ALEX BIJLSMA

Algebraic points of abelian functions in two variables


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Résumé : On donne une mesure d'indépendance linéaire pour les coordonnées des points algébriques de fonctions abéliennes de deux variables. On en déduit un analogue abélien du théorème de Franklin-Schneider.

Summary : A linear independence measure is given for the coordinates of algebraic points of abelian functions in two variables. From this an abelian analogue of the Franklin-Schneider theorem is deduced.

Let $A$ be a simple abelian variety defined over the field of algebraic numbers and let $\Theta : \mathbb{C}^2 \to A[\mathbb{C}]$ be a normalised theta homomorphism (cf. [12], § 1.2). Let $\vartheta_0, \ldots, \vartheta_\nu$ be entire functions such that $(\vartheta_0(z), \ldots, \vartheta_\nu(z))$ forms a system of homogeneous coordinates for the point $\Theta(z)$ in projective $\nu$-space. Put $f_i := \vartheta_i / \vartheta_0$. Assume that $\vartheta_0(0) \neq 0$; then $f_i(0)$ is algebraic for all $i$. A point $u$ in $\mathbb{C}^2$ with $\vartheta_0(u) \neq 0$ is by definition an algebraic point of $\Theta$ if and only if $f_i(u)$ is algebraic for all $i$. The field of abelian functions associated with $\Theta$ is $\mathbb{C}(f_1, \ldots, f_\nu)$.

If $(u_1, u_2)$ is a non-zero algebraic point of $\Theta$, the coordinates $u_1$ and $u_2$ are linearly independent over the algebraic numbers (cf. [12], Théorème 3.2.1); the proof uses the Schneider-Lang criterion (cf. [5], Chapter III, Theorem 1). It is the main purpose of this paper to obtain, by means of Gel'fond's method, a quantitative refinement of this statement.
THEOREM 1. For every compact subset $K$ of $\mathbb{C}^2 \setminus \{0\}$ that contains no zeros of $\partial^0$ there exists an effectively computable $C$ with the following property. Let $u$ be an algebraic point of $\Theta$ that lies in $K$, and let $\beta$ be an algebraic number. Let $A$ be an upper bound for the (classical) heights of the numbers $f_i(u)$, let $B$ be an upper bound for the height of $\beta$ and take $D := [\mathbb{Q}(f_1(u), \ldots, f_\nu(u), \beta) : \mathbb{Q}]$; assume $A > B > e$. Then

$$|\beta u_1 - u_2| > \exp(-CD^6 \log^2 A \log^4 (DB \log A) \log^{-5}(D \log A)),$$

where $u = (u_1, u_2)$.

The dependence of this lower bound on $B$ was first studied in [3]. Moreover, in an unpublished 1979 investigation, Y.Z. Flicker and D.W. Masser also studied the dependence on $B$ and obtained $\log^4 B$ in the exponent. I wish to thank Dr. Masser for making available to me a report of this study, to which several improvements in the present paper are due.

The proof of Theorem 1 resembles that of Lemma 1 of [1]; in parts where this resemblance is particularly strong, the exposition will be brief. The proof is preceded by a lemma that may be called, in Masser’s terminology, a 'safe addition formula' for abelian functions.

LEMMA. There exists an effectively computable $C'$ with the following property. If $w_1$ and $w_2$ are points of $\mathbb{C}^2$ such that $\partial^0(w_1) \neq 0$, $\partial^0(w_2) \neq 0$, $\partial^0(w_1 + w_2) \neq 0$, then for every $i$ in $\{1, \ldots, \nu\}$ there exist polynomials $\Phi_i, \Phi_i^*$ of total degree at most $C'$ and a neighbourhood $N$ of $(w_1, w_2)$ such that

$$f_i(z_1 + z_2) = \frac{\Phi_i^*}{\Phi_i}(f_1(z_1), \ldots, f_\nu(z_1), f_1(z_2), \ldots, f_\nu(z_2))$$

for all $(z_1, z_2)$ in $N$; the denominator is non-zero on $N$. The coefficients of these polynomials are algebraic integers in a field of degree at most $C'$. Their size (i.e., the maximum of the absolute values of their conjugates) is also bounded by $C'$.

