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Linear forms in two logarithms and Schneider's method (III)

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Abstract. — We apply Schneider's method to get lower bounds for linear forms of two logarithms of algebraic numbers. Here we consider only the rational case. In the first part, we refine the estimates which we proved in the second paper of this series. The end of this paper is devoted to the case when one of these logarithms is equal to \(i\pi\).

Introduction

We refine the lower bound which was obtained in our previous paper [M. W. 2] (which will be denoted [*] in the sequel). We consider an homogenous linear combination of two logarithms of algebraic numbers with integer coefficients.

\[ b_1 \log \alpha_1 - b_2 \log \alpha_2. \]

We combine the method of [*] with a technique which already appeared in [M. W. 1]. We improve the numerical results, which is relevant in several circumstances (see e.g. [C.K.T.], [C. W.], [C.F.]). We treat the case of linear dependent logarithms which was only tackled in [*] and we pay special attention to the case when one of the algebraic numbers is a root of unity.

Since we use intensively [*], we keep the numerotation of the sections up to §8 and very often we only give the modifications which we introduce here (this is the reason why there is no §4 here).

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§1. A lower bound for linear forms in two logarithms

We first give here a simple statement: better estimates will be proved later (especially theorem 5.11 in §5).

For the convenience of the reader, we recall the definition of Weil’s absolute logarithmic height $h(\alpha)$ of algebraic numbers. Namely, if $\alpha$ is algebraic of degree $d$ over $\mathbb{Q}$, with conjugates $\sigma_1 \alpha, \ldots, \sigma_d \alpha$, and minimal polynomial

$$c_0 X^d + \cdots + c_d = c_0 \prod_{i=1}^{d} (X - \sigma_i \alpha), \quad (c_0 > 0)$$

then

$$h(\alpha) = \frac{1}{d} (\log c_0 + \sum_{i=1}^{d} \log \max \{1, |\sigma_i \alpha|\})$$

Let $\alpha_1, \alpha_2$ be two non-zero algebraic numbers of exact degrees $D_1, D_2$. Let $D$ denote the degree over $\mathbb{Q}$ of the field $\mathbb{Q}(\alpha_1, \alpha_2)$. For $j = 1, 2$, let $\log \alpha_j$ be any non-zero determination of the logarithm of $\alpha_j$.

Further let $b_1, b_2$ be two positive rational integers such that

$$b_1 \log \alpha_1 \neq b_2 \log \alpha_2.$$  

Define $B = \max \{b_1, b_2\}$ and choose two positive real numbers $a_1, a_2$ satisfying

$$a_j \geq 1, a_j \geq h(\alpha_j) + \log 2, a_j = f |\log \alpha_j| / D \text{ for } j = 1, 2 \text{ and } f \geq 2e.$$  

Then theorem 5.11 implies the following result.

**Corollary 1.1.** — Under the above hypotheses, we have

$$|b_1 \log \alpha_1 - b_2 \log \alpha_2| \geq \exp \{-270D^4 \cdot a_1 a_2 \cdot (7.5 + \log B)^2\}.$$  

At the end of this paper, we study $|\beta \log \alpha - i\pi|$.

The fact that we get a sharper estimate than in our previous work [*] comes from two modifications. Firstly, we look more closely at the
conditions which are to be verified by the parameters of the auxiliary function. Secondly, we conclude in two steps:

(i) like in [*], we show that the polynomial \( \varphi \) which occurs in the construction of the auxiliary function is zero at integer points \((u, v)\) in a rectangle of average size (this rectangle is not as “big” as in [*]),

(ii) then (like in [M. W. 1]), we prove that \( \varphi(u/2, v/2) = 0 \) for integer points in a big rectangle.

The plan of this paper is the following:

§1. A lower bound for linear forms in two logarithms
§2. Auxiliary lemmas
§3. Interpolation formula
§4. Zero estimate
§5. The main result
§6. Numerical examples
§7. A consequence of the main results
§8. Proof of corollary 1.1
§9. Examples
   a) Class number one
   b) Quotient of two pure powers
   c) Ray class-field
§10. The case of a root of unity
§11. Numerical examples for theorem 10.1
§12. A consequence of theorem 10.1
§13. A corollary of theorem 10.1
§14. An example of a measure of irrationality

References
Appendix: A lower bound for the Euler function.

§2. Auxiliary lemmas

We keep the auxiliary lemmas given in [*], §2, except for the following one.

