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Generalized Hopf manifolds with flat local Kaelher metrics


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Generalized Hopf manifolds with flat local Kähler metrics

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Résumé. — On donne un résultat du type B.Y. Chen et M. Okumura (voir [3]) sur la courbure scalaire d'une sous-variété $M$ d'une variété de Vaisman (c'est-à-dire une variété localement conformément Kähleriennes ayant la forme de Lee parallèle et les métriques locales Kählériennes plates). Si $M$ est une sous-variété de Cauchy-Riemann Levi-plate (d'une variété de Vaisman), alors on calcule les courbures sectionnelles complexes de $M$.

Abstract. — We give a B.Y. Chen and M. Okumura (see [3]) type result on the scalar curvature of a submanifold $M$ of a Vaisman manifold (i.e. a locally conformal Kähler manifold having a parallel Lee form and flat local Kähler metrics). If $M$ is a Levi-flat Cauchy-Riemann submanifold (of a Vaisman manifold), the complex sectional curvatures of $M$ are estimated.

1. Introduction and statement of results

Let $(M, g, J)$ be a Hermitian manifold of complex dimension $n$, with the complex structure $J$ and the Hermitian metric $g$. It is locally conformal Kähler (l.c.K.) if there exists an open covering $(U_i)_{i \in I}$ of $M$ and a family $(f_i)_{i \in I}$ of real valued smooth functions $f_i \in C^\infty(U_i)$ such that each $g_i = \exp(-f_i)g$ is a Kähler metric on $U_i$, $i \in I$.

The local 1-forms $df_i$ of a l.c.K. manifold $M$ are known to glue up to a globally defined (closed) 1-form $\omega$ on $M$, namely the Lee form.

A l.c.K. manifold is a generalized Hopf (g.H.) manifold if its Lee form is parallel with respect to the Riemannian connection of $(M, g)$. Typical
examples of g.H. manifolds are products $S \times \mathbb{R}$ between a Sasaki manifold $S$ and the real line, see [10], p. 614.

Let $M$ be a g.H. manifold. It is said to be a Vaisman manifold if the local Kaehler metrics $g_i, i \in I$, of $M$ are flat. Each complex Hopf manifold $CH^n = W/G_d, W = C^n - \{0\}, G_d = \{d^m I : m \in z\}, d \in C - \{0\}, |d| \neq 1$, is a Vaisman manifold in a natural way. Indeed, let $g_0 = |z|^{-2}\delta_{ij}dz^i \otimes dz^j$, where $|z|^2 = \delta_{ij}z^i z^j$ and $(z^1, \ldots, z^n)$ are the natural complex analytic coordinates on $W$. Note that $g_0$ is $G_d$-invariant, thus giving rise to a (globally defined) l.c.K. metric on $CH^n$.

Let $M$ be a Vaisman manifold. Since the Lee form $\omega$ is parallel, its norm is constant; set $\|\omega\| = 2c, c \in \mathbb{R} - \{0\}$. The local structure of Vaisman manifolds is completely understood due to a deep result of I. Vaisman, (thus justifying our terminology), i.e. the theorem 3.8. in [12], p. 277, asserting that the universal covering of $M$ is $W$ with the metric $\rho^2 g_0, \rho = \frac{1}{c}$.

The curvature form of a Vaisman manifold is expressed by

\[ R(X, Y)Z = \frac{1}{4} \{ [\omega(X)Y - \omega(Y)X] \omega(Z) + 
\quad + [g(X, Z)\omega(Y) - g(Y, Z)\omega(X)]B \} + \n\quad + \frac{1}{4}\|\omega\|^2 \{ g(Y, Z)X - g(X, Z)Y \} \]

for any tangent vector fields $X, Y, Z$ on $M$, see (2.1) of [13], p. 441. Here $B = \omega#$ is the Lee field of $M$, while $#$ denotes raising of indices with respect to $g$. As a consequence of (1.1) one obtains the following results:

**Theorem 1.** — Let $M$ be an $n$-dimensional submanifold of a Vaisman manifold. If the scalar curvature $\rho$ of $M$ is subject to:

\[ \rho \geq (n - 2)\|h\|^2 + (n - 2)(n - 1)c^2 + 2(n - 1)A \]

at a point $x \in M$ for some $A \in \mathbb{R}$ then the sectional curvatures of $M$ are $\geq A$ at the point $x$.

