SORIN DRAGOMIR

RENATA GRIMALDI

Generalized Hopf manifolds with flat local Kaelher metrics

Annales de la faculté des sciences de Toulouse 5e série, tome 10, nº 3 (1989), p. 361-368

<http://www.numdam.org/item?id=AFST_1989_5_10_3_361_0>

Annales Faculté des Sciences de Toulouse Vol. X, n°3, 1989

Generalized Hopf manifolds with flat local Kaelher metrics

SORIN DRAGOMIR[1] AND RENATA GRIMALDI[2]

RÉSUMÉ. — On donne un résultat du type B.Y. Chen et M. Okumura (voir [3]) sur la courbure scalaire d'une sous-variété M d'une variété de Vaisman (c'est-à-dire une variété localement conformément Kaehlérienne ayant la forme de Lee parallèle et les métriques locales Kaehlériennes plates). Si M est une sous-variété de Cauchy-Riemann Levi-plate (d'une variété de Vaisman), alors on calcule les courbures sectionnelles complexes de M.

ABSTRACT. — We give a B.Y. Chen and M. Okumura (see [3]) type result on the scalar curvature of a submanifold M of a Vaisman manifold (i.e. a locally conformal Kaehler manifold having a parallel Lee form and flat local Kaehler metrics). If M is a Levi-flat Cauchy-Riemann submanifold (of a Vaisman manifold), the complex sectional curvatures of M are estimated.

1. Introduction and statement of results

Let (M, g, J) be a Hermitian manifold of complex dimension n, with the complex structure J and the Hermitian metric g. It is *locally conformal Kaehler* (l.c.K.) if there exists an open covering $(U_i)_{i \in I}$ of M and a family $(f_i)_{i \in I}$ of real valued smooth functions $f_i \in C^\infty(U_i)$ such that each $g_i = \exp(-f_i)g$ is a Kaehler metric on U_i, $i \in I$.

The local 1-forms df_i of a l.c.K. manifold M are known to glue up to a globally defined (closed) 1-form ω on M, namely the *Lee form*.

A l.c.K. manifold is a *generalized Hopf* (g.H.) manifold if its Lee form is parallel with respect to the Riemannian connection of (M, g). Typical

(1) Università degli Studi di Bari, Dipartimento di Matematica, Via G. Fortunato, Campus Universitario, 70125 Bari, Italia.
(2) Università di Palermo, Dipartimento di Matematica e Applicazioni, Via Archirafi 34, 90123 Palermo, Italia.

examples of g.H. manifolds are products $S \times I\!R$ between a Sasaki manifold S and the real line, see [10], p. 614.

Let M be a g.H. manifold. It is said to be a *Vaisman manifold* if the local Kaehler metrics $g_i, i \in I$, of M are flat. Each complex Hopf manifold $CH^n = W/G_d$, $W = C^n - \{0\}$, $G_d = \{d^m I : m \in z\}$, $d \in C - \{0\}$, $|d| \neq 1$, is a Vaisman manifold in a natural way. Indeed, let $g_0 = |z|^{-2}\delta_{ij} dz^i \otimes d\overline{z}^j$, where $|z|^2 = \delta_{ij} z^i \overline{z}^j$ and $(z^1, \dots, z^n)$ are the natural complex analytic coordinates on W. Note that g_0 is G_d-invariant, thus giving rise to a (globally defined) l.c.K. metric on CH^n.

Let M be a Vaisman manifold. Since the Lee form ω is parallel, its norm is constant; set $\|\omega\| = 2c$, $c \in I\!R - \{0\}$. The local structure of Vaisman manifolds is completely understood due to a deep result of I. Vaisman, (thus justifying our terminology), i.e. the theorem 3.8. in [12], p. 277, asserting that the universal covering of M is W with the metric $\rho^2 g_0$, $\rho = \dfrac{1}{c}$.