Proof. Let $(w_1, w_2)$ be any point in $\mathbb{C}^4$. Define $\sigma : \mathbb{C}^4 \rightarrow \mathbb{P}^{2+2\nu}(\mathbb{C})$ by $\sigma(z_1, z_2) := \psi(\Theta(z_1), \Theta(z_2))$, where $\psi$ is the Segre embedding (cf. [9], (2.12)) of $\mathbb{P}^{2}(\mathbb{C}) \times \mathbb{P}^{\nu}(\mathbb{C})$ into projective space. By the regularity of the addition in $\mathcal{X}$, we find projective coordinates for $\Theta(z_1 + z_2)$ of the form

$$H_i(\Theta(z_1), \Theta(z_2)) \quad (0 \leq i \leq \nu)$$

for all $(z_1, z_2)$ with the property that $\sigma(z_1, z_2)$ lies in a certain Zariski neighbourhood of $\sigma(w_1, w_2)$; here the polynomials $H_i$ have algebraic coefficients. The continuity of $\sigma$ now proves this for all $(z_1, z_2)$ in a neighbourhood of $(w_1, w_2)$. Let $P$ be a fundamental region for $\mathbb{C}^2 / P$; covering the compact set $P^2$ with a finite number of these neighbourhoods shows that we can bound the
degrees of the polynomials $H_i$, the sizes of their coefficients, the degree of the field generated by these coefficients and their common denominator independently of $(w_1, w_2)$. In particular, it is no restriction to assume the coefficients to be algebraic integers.

Finally, if $\vartheta_0(w_1) \neq 0$, $\vartheta_0(w_2) \neq 0$, $\vartheta_0(w_1 + w_2) \neq 0$, these also hold on some neighbourhood of $(w_1, w_2)$; hence

$$H_0(\Theta(z_1), \Theta(z_2)) \neq 0$$

on some neighbourhood of $(w_1, w_2)$, which now proves (2).

Proof of Theorem 1. In this proof $c_1, c_2, \ldots$ will denote effectively computable real numbers greater than 1 that depend only on $\Theta$ and $K$. Let $x$ be some large real number; further conditions on $x$ will appear at later stages of the proof. Put $B' := xDB \log A, E := 4D^{1/2} \log^{1/2} A$ and assume

$$|\beta u_1 - u_2| \leq \exp(-x^{16}D^6 \log^2 A \log^4 B' \log^{-5} E).$$

This will lead to a contradiction, which will prove (1).

The field $\mathbb{C}(f_1, \ldots, f_\nu)$ has transcendence degree 2 over $\mathbb{C}$ (cf. [10], § 6); assume, without loss of generality, that $f_1$ and $f_2$ are algebraically independent over $\mathbb{C}$. As in [8], § 4.2, we choose a system $\xi_0, \ldots, \xi_{D-1}$ of generators of $\mathbb{Q}(f_1(u), \ldots, f_\nu(u), \beta)$ of the form

$$\xi_\delta = f_1^j(\delta) \ldots f_\nu^j(\delta) (u) \beta^{j+1}(\delta),$$

where the $j_\delta(\delta)$ are non-negative integers satisfying $j_1(\delta) + \ldots + j_{\nu+1}(\delta) \leq D-1$. Put

$$L := [x^8D^3 \log A \log^2 B' \log^{-3} E]$$

and consider the auxiliary functions

$$F(z) := \sum_{\lambda_1 = 0}^L \sum_{\lambda_2 = 0}^L \sum_{\delta = 0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_\delta f_1^{\lambda_1} f_2^{\lambda_2}(z, \beta z),$$

$$F_\varepsilon(z) := \sum_{\lambda_1 = 0}^L \sum_{\lambda_2 = 0}^L \sum_{\delta = 0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_\delta f_1^{\lambda_1} f_2^{\lambda_2}(z, \beta z - \varepsilon),$$

where $\varepsilon := \beta u_1 - u_2$. As $K$ is compact and the zero set of $\vartheta_0$ is closed, these sets have a distance at least $c_1^{-1}$. The functions $f_1, \ldots, f_\nu$ are continuous on the set $K'$ of points $z$ satisfying $\text{dist}(z, K) \leq \frac{1}{2} c_1^{-1}$; hence their absolute values are bounded by some $c_2$ on $K'$ and a fortiori on
the ball \( U \) with radius \( \frac{1}{4} c_1^{-1} \) centred at \( u \). Now put

\[
S := [x^3 D \log B' \log^{-1} E].
\]

As in § 4 of [6], an application of the box principle shows that there is a subset \( V \) of \( \{1, ..., S\} \) such that \( \# V \geq c_3^{-1} S \) with the property that \((su_1, su_2)\) and \((su_1, s\beta u_1)\) lie in \( U + \Omega \) for all \( s \) in \( V \), where \( \Omega \) is the period lattice of \( \Theta \). Put

\[
T := [x^{12} D^5 \log^2 A \log^3 B' \log^{-5} E]
\]

and consider the system of linear equations

\[
(6) \quad f_s^{(t)}(su_1) = 0 \quad (s \in V, \ t = 0, ..., T-1)
\]

in the \( p(\lambda_1, \lambda_2, \delta) \).