Lemma 2.1. — (Siegel’s lemma). Let \( \alpha_1, \ldots, \alpha_q \) be algebraic numbers of absolute heights \( a_1, \ldots, a_q \) respectively. Define \( D = [\mathbb{Q}(\alpha_1, \ldots, \alpha_q) : \mathbb{Q}] \).

Let

\[
P_{ij} \in \mathbb{Z}[X_1, \ldots, X_q], (1 \leq i \leq \nu, 1 \leq j \leq \mu)
\]
be polynomials (not all zero) of degree at most \( N_{j,h} \) in \( X_h \) (for \( 1 \leq h \leq q \)).

Define

\[
L_j = (\Sigma_i L^2(P_{ij}))^{1/2} \quad \text{and} \quad \gamma_{ij} = P_{ij}((\alpha_1, \ldots, \alpha_q), (1 \leq i \leq v, 1 \leq j \leq \mu).
\]

If \( \nu > \mu D \), then there exist rational integers \( x_1, \ldots, x_\nu \) not all of which are zero, such that

\[
\sum_{i=1}^{\nu} \gamma_{i,j} x_i = 0, \quad (1 \leq j \leq \mu),
\]

and \( \max |x_i| \leq ((V_1 \ldots V_\nu)^{D/(\nu - \mu D)} \), where \( V_j = L_j \cdot \exp\left(\sum_{h=1}^{q} N_{j,h} a_h\right)\).

**Proof.** — Apply [B. V.] theorem 12.

§3. Interpolation formula

We replace lemma 3.2 of [*] by the following result.

**Lemma 3.2.** — Let \( \beta \) be a rational number, \( \beta = b_1/b_2, b_1, b_2 \in \mathbb{Z}, \quad (b_1, b_2) = 1. \)

Let \( U \) and \( V \) be two positive integers. Put

\[
\Gamma = \{ u + v\beta; (u, v) \in \mathbb{Z} \times \mathbb{Z}, |u| \leq U, |v| \leq V \}
\]

and

\[
\Delta = \min_{\gamma \in \Gamma} \prod_{\gamma' \in \Gamma, \gamma' \neq \gamma} |\gamma' - \gamma|.
\]

We suppose

\( (H) \) the points \( (u + v\beta), |u| \leq 4U \text{ and } |v| \leq 4V \) are pairwise distinct.

Then we have

\[
\Delta \geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{-14\pi^2/27(V + 1)^3 b_2^{-2}\}.
\]

**Proof.** — The proof is the same as in [*] except that we notice that our new hypothesis \( (H) \) implies now that each value of \( x_v \) can be obtained only once, where – for \( v \neq v_0 \) fixed – we denote by \( x_v \) be the minimum of |u - u_0 + \beta(v - v_0)|.
Thus, arguing as in [*], we get
\[ \Delta \geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{- (7\pi^2/36)(1 + 2^2 + \cdots + (2V)^2)b_2^{-2}\} \]
\[ \geq (2V)! \cdot b_2^{-2V} \cdot (U!)^{2(2V+1)} \cdot \exp\{- (7\pi^2/108)(2V + 1)^3b_2^{-2}\}. \]

This implies lemma 3.2.

§4 Zero estimate

There is no section 4 here because we shall just apply the zero estimate of [*] §4.

§5. The main result

5.1. Common notations and hypotheses for §§5, 6 and 7

Let \( \alpha_1, \alpha_2 \) be two non-zero algebraic numbers of respective degrees equal to \( D_1 \) and \( D_2 \), the total degree of the field we are working in is \( D = [Q(\alpha_1, \alpha_2) : Q] \), \( \log \alpha_j \) is any non-zero determination of the logarithm of \( \alpha_j \), \( l_j = |\log \alpha_j| \), for \( j = 1, 2 \).

Moreover let \( \beta = b_1/b_2 \) be a rational number, where \( b_1, b_2 \in \mathbb{Z}, 0 < b_1, b_2 \), and \( (b_1, b_2) = 1 \), such that

\[ \Lambda = \beta \log \alpha_1 - \log \alpha_2 \]

does not vanish.