If $\overline{M}$ is a Vaisman manifold and $j : M \rightarrow \overline{M}$ the given immersion of $M$ in $\overline{M}$, then $h$ denotes the second fundamental form of $j$. Let $\omega_0$ be the Lee form of $\overline{M}$ and $\omega = j^*\omega_0$. Since $\omega$ is closed, the distribution $\text{Ker}(\omega)$ is integrable thus defining a canonical foliation $\mathcal{F}$ on $M$, see also [4].

**Theorem 2.** — Let $M$ be a Levi flat Cauchy-Riemann submanifold of a Vaisman manifold. Let $p \in G_2(M), p \subseteq D_{\pi(p)}, J(p) = p$. Then the complex
sectional curvature \( k_C \) of \( M \) verifies:

\[
(1.3) \quad k_C(p) \leq c^2 - \omega_0(h(X,X))
\]

for any \( X \in p, \|X\| = 1 \). The equality holds if and only if \( p \) is tangent to some leaf of \( \mathcal{F} \) passing through \( x \) and \( h_x = 0 \) on \( p \times p \).

Here \( \pi : G_2(M) \to M \) denotes the Grassman bundle of all 2-planes tangent to \( M \). Also \( D \) stands for the Levi distribution of the C.R. submanifold \( M \), (i.e. \( D_x \) is the maximal holomorphic subspace of \( T_x(M), x \in M \)).

For other results concerning the geometry of (the second fundamental form of) submanifolds in l.c.K. manifolds see [4], [5], [6], [7], [8].

2. Scalar curvature of submanifolds in Vaisman manifolds

Let \( M \) be an \( n \)-dimensional submanifold of a Vaisman manifold \((\overline{M}, \overline{g}, J)\).

By (2.8) in [4], p. 203, the Gauss equation of \( M \) in \( \overline{M} \) is given by:

\[
(2.1) \quad R(X,Y)Z = A_{h(Y,Z)}X - A_{h(X,Z)}Y + \frac{1}{4} \{[\omega(x)Y - \omega(Y)X]\omega(Z) + \{[g(X,Z)\omega(Y) - g(Y,Z)\omega(X)]B\} + \frac{1}{4} \|\omega_0\|^2 \{g(Y,Z)X - g(X,Z)Y\}
\]

for any tangent vector fields \( X, Y, Z \) on \( M \). Here \( g = j^*\overline{g} \). Moreover \( A_{\xi} \) is the Weingarten operator (associated with the normal section \( \xi \)). Suitable contraction of indices in (2.1) leads to the expression of the Ricci tensor of \((M,g)\), i.e.

\[
(2.2) \quad R_{jk} = h^a_{jk} \text{Trace} (A_a) - g^{is}h^a_{ik}h^b_{js}\delta_{ab} + c^2(n-2)g_{jk} - \frac{n-2}{4}\omega_j\omega_k
\]

Indices \( i, j, k, \ldots \) run from 1 to \( n \), while \( a, b, c, \ldots \) from 1 to \( \text{codim}(M) = 2m - n \). Further contraction of indices in (2.2) gives:

\[
(2.3) \quad \rho = n^2\|H\|^2 - \|h\|^2 + c^2(n-1)(n-2)
\]

Here \( \rho, H \) denote respectively the scalar curvature of \((M,g)\) and the mean curvature vector (i.e. \( H = \frac{1}{n} \text{Trace} (h) \)) of the given immersion \( j \).
Let \( k : G^2(M) \to \mathbb{R} \) be the sectional curvature of \((M, g)\). Let \( p \in G^2(M) \) and \( \{X, Y\} \) an orthonormal basis in \( p \). By (2.1) one obtains:

\[
(2.4) \quad k(p) = \overline{g}(h(X, X), h(Y, Y)) - \|h(X, Y)\|^2 + c^2 - \frac{1}{4} \{\omega(X)^2 + \omega(Y)^2\}
\]

At this point we may prove our Theorem 1. To this end, let \( x \in M \) and \((U, x^i)\) be normal coordinates at \( x \).