Take \( \lambda \leq i \leq \nu \). Lemma 7.2 of [6], part of which remains valid without complex multiplication, states that for every integer \( s \) there exist polynomials \( \Psi_{s,i}, \psi_{s,i} \) of total degree \( N_s \leq c_4^{\nu^2} \) such that, if \( \vartheta_0(su) \neq 0 \), then

\[
f_i(su) = \frac{\Psi_{s,i}}{\psi_{s,i}} (f_1(u), ..., f_{\nu}(u))
\]

and \( \Psi_{s,i}, f_1(u), ..., f_{\nu}(u) \neq 0 \). The coefficients of these polynomials are algebraic numbers in a field of degree at most \( c_5 \), of size at most \( c_6^{\nu^2} \) and with a common denominator at most \( c_7^{\nu^2} \).

According to the preceding Lemma, there also exist polynomials \( \Phi_i, \Phi_{s,i} \) of total degree at most \( c_8 \) and a neighbourhood \( N \) of the origin such that

\[
f_i(u + z) = \frac{\Phi_{s,i}}{\Phi_i} (f_1(u), ..., f_{\nu}(u), f_1(z), ..., f_{\nu}(z))
\]

for all \( z \) in \( N \), with non-zero denominator, the coefficients are algebraic integers in a field of degree at most \( c_9 \), whose sizes are also bounded by \( c_9 \).

Now define

\[
\Phi := \prod_{i=1}^\nu \Phi_i,
\]

\[
\psi_{s,i}(z) := \Phi^{N_s} (f_1(u), ..., f_{\nu}(u), f_1(z), ..., f_{\nu}(z)) \psi_{s,i} (f_1(u + z), ..., f_{\nu}(u + z)),
\]

\[
\psi_{s,i}(z) := \Phi^{N_s} (f_1(u), ..., f_{\nu}(u), f_1(z), ..., f_{\nu}(z)) \psi_{s,i} (f_1(u + z), ..., f_{\nu}(u + z)).
\]
Note that on a neighbourhood of the origin $\varphi_{s,1}$ and $\psi_{s,1}$ are holomorphic and $\varphi_{s,1}$ is non-zero. As Leibniz' rule shows that we have found a solution of (6) if we choose the $p(\lambda_1, \lambda_2, \delta)$ in such a way that

\[ f_{s,t} = 0 \quad (s \in \mathcal{V}, t = 0, \ldots, T-1), \]

where

\[ f_{s,t} = \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1, \lambda_2, \delta) \xi_s \frac{d^t}{dz^t} \left( \varphi_{s,1} \psi_{s,1} \varphi_{s,2} \psi_{s,2} \left( z, \beta z \right) \right)_{z=0}. \]

The number of equations in (7) is at most

\[ ST \leq c_9 x^{15} D^6 \log^2 A \log^4 B \log^{-6} E, \]

while the number of unknowns is

\[ (L + 1)^2 D \geq c_{10} x^{16} D^7 \log^2 A \log^4 B \log^{-6} E. \]

From the above estimates it follows that $\varphi_{s,1}(z)$ can be written as a polynomial in $f_1(u), \ldots, f_{t}(u)$ of total degree at most $c_{11} \lambda_1 s^2$; the coefficients are algebraic numbers in a field of degree at most $c_{12}$, whose sizes and common denominator are bounded by $c_{13}$. With the aid of Lemma 5.1 of [6] it is now easy to see that the expression

\[ \left. \frac{d^t}{dz^t} \varphi_{s,1}(z, \beta z) \right|_{z=0} \]

is a polynomial in $f_1(u), \ldots, f_{t}(u)$ of total degree at most $c_{14}(\lambda_1^2 + t)$; the coefficients are algebraic numbers in a field of degree at most $c_{16}$ over $\mathbb{Q}(\beta)$, whose sizes and common denominator $\lambda s^2 + t \log t + t \log B$ are bounded by $c_{17}$. A similar statement holds for

\[ \left. \frac{d^t}{dz} \psi_{s,1}(z, \beta z) \right|_{z=0}. \]