We define many parameters as follows.

a) parameters depending on \( \alpha_1 \) and \( \alpha_2 \) (namely \( a_1, a_2, a'_1, a'_2, a', \sigma, \sigma_1, \sigma_2 \), \( f, \nu \)) :

We define
\[ a'_j = h(\alpha_j)(j = 1, 2), a' = \max\{a'_1, a'_2\}, a'' = \max\{|\log \alpha_1|, |\log \alpha_2|\}, \]
so that \( a'' \leq Da' \),
\[ \sigma_j = a'_j/\alpha_j, j = 1, 2, \) (so that \( 0 \leq \sigma_j \leq 1 \)) \( , \sigma = (\sigma_1 + \sigma_2)/2 \) (so that \( 0 \leq \sigma \leq 1 \)).
Now we notice that for any non-zero algebraic number $\alpha$ and any non-zero determination $\log \alpha$ of its logarithm we have

$$|\log \alpha| \geq \exp(-\deg(\alpha)(1+h(\alpha))),$$

therefore we can choose a real number $f$ with $1 \leq f \leq 2e^{D(a'+1)}$, such that the numbers $a_j = fD^{-1}i_j$ satisfy

$$a_j \geq 1/D, \text{ for } j = 1, 2.$$ 

We also assume

$a_j \geq a_j'$ for $j = 1, 2$.

Finally we put $\nu = 1$ if $D > 1$ and $\nu = 0$ if $D = 1$.

b) parameters depending on $\beta$ (namely $B, G, G_0, G', \mu, \rho$):

We put

$$B = \max\{b_1, b_2\},$$

$$G_0 = 0.59 + \log B + \log \log 2B,$$

$$0.09 + \log B + \log \log 2B = (1 + \mu) \log B.$$

Let $\rho$ be a positive number (further conditions on $\rho$ will be required in §5.3 below), we define

$$G' = 1 + \log (0.5 + \rho/l_1), G = G_0 + \max\{0.41, G'/D\}.$$ 

c) the parameters $Z, \theta, \theta_0, \theta_1, \theta_2$:

Let

$$\theta \geq 10$$

be a real number and let $Z$ be a positive number which satisfies

$$1 \leq Z \leq \min\{DG/\theta, Da_1, Da_2, \log (ef)\} \text{ (as usual $e$ is defined by $\log e = 1$),}$$

$$Z \leq \sqrt{DG}/10. \quad (5.0)$$

We put

$$\varepsilon = Z/\log (ef) \quad \text{(so that $\varepsilon \leq 1$).}$$

Further we define

$$\theta_1 = \theta(Da_2/Z), \theta_2 = \theta(Da_1/Z), \text{ (so that $\theta_1, \theta_2 \geq \theta$, $\theta_0 = \max\{\theta_1, \theta_2\}$.}$$
d) the main parameter $U$:

As in [\*], we define

$$U = D^4 \alpha_1 \alpha_2 G^2 Z^{-3}.$$  

Notice that (5.0) and the conditions on $Z$ imply

$$U \geq \max\{\theta DG, 100D^2 a_1, 100D^2 a_2, \theta^2 Da_1, \theta^2 Da_2, 10D^{3/2} G\}.$$  

5.2 Notations and hypotheses for §5 and §6

Let $c_0, c_1, c, \chi_0, \chi_1, \chi_2, X, C, \eta^*, \mu, \rho, p^*, \xi, \xi_0$ be positive real numbers.

Assume

\begin{align*}
(5.1) \quad & c_0 \theta_0 \geq 190, c_1 \theta \geq 20, c_1 \theta \geq 20, c_1 \theta \geq 12, c_1 \leq 5c \leq c_0 - 1/\theta, \\
& 2c_1 + c_0 / c_\theta + 2 \left(1 - 1/\theta c_0\right) \leq 4c_\xi, 6c_1 \leq c_0 - 1/\theta, 1 \leq X \leq e, \\
& (2c - 1/\theta) \xi \geq c_1,
\end{align*}

\begin{align*}
(5.2) \quad & (c_0 - 1/\theta)(c_1 - 1/\theta) \geq 3.65(c + 1/\theta)^2, c_0(c_1 + 1/\theta) \leq 4.85(c - 1/\theta)^2,
\end{align*}

\begin{align*}
(5.3) \quad & \eta^* = \frac{(2c + 1/\theta_1)(2c + 1/\theta_2)}{c_0(2c_1 - 1/\theta) - (2c + 1/\theta_1)(2c + 1/\theta_2)}, \\
& \rho = \frac{(2c + 1/\theta_1)(2c + 1/\theta_2)}{(c_0 - 1/\theta_0)(c_1 - 1/\theta)},
\end{align*}

\begin{align*}
(5.4) \quad & p^* = \eta^* \left\{c_0 + (c_1 - 1/2\theta)(\sigma_1(c - 1/2