula (1.7) gives:

$$
\frac{\delta}{2} |w|^2 + \frac{1}{2} \sum_{k=1}^{n} \left( \sum_{k=1}^{n} b_k(x) w^2 d\Gamma \leq k_1 \left( |u_0 - v_0|^2 + \frac{1}{2} |f-g|^2 + k_2 \int_0^t |w|^2 ds \right)
$$

and Gronwall’s lemma then implies that:

$$
|w|^2 \leq k_3 \left( |f-g|^2 + |u_0 - v_0| \right)
$$

(4.3)

at last, (4.1) and (4.3) give the inequality:

$$
|T_{\varepsilon}(f,u_0,u_1) - T_{\varepsilon}(g,v_0,v_1)|_2 \leq k_4 \left( |f-g|^2 + \|u_0 - v_0\|^2 + |u_1 - v_1| \right)^{1/2}
$$

$$
\forall (f,u_0,u_1) \in A_1, \forall (g,v_0,v_1) \in A_1
$$

(b) If we consider $T_{\varepsilon} : A_0 \to B_0$, it results from remark 3.2.5 that:

$$
\forall (f,u_0,u_1) \in A_0 \quad |T_{\varepsilon}(f,u_0,u_1)|_2 \leq K_5 \varepsilon^{1/2}
$$

(c) The hypotheses of the theorem of non linear interpolation are satisfied thanks to (a) and (b), with $\alpha = 1$, $\beta = 1$, $f(r,s) = k_4$, $g(r) = (C_5 \varepsilon)^{1/2}$, $p = 2$

and the application of this theorem allows us to assert that

if $(f,u_0,u_1) \in [A_0,A_1]_\theta$, then $T_{\varepsilon}(f,u_0,u_1) \in L^\infty(0,T;L^2(\Omega))$ and

$$
|T_{\varepsilon}(f,u_0,u_1)|_2 \leq C_6 \varepsilon^{\frac{1-\theta}{2}} \left\| (f,u_0,u_1) \right\|_{[A_0,A_1]_\theta}
$$
5. REMARK ABOUT CORRECTORS

We can define under hypothesis $H_1$ and condition (A) correctors in the sense of J.L. Lions [6].

Let $g_\varepsilon \in L^2(Q)$ given and $\theta_\varepsilon$ defined by

$$
\begin{cases}
\varepsilon \left((\theta_\varepsilon + u)'' + \varepsilon \alpha(\theta_\varepsilon + u, v) + (L_1(\theta_\varepsilon + u), v) + (G(\theta_\varepsilon + u), v) = (f, v) + \varepsilon^{1/2}(g_\varepsilon, v) \right) \\
(\theta_\varepsilon + u)(x,0) = u_0, \quad (\theta_\varepsilon + u)'(x,0) = u_1.
\end{cases}
$$

\forall v \in H^1_0(\Omega) \quad \text{a.e. on } t \in [0,T[.

The theorem 1.2 ensures the existence and uniqueness of $\theta_\varepsilon + u$ such that:

$$
\theta_\varepsilon + u \in L^\infty(0,T;H^1_0(\Omega)) ; (\theta_\varepsilon + u)' \in L^\infty(0,T;L^2(\Omega)).
$$

Then $\theta_\varepsilon$ is a corrector in the following sense:

**THEOREM 5.1.** Under hypothesis $H_1$, condition (A), if $g_\varepsilon \in L^2(Q)$ with $\|g_\varepsilon\|_2$ bounded independently of $\varepsilon$, we have:

$$
|u_\varepsilon - (\theta_\varepsilon + u)|_2 \leq K\sqrt{\varepsilon} \quad \text{where } K \text{ is a positive constant independent of } \varepsilon.
$$

$$
\begin{align*}
u_\varepsilon - (\theta_\varepsilon + u) &\rightharpoonup 0 \text{ weakly in } L^2(0,T;H^1_0(\Omega)) \\
u_\varepsilon' - (\theta_\varepsilon + u') &\rightharpoonup 0 \text{ weakly in } L^2(Q).
\end{align*}
$$

**Proof.** We consider $w_\varepsilon = u_\varepsilon - (\theta_\varepsilon + u)$ which verifies

$$
\begin{cases}
\varepsilon \left((w_\varepsilon'') + \varepsilon \alpha(w_\varepsilon', v) + (L_1 w_\varepsilon, v) + (G(w_\varepsilon) - G(\theta_\varepsilon + u), v) = -\varepsilon^{1/2}(g_\varepsilon, v) \right) \\
w_\varepsilon(x,0) = 0, \quad w_\varepsilon'(x,0) = 0
\end{cases}
$$

and we follow once more the method of the proof of theorem 2.1.1.

We first suppose that $g_\varepsilon' \in L^2(Q)$ and hypothesis $H_2$.

We obtain by the same arguments, taking into account $|G(u_\varepsilon) - G(\theta_\varepsilon + u)| \leq \ell \|w_\varepsilon\|$ the inequality:
It is enough then to approximative in $L^2(Q) \times L^2(Q) \times H^1_0(Q) \times L^2(Q)$ by $(f; g; u; u_1)$ satisfying hypothesis $H_2$ with $G \in L^2(Q)$ to assert that (4.4) is still valid under hypotheses of the theorem which thus is proved.
REFERENCES


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