The curvature form of a Vaisman manifold is expressed by

$$\begin{aligned} R(X,Y)Z = {} & \frac{1}{4}\{[\omega(X)Y - \omega(Y)X]\omega(Z) + \\ & + [g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\} + \\ & + \frac{1}{4}\|\omega\|^2\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{1.1}$$

for any tangent vector fields X, Y, Z on M, see (2.1) of [13], p. 441. Here $B = \omega^{\#}$ is the *Lee field* of M, while $\#$ denotes raising of indices with respect to g. As a consequence of (1.1) one obtains the following results :

THEOREM 1.— *Let M be an n-dimensional submanifold of a Vaisman manifold. If the scalar curvature ρ of M is subject to :*

$$\rho \geq (n-2)\|h\|^2 + (n-2)(n-1)c^2 + 2(n-1)A \tag{1.2}$$

at a point $x \in M$ for some $A \in I\!R$ then the sectional curvatures of M are $\geq A$ at the point x.

If $\overline{M}$ is a Vaisman manifold and $j : M \to \overline{M}$ the given immersion of M in $\overline{M}$, then h denotes the second fundamental form of j. Let ω_0 be the Lee form of $\overline{M}$ and $\omega = j^*\omega_0$. Since ω is closed, the distribution $\mathrm{Ker}(\omega)$ is integrable thus defining a canonical foliation $\mathcal{F}$ on M, see also [4].

THEOREM 2.— *Let M be a Levi flat Cauchy-Riemann submanifold of a Vaisman manifold. Let $p \in G_2(M)$, $p \subseteq D_{\pi(p)}$, $J(p) = p$. Then the complex* *sectional curvature* k_C *of* M *verifies :*

$$k_C(p) \leq c^2 - \omega_0(h(X,X)) \tag{1.3}$$

for any $X \in p, \|X\| = 1$. *The equality holds if and only if* p *is tangent to some leaf of* $\mathcal{F}$ *passing through* x *and* $h_x = 0$ *on* $p \times p$.

Here $\pi : G_2(M) \to M$ denotes the Grassman bundle of all 2-planes tangent to M. Also D stands for the Levi distribution of the C.R. submanifold M, (i.e. D_x is the maximal holomorphic subspace of $T_x(M)$, $x \in M$).

For other results concerning the geometry of (the second fundamental form of) submanifolds in l.c.K. manifolds see [4], [5], [6], [7], [8].

2. Scalar curvature of submanifolds in Vaisman manifolds

Let M be an n-dimensional submanifold of a Vaisman manifold $(\overline{M}, \overline{g}, J)$. By (2.8) in [4], p. 203, the Gauss equation of M in $\overline{M}$ is given by :

$$\begin{aligned} R(X,Y)Z = {} & A_{h(Y,Z)}X - A_{h(X,Z)}Y + \\ & + \frac{1}{4}\{[\omega(x)Y - \omega(Y)X]\omega(Z) + \\ & + [g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\} + \\ & + \frac{1}{4}\|\omega_0\|^2\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{2.1}$$

for any tangent vector fields X, Y, Z on M. Here $g = j^*\overline{g}$. Moreover A_ξ is the Weingarten operator (associated with the normal section ξ). Suitable contraction of indices in (2.1) leads to the expression of the Ricci tensor of (M, g), i.e.

$$\begin{aligned} R_{jk} = {} & h^a_{jk} \text{ Trace } (A_a) - g^{is} h^a_{ik} h^b_{js} \delta_{ab} + \\ & + c^2(n-2) g_{jk} - \frac{n-2}{4}\omega_j \omega_k \end{aligned} \tag{2.2}$$

Indices $i, j, k, \ldots$ run from 1 to n, while $a, b, c, \ldots$ from 1 to $\text{codim}(M) = 2m - n$. Further contraction of indices in (2.2) gives :

$$\rho = n^2\|H\|^2 - \|h\|^2 + c^2(n-1)(n-2) \tag{2.3}$$

Here ρ, H denote respectively the scalar curvature of (M, g) and the mean curvature vector (i.e. $H = \frac{1}{n}$ Trace (h)) of the given immersion j.