Substitution from (2.3) into (1.2) furnishes:

\[
(2.5) \quad n^2\|H\|^2 \geq (n - 1)\|h\|^2 + 2(n - 1)A
\]

Let \( \xi_a, 1 \leq a \leq 2m - n, \dim(M) = 2m \), be an orthonormal frame in the normal bundle \( T(M)^\perp \) of the given immersion. For simplicity, we may choose \( \xi_1 \) to be collinear with \( H \) at \( x \) (if \( H_x \neq 0 \), and arbitrary if \( H_x = 0 \) occurs). Let \( X_i = \frac{\partial}{\partial x^i}, 1 \leq i \leq n \). We set \( h(X_i, X_j) = h^a_{ij}\xi_a \). Also \( h^a_{ij} = g^{ik}h^a_{jk}, h^a_{ij} = g^{ik}g^{is}h^a_{ks} \). Clearly \( h^a_{ij} = h^a_{ji} \). All computations are carried out at \( x \) (where \( g_{ij} = \delta_{ij} \)) so that \( h^a_{ij} = h^a_{ji} = h^a_{ij} \) at \( x \). Let us put \( h_{ij} = h^1_{ij} \). Then:

\[
(2.6) \quad n^2\|H\|^2 = \left( \sum_{i=1}^{n} h_{ii} \right)^2
\]

Substitution from (2.6) into (2.5) gives:

\[
(2.7) \quad \left( \sum_{i=1}^{n} h_{ii} \right)^2 \geq (n - 1) \left\{ \sum_{i=1}^{n} (h_{ii})^2 + \sum_{i \neq j} (h_{ij})^2 + \sum_{a \geq 2} \sum_{1 \leq i, j \leq n} (h^a_{ij})^2 \right\} + 2(n - 1)A
\]

since \( \|h\|^2 = h^a_{ij}h^a_{ij} \). We shall need the following:

**Lemma.** — \( (B. Y. Chen and M. Okumura, [3]) \)

Let \( a_1, \ldots, a_n, b \) be real numbers, \( n > 1 \), with the property:

\[
\left( \sum_{i=1}^{n} a_i \right)^2 \geq (n - 1) \sum_{i=1}^{n} (a_i)^2 + b
\]
Then for any $i \neq j$ one has $2a_ia_j \geq \frac{b}{n-1}$.

At this point we may use (2.7) and the Lemma (for $a_i = h_{ii}$) such as to yield:

\begin{equation}
\begin{align*}
(2.8) \quad h_{ii}h_{jj} - (h_{ij})^2 & \geq \\
& \geq \sum_{a=2}^{2m-n} \{|h_{ii}^a h_{jj}^a| + (h_{ij}^a)^2\} + A
\end{align*}
\end{equation}

for any $i \neq j$. Set $\omega_i = \omega(X_i)$. Then $\omega_i^2 + \omega_j^2 \leq \sum_{i=1}^n \omega_i^2 = ||\omega||^2 = 4c^2$. Let $p_{ij} \in G_2(M)$ be spanned by $X_i, X_j, i \neq j$. Finally, using (2.4) and (2.8) we have:

\begin{equation}
k(p_{ij}) = \sum_{a=1}^{2m-n} (h_{ii}^a h_{jj}^a - (h_{ij}^a)^2) + \\
+ c^2 - \frac{1}{4}(\omega_i^2 + \omega_j^2) \geq A
\end{equation}

Q.E.D.

This extends Theorem 4.1. in [1], p. 55, to the case of submanifolds in Vaisman manifolds.

3. Cauchy-Riemann submanifolds of Vaisman manifolds

Let $(\overline{M}, \overline{g}, J)$ be a Vaisman manifold of complex dimension $m$ and $M$ a real $n$-dimensional Cauchy-Riemann (C.R.) submanifold of $\overline{M}$. That is $M$ carries a pair of orthogonal (complementary) distributions $D, D^\perp$ such that $D$ is holomorphic, i.e. $J_x(D_x) = D_x, x \in M$, while $D^\perp$ is totally-real, i.e. $J_x(D^\perp_x) \subseteq T_x(M)^\perp, x \in M$. See also [15], p. 83. Hereafter $D$ is called the Levi distribution of $M$. Moreover, if $D$ is integrable, the C.R. submanifold $M$ is said to be Levi flat.

Let $B_0 = \omega_0^\#$ and $A_0 = -J B_0$ be the Lee, respectively the anti-Lee vector fields of $\overline{M}$. Also $\theta_0 = \omega_0 \circ J$ will denote the anti-Lee form.