Thus the coefficients of the system of linear equations (7) lie in a field of degree at most $c_{17} D$ and their size and common denominator are bounded by
According to Lemme 1.3.1 of [11], if $x > 2c_9c_{10}$, this implies the existence of rational integers $p(\lambda_1, \lambda_2, \delta)$, not all zero, such that (7) and thereby (6) hold, while

$$P := \max |p(\lambda_1, \lambda_2, \delta)| \leq \exp(c_{21} \times 14^D \log^2 A \log^4 B' \log^{-5} E).$$

Take $s \in V, \eta \in IR, z \in C$ such that $|z - su_1| = \eta$. Then the distance between $(z, \beta z)$ and $(su_1, \beta su_1)$ is bounded by $2B\eta$; if $\eta = (8c_1 B)^{-1}$, it follows that $(z, \beta z)$ lies in $U' + \Omega$, where $U'$ is the ball with radius $\frac{1}{2} c_1^{-1}$ centred at $u$. Similarly $(z, \beta z - se) \in U'$. Note that $U' \subset K'$ and therefore $|f(z)| \leq c_2$ for all $z$ in $U'$. Comparison of the definitions of $F$ and $F_s$ now gives

$$\sup_{|z-su_1| = \eta} |F(z) - F_s(z)| \leq Pc_{22}^{D+L} S |\epsilon|^L.$$

By Cauchy's inequality this implies

$$|F(t)(su_1) - F_s(t)(su_1)| \leq t^{c_{23}^T} B^T Pc_{24}^{D+L} S |\epsilon|^L.$$

If $t \leq T-1$, it now follows from (6) that

$$|F(t)(su_1)| \leq \exp(-c_{25}^{-1} x^6 \log^2 A \log^4 B' \log^{-5} E).$$

Define the entire function $G$ by

$$G(z) := g(z)F(zu_1),$$

where

$$g(z) := g_0^{2L}(zu_1, \beta zu_1).$$

By Lemma 1 of [7], the function $g$ satisfies

$$|g(z)| \leq \exp(c_{26}L |z|^L);$$

also the definition of $V$ gives

$$|g(s)| \geq \exp(-c_{27} LS^2) \quad (s \in V).$$

Formulas (8), (9) and (10) form the starting-point for an extrapolation procedure on $G$, analogous to that in [1], which yields

$$F_s^{(t)}(su_1) = 0 \quad (s \in V, t = 0, \ldots, T' - 1),$$

$$\sum_{i=1}^{\nu} (H(f_i(u)) + 1)^{c_19(D+LS^2)} \leq \exp(c_{20} x^{14^D} \log^2 A \log^4 B' \log^{-5} E).$$
II. By Proposition 1.2.3 of [12], the partial derivatives of $f_1, \ldots, f_v$ are polynomials in $f_1, \ldots, f_v'$. Therefore there exist polynomials $P_1, \ldots, P_v$ such that the functions $h_{i,s}$, defined by

$$h_{i,s}(z) := f_i(z + su_1, z + su_2)$$

satisfy

$$h'_i(z) = P_i(h_{1,s}, \ldots, h_{v,s})$$

and

$$h_{i,s}(0) = f_i(su_1, su_2).$$

Define

$$Q_1(x_1, \ldots, x_v) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} \sum_{\delta=0}^{D-1} p(\lambda_1, \lambda_2, \delta) x_1^{\lambda_1} x_2^{\lambda_2}. $$

As

$$h^{(t)}_{i,s}(0) = \frac{d^t}{dz^t} f_i(z, z - se) \bigg|_{z=su_1},$$

(11) shows

$$\frac{d^t}{dz^t} Q_1(h_{1,s}(z), \ldots, h_{v,s}(z)) \bigg|_{z=0} = 0 \quad (s \in V, t = 0, \ldots, T' - 1),$$

i.e.