Let $k : G_2(M) \to I\!R$ be the sectional curvature of (M,g). Let $p \in G_2(M)$ and $\{X,Y\}$ an orthonormal basis in p. By (2.1) one obtains :

$$k(p) = \overline{g}(h(X,X),h(Y,Y)) - \|h(X,Y)\|^2 + \\ + c^2 - \frac{1}{4}\{\omega(X)^2 + \omega(Y)^2\} \tag{2.4}$$

At this point we may prove our Theorem 1. To this end, let $x \in M$ and (U, x^i) be normal coordinates at x.

Substitution from (2.3) into(1.2) furnishes :

$$n^2\|H\|^2 \geq (n-1)\|h\|^2 + 2(n-1)A \tag{2.5}$$

Let ξ_a, $1 \leq a \leq 2m-n$, $\dim(\overline{M}) = 2m$, be an orthonormal frame in the normal bundle $T(M)^\perp$ of the given immersion. For simplicity, we may choose ξ_1 to be collinear with H at x (if $H_x \neq 0$, and arbitrary if $H_x = 0$ occurs). Let $X_i = \dfrac{\partial}{\partial x^i}$, $1 \leq i \leq n$. We set $h(X_i,X_j) = h^a_{ij}\xi_a$. Also $h^i_{ja} = g^{ik}h^a_{jk}, h^{ij}_a = g^{jk}g^{is}h^a_{ks}$. Clearly $h^a_{ij} = h^a_{ji}$. All computations are carried out at x (where $g_{ij} = \delta_{ij}$) so that $h^a_{ji} = h^i_{ja} = h^{ji}_a$ at x. Let us put $h_{ij} = h^1_{ij}$. Then :

$$n^2\|H\|^2 = \left(\sum_{i=1}^n h_{ii}\right)^2 \tag{2.6}$$

Substitution from (2.6) into (2.5) gives :

$$\left(\sum_{i=1}^n h_{ii}\right)^2 \geq (n-1)\left\{\sum_{i=1}^n (h_{ii})^2 + \right. \\ \left. + \sum_{i\neq j}(h_{ij})^2 + \sum_{\substack{a\geq 2\\ 1\leq i,j,\leq n}} (h^a_{ij})^2 \right\} + \\ + 2(n-1)A \tag{2.7}$$

since $\|h\|^2 = h^a_{ji}h^{ji}_a$. We shall need the following :

LEMMA .— *(B.Y. Chen and M. Okumura, [3])*

Let $a_1,\dots,a_n,b$ be real numbers, $n > 1$, with the property :

$$\left(\sum_{i=1}^n a_i\right)^2 \geq (n-1)\sum_{i=1}^n (a_i)^2 + b$$

Then for any $i \neq j$ *one has* $2a_i a_j \geq \dfrac{b}{n-1}$.

At this point we may use (2.7) and the Lemma (for $a_i = h_{ii}$) such as to yield :

$$\begin{aligned} (2.8) \qquad & h_{ii}h_{jj} - (h_{ij})^2 \geq \\ & \geq \sum_{a=2}^{2m-n} \{|h^a_{ii}h^a_{jj}| + (h^a_{ij})^2\} + A \end{aligned}$$

for any $i \neq j$. Set $\omega_i = \omega(X_i)$. Then $\omega_i^2 + \omega_j^2 \leq \sum_{i=1}^{n} \omega_i^2 = \|\omega\|^2 = 4c^2$. Let $p_{ij} \in G_2(M)$ be spanned by X_i, X_j, $i \neq j$. Finally, using (2.4) and (2.8) we have :

$$\begin{aligned} k(p_{ij}) = & \sum_{a=1}^{2m-n} (h^a_{ii}h^a_{jj} - (h^a_{ij})^2) + \\ & + c^2 - \frac{1}{4}(\omega_i^2 + \omega_j^2) \geq A \end{aligned}$$

Q.E.D.

This extends Theorem 4.1. in [1], p. 55, to the case of submanifolds in Vaisman manifolds.