Let $X, \xi$ be respectively a tangent vector field on $M$ and a normal section. We set $PX = \tan(J, X), FX = \nor(JX)$, $t\xi = \tan(J\xi), f\xi = \nor(J\xi)$.

Here $\tan_x, \nor_x$ denote the natural projections associated with the direct sum decomposition $T_x(\overline{M}) = T_x(M) \oplus T_x(M)^\perp$, for any $x \in M$. Note that
P is $D$-valued. Also $F = 0$ on $D$. Moreover the following identities hold:

\begin{align}
F \circ P &= 0, f \circ F = 0 \\
P^2 + t \circ F &= -I \\
t \circ f &= 0, P \circ t = 0 \\
f^2 + F \circ t &= -I
\end{align}

(3.1)

Let $B = \tan(B_0), B^\perp = \text{nor}(B_0), A = \tan(A_0)$ and $A^\perp = \text{nor}(A_0)$. Note that:

\begin{align}
A &= -PB - tB^\perp, A^\perp = -FB - fB^\perp
\end{align}

(3.2)

The complex structure $J$ is not parallel with respect to the Levi-Civita connection $\nabla$ of $M$. Nevertheless $M$ admits a significant almost complex connection $\overline{D}$, namely the Weyl connection, i.e.

\begin{align}
\overline{D}_X Y &= \overline{\nabla}_X Y - \frac{1}{2}\{\omega_0(X)Y + \omega_0(Y)X - \\
&\quad - \overline{g}(X,Y)B_0\}
\end{align}

(3.3)

Since $\overline{D}J = 0$, (3.3) yields:

\begin{align}
\overline{\nabla}_X JY &= J\overline{\nabla}_X Y + \\
&\quad + \frac{1}{2}\{\theta_0(Y)X + \omega_0(Y)JX - \\
&\quad - \overline{\Omega}(X,Y)B_0 - \overline{g}(X,Y)A_0\}
\end{align}

(3.4)

Here $\overline{\Omega}$ denotes the Kaehler 2-form of $\overline{M}$. By (3.4) and the Gauss formula (1.10) of [1], p. 38, one obtains:

\begin{align}
\nabla_X JY &= P\nabla_X Y + th(X,Y) + \\
&\quad + \frac{1}{2}\{\theta(Y)X + \omega(Y)JX - \\
&\quad - \Omega(X,Y)B - g(X,Y)A\}
\end{align}

(3.5)

\begin{align}
h(X, JY) &= f h(X, Y) + F \nabla_X Y - \\
&\quad - \frac{1}{2}\{\Omega(X,Y)B^\perp + g(X,Y)A^\perp\}
\end{align}

(3.6)

for any $X, Y \in D$. Here $\nabla$ denotes the Levi-Civita connection of $(M, g)$ and $\theta = j^*\theta_0$, $\Omega = j^*\overline{\Omega}$. 

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Let us denote by $k_C$ the restriction of the sectional curvature $k$ of $M$ to the holomorphic 2-planes $p \in G_2(M)$, $J(p) = p$, with the property $p \subseteq D_x$, $x = \pi(p)$, $x \in M$. Then $k_C$ is called the complex sectional curvature of the C.R. submanifold $M$.

At this point we may prove our Theorem 2. As the Levi distribution $D$ is integrable, one has $F\nabla_X Y = F\nabla_Y X$, for any $X, Y \in D$. By (3.5)-(3.6) one obtains:

\[(3.7) \quad h(JX, JY) = -h(X, Y) - g(X, Y)B \perp \]

for any $X, Y \in D$. Let us apply (2.4) for the 2-plane $p \in G_2(M)$ spanned by $\{X, JX\}$, $X \in D_x, \|X\| = 1, x = \pi(p)$. It follows:

\[(3.8) \quad k_C(p) = \overline{g}(h(X, X), h(JX, JX)) - \|h(X, JX)\|^2 + c^2 - \frac{1}{4}\{\omega(X)^2 + \theta(X)^2\} \]

Finally (3.7)-(3.8) lead to (1.3). If equality holds, then $h(X, X) = 0, \omega(X) = 0, \omega(JX) = 0$, (and actually (1.3) reads $k_C(p) = c^2$). The converse is obvious, Q.E.D.

Références


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