(12)  $$\sum_{s \in V} \text{ord}_{z=0} Q_1(h_{1,s}(z), \ldots, h_{v,s}(z)) \geq c_3^{-1} ST' \geq c_3^{-1} \frac{1}{28} x_1^{17} D^6 \log^2 A \log^4 B \log^{-6} E.$$  

Let $Q_2, \ldots, Q_n$ be generators of the ideal of $C[X_1, \ldots, X_v]$ corresponding to the affine part of $A$. Then

(13)  $$Q_j(f_1(w), \ldots, f_v(w)) = 0 \quad (j = 2, \ldots, n)$$

for every $w$ that is not a zero of $\theta_0$; thus in particular

(14)  $$\text{ord}_{z=0} Q_j(h_{1,s}(z), \ldots, h_{v,s}(z)) = \infty \quad (s \in V, j = 2, \ldots, n).$$

Put $W := \{ \Theta(z, \beta z) | z \in C \}$. Then $W$, with the addition of $A$, forms a subgroup of $A$; it follows
that the Zariski closure of \( W \), with the addition of \( A \), forms an algebraic subgroup of \( A \). Small values of \( z \) are separated, thus \( W \) is infinite. As \( A \) is simple, this implies that \( \overline{W} = A_{\mathbb{C}} \). Therefore the Zariski closure of

\[
\{ \Theta(z + su_1, \beta z + su_2) \mid z \in \mathbb{C}, \partial_0(z + su_1, \beta z + su_2) \neq 0 \}
\]

is also equal to \( A_{\mathbb{C}} \). Now suppose for a moment that for some \( s \) in \( V \). By continuity, this implies that (13) also holds if \( j = 1 \). But that contradicts either the algebraic independence of \( f_1 \) and \( f_2 \) or the linear independence of \( \xi_0, \ldots, \xi_{D-1} \). Thus

\[
\text{ord}_z Q_1(h_{1,s}(z), \ldots, h_{p,s}(z)) = \infty \quad (s \in V).
\]

for some \( s \) in \( V \). By continuity, this implies that (13) also holds if \( j = 1 \). But that contradicts either the algebraic independence of \( f_1 \) and \( f_2 \) or the linear independence of \( \xi_0, \ldots, \xi_{D-1} \). Thus

\[
\text{ord}_z Q_1(h_{1,s}(z), \ldots, h_{p,s}(z)) < \infty \quad (s \in V).
\]

The set of common zeros of \( Q_2, \ldots, Q_n \) has algebraic dimension two (cf. [9], (2.7)). As, by (14) and (15), \( Q_1 \) is not in the ideal generated by \( Q_2, \ldots, Q_n \), the set of common zeros of \( Q_1, \ldots, Q_n \) has algebraic dimension at most one (cf. [9], (1.14)). It is no restriction to assume \( n > \nu \). Then the Main Theorem of [2] implies that either

\[
\sum_{s \in V} \text{ord}_z Q_1(h_{1,s}(z), \ldots, h_{p,s}(z)) \leq \text{exp}(c_{31} \times 16 D^6 \log^2 A \log^4 B' \log^{-6} E),
\]

which contradicts (12) if \( x > c_{32} c_{31} \), or the points \( \Theta(su) \) are not all different. As \( \Theta \) induces an isomorphism between \( \mathbb{C}^2 / \Omega \) and \( A_{\mathbb{C}} \), the equality of \( \Theta(su) \) and \( \Theta(s'u) \), say, shows that there is an \( \omega \in \Omega \) with

\[
su = s'u + \omega.
\]

Therefore we have now proved the theorem under the hypothesis

\[
\forall m \leq S \text{ } mu \notin \Omega.
\]

III. It now remains to prove the theorem in the case where \( mu \in \Omega \) for some \( m \leq S \). In particular, let \( m \) be the smallest positive integer with this property; then the points \( \Theta(y), \Theta(2y), \ldots, \Theta(mu) \) are all different. As before, we can choose a subset \( V' \) of \( \{1, \ldots, m\} \) such that \( \#V' \geq c_{32}^{-1} m \) with the property that \( (su_1, su_2) \) and \( (su_1, s'u_1) \) lie in \( U + \Omega \) for all \( s \) in \( V' \). Put
where \( E, B' \) retain their earlier meaning, and let \( F \) and \( F_s \) be defined again by (4) and (5). Put

\[
T := \left[ x^9 \, m^4 \log^2 A \log^2 B' \log^{-4} E \right]
\]

and consider the system of linear equations

\[
F_s^{(t)}(u_1) = 0 \quad (s \in V', \ t = 0, \ldots, T-1).
\]  