3. Cauchy-Riemann submanifolds of Vaisman manifolds

Let $(\overline{M}, \overline{g}, J)$ be a Vaisman manifold of complex dimension m and M a real n-dimensional Cauchy-Riemann (C.R.) submanifold of $\overline{M}$. That is M carries a pair of orthogonal (complementary) distributions $D, D^\perp$ such that D is holomorphic, i.e. $J_x(D_x) = D_x$, $x \in M$, while $D^\perp$ is totally-real, i.e. $J_x(D_x^\perp) \subseteq T_x(M)^\perp$, $x \in M$. See also [15], p. 83. Hereafter D is called the *Levi distribution* of M. Moreover, if D is integrable, the C.R. submanifold M is said to be *Levi flat*.

Let $B_0 = \omega_0^\#$ and $A_0 = -JB_0$ be the Lee, respectively the *anti-Lee vector* fields of $\overline{M}$. Also $\theta_0 = \omega_0 \circ J$ will denote the *anti-Lee form*.

Let X, ξ be respectively a tangent vector field on M and a normal section. We set $PX = \tan(J, X)$, $FX = \mathrm{nor}(JX)$, $t\xi = \tan(J\xi)$, $f\xi = \mathrm{nor}(J\xi)$.

Here $\tan_x$, nor_x denote the natural projections associated with the direct sum decomposition $T_x(\overline{M}) = T_x(M) \oplus T_x(M)^\perp$, for any $x \in M$. Note that

P is D-valued. Also $F = 0$ on D. Moreover the following identities hold :

$$
\begin{aligned}
F \circ P = 0, f \circ F = 0 \\
P^2 + t \circ F = -I \\
t \circ f = 0, P \circ t = 0 \\
f^2 + F \circ t = -I
\end{aligned} \tag{3.1}
$$

Let $B = \tan(B_0)$, $B^\perp = \mathrm{nor}(B_0), A = \tan(A_0)$ and $A^\perp = \mathrm{nor}(A_0)$. Note that :

$$
A = -PB - tB^\perp, A^\perp = -FB - fB^\perp \tag{3.2}
$$

The complex structure J is not parallel with respect to the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$. Nevertheless $\overline{M}$ admits a significant almost complex connection $\overline{D}$, namely the *Weyl connection*, i.e.

$$
\begin{aligned}
\overline{D}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\{\omega_0(X)Y + \omega_0(Y)X - \\
- \overline{g}(X,Y)B_0\}
\end{aligned} \tag{3.3}
$$

Since $\overline{D}J = 0$, (3.3) yields :

$$
\begin{aligned}
\overline{\nabla}_X JY = J\overline{\nabla}_X Y + \\
+ \frac{1}{2}\{\theta_0(Y)X + \omega_0(Y)JX - \\
- \overline{\Omega}(X,Y)B_0 - \overline{g}(X,Y)A_0\}
\end{aligned} \tag{3.4}
$$

Here $\overline{\Omega}$ denotes the Kaehler 2-form of $\overline{M}$. By (3.4) and the Gauss formula (1.10) of [1], p. 38, one obtains :

$$
\begin{aligned}
\nabla_X JY = P\nabla_X Y + th(X,Y) + \\
+ \frac{1}{2}\{\theta(Y)X + \omega(Y)JX - \\
- \Omega(X,Y)B - g(X,Y)A\}
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
h(X,JY) = fh(X,Y) + F\nabla_X Y - \\
- \frac{1}{2}\{\Omega(X,Y)B^\perp + g(X,Y)A^\perp\}
\end{aligned} \tag{3.6}
$$

for any $X, Y \in D$. Here ∇ denotes the Levi-Civita connection of (M, g) and $\theta = j^*\theta_0$, $\Omega = j^*\overline{\Omega}$.

Let us denote by k_C the restriction of the sectional curvature k of M to the holomorphic 2-planes $p \in G_2(M)$, $J(p) = p$, with the property $p \subseteq D_x$, $x = \pi(p)$, $x \in M$. Then k_C is called the *complex sectional curvature* of the C.R. submanifold M.