By the same method used earlier, it is proved that the coefficients \( p(\lambda_1, \lambda_2, \delta) \) may be chosen in such a way that they are not all zero and (16) holds. Now let \( V \) be the set of all \( s \in \{1, \ldots, S\} \) that differ by a multiple of \( m \) from an element of \( V' \); here \( S \) has the same meaning as before. Then

\[
\#V > c_{33}^{-1} S ; \quad \text{as \( mu \) is a period of every \( f_i \), (16) implies}
\]

\[
F_s^{(t)}(u_1) = 0 \quad (s \in V, \ t = 0, \ldots, T-1).
\]

Repeating the extrapolation procedure gives

\[
F_s^{(t)}(u_1) = 0 \quad (s \in V, \ t = 0, \ldots, T'-1)
\]

where \( T' := [x^2 T] \). Define \( Q_i \) and \( h_{i,s} \) as before; then

\[
\sum_{s \in V'} \text{ord}_z Q_1(h_{1,s}(z), \ldots, h_{n,s}(z)) \geq c_{32}^{-1} m T' \geq c_{34}^{-1} x^{11} m^2 D^4 \log^2 A \log^2 B' \log^{-4} E.
\]

Another application of the Main Theorem of [2] gives the desired contradiction. Note that for this special case of the theorem we may replace (1) with

\[
|\beta u_1 - u_2| > \exp(-CmD^5 \log^2 A \log^3 (DB \log A)\log^{-4} (D \log A)),
\]

which is sharper if \( m \) is small compared to \( S \).

As a corollary to Theorem 1, an abelian analogue of the Franklin-Schneider theorem is easily obtained. It should be noted that the assumption as to the nature of \( \beta \), necessary in the exponential and elliptic versions of this result (cf. [1]) does not occur here.

**THEOREM 2.** For every point \( \mathbf{a} \) in \( \mathbb{C}^2 \setminus \{0\} \) such that \( \delta_0(\mathbf{a}) \neq 0 \), there exists an effectively computable \( C^* \) with the following property. Let \( \alpha_1, \ldots, \alpha_p, \beta \) be algebraic numbers, let \( A > e^0 \) be an upper bound for the heights of \( \alpha_1, \ldots, \alpha_p \) and let \( B > e \) be an upper bound for the height of \( \beta \).
Then if \( D = \{ Q(\alpha_1, \ldots, \alpha_p) : Q \} \), we have

\[
(17) \quad \sum_{i=1}^{\nu} |f_i(a) - \alpha_i| + |\beta a_1 - a_2| > \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A)).
\]

**Proof.** Let \( Q_2, \ldots, Q_n \) be generators of the ideal of \( \mathbb{C}[X_1, \ldots, X_p] \) corresponding to the affine part of \( A \). If \( Q_j(\alpha_1, \ldots, \alpha_p) \neq 0 \) for some \( j \) with \( 2 \leq j \leq n \), then the result is trivial, as \( Q_j(f_1(a), \ldots, f_p(a)) = 0 \). Thus we may assume \( (\alpha_1, \ldots, \alpha_p) \) to be on the affine part of \( A \). By the smoothness of \( A \) at \( \Theta(a) \), the matrix of partial derivatives of \( (f_1, \ldots, f_p) \) at \( a \) has rank 2. Thus there exist \( k \) and \( \ell \) such that the matrix of partial derivatives of \( (f_k, f_\ell) \) at \( a \) has rank 2. According to Theorem 7.4 in Chapter I of [4], there are open neighbourhoods \( U \) of \( a \) and \( V \) of \( (f_k(a), f_\ell(a)) \) such that \( (f_k, f_\ell) \) induces a biholomorphic mapping from \( U \) onto \( V \). If \( C'' \) is sufficiently large, the negation of (17) implies that \( f_\ell(0) \) belongs to \( V \) for some \( u \in U \) and \( c \) that depends only on \( a \) and \( \Theta \). Thus

\[
|a - u| \leq c \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A))
\]

for some \( c \) that depends only on \( a \) and \( \Theta \). Thus

\[
(18) \quad |\beta u_1 - u_2| \leq |\beta a_1 - \beta u_1| + |a_2 - u_2| + |\beta a_1 - a_2| \leq (|\beta| c + c + 1) \exp(-C''D^6 \log^2 A \log^4(DB \log A) \log^{-5}(D \log A)).
\]

Let \( K \) be a compact subset of \( \mathbb{C}^2 \setminus \{0\} \) containing a neighbourhood of \( a \) but no zeros of \( \partial_\Theta ; \) by Theorem 1, (18) is impossible if \( C'' \) is sufficiently large in terms of \( c \) and \( K \).
REFERENCES


(Manuscrit reçu le 26 juin 1981)