At this point we may prove our Theorem 2. As the Levi distribution D is integrable, one has $F\nabla_X Y = F\nabla_Y X$, for any $X, Y \in D$. By (3.5)-(3.6) one obtains :

$$h(JX, JY) = -h(X,Y) - g(X,Y)B^{\perp} \tag{3.7}$$

for any $X, Y \in D$. Let us apply (2.4) for the 2-plane $p \in G_2(M)$ spanned by $\{X, JX\}$, $X \in D_x$, $\|X\| = 1$, $x = \pi(p)$. It follows :

$$\begin{aligned} k_C(p) = \overline{g}(h(X,X), h(JX,JX)) - \|h(X,JX)\|^2 + \\ + c^2 - \frac{1}{4}\{\omega(X)^2 + \theta(X)^2\} \end{aligned} \tag{3.8}$$

Finally (3.7)-(3.8) lead to (1.3). If equality holds, then $h(X,X) = 0$, $\omega(X) = 0$, $\omega(JX) = 0$, (and actually (1.3) reads $k_C(p) = c^2$). The converse is obvious, Q.E.D.

Références

[1] CHEN (B.Y.). — *Geometry of submanifolds*, M. Dekker, Inc., New York, 1973.

[2] CHEN (B.Y.). — Cohomology of C.R. submanifolds, *Ann. Fac. Sc. Toulouse*, Math **3** 1981, p. 167-172.

[3] CHEN (B.Y.), OKUMURA (M.). — Scalar curvature, inequality and submanifold, *Proc. A.M.S.*, **38** 1973, p. 605-607.

[4] CHEN (B.Y.), PICCINNI (P.). — The canonical foliations of a locally conformal Kaehler manifold, *Ann. di Matem. pura appl.*, **141** 1985, p. 283-305.

[5] DRAGOMIR (S.). — On submanifolds of Hopf manifolds Israel J. Math., (2) **61** 1988, p. 199-210.

[6] DRAGOMIR (S.). — *Cauchy-Riemann submanifolds of locally conformal Kaehler manifolds, I-II*, Geometriae Dedicata **28** 1988, p. 181-197, Atti Sem. Mat. Fis. Univ. Modena, **37** 1989, p. 1-11.

[7] IANUS (S.), MATSUMOTO (K.), ORNEA (L.). — Complex hypersurfaces of a generalized Hopf manifold, *Publication de l'Inst. Mathém., N.S.*, **42** 1987, p. 123-129.

[8] MATSUMOTO (K.). — On submanifolds of locally conformal Kaehler manifolds, *Bull. Yamagoto Univ., N.S.*, **11** 1984, p. 33-38.

[9] TRICERRI (F.). — Some examples of locally conformal Kaehler manifolds, *Rend. Sem. Mat. Politecn. Torino* **40** 1982, p. 81-92.

[10] TSUKADA (K.). — Hopf manifolds and spectral geometry, *Trans. A.M.S.*, **(2)270** 1982, p. 609-621.

[11] VAISMAN (I.). — A theorem on compact locally conformal Kaehler manifolds, *Proc. A.M.S.*, **(2)75** 1979, p. 279-283.

[12] VAISMAN (I.). — *Locally conformal Kaehler manifolds with parallel Lee form*, Rendiconti di Matem., Roma **12** 1979, p. 263-284.

[13] VAISMAN (I.). — Some curvature properties of locally conformal Kaehler manifolds *Trans A.M.S.*, **(2)259** 1980, p. 439-447.

[14] VAISMAN (I.), GOLDBERG (S.). — On compact locally conformal Kaehler manifolds with non-negative sectional curvature, *Ann. Fac. Sci. Toulouse, Math.* **2** 1980, p. 117-123.

[15] YANO (K.), KON (M.). — *C.R. submanifolds of Kaehlerian and Sasakian manifolds*, Progress in Math., vol. 30, Birkhauser, Boston-Basel-Stuttgart, 1983.

(Manuscrit reçu le 30 avril 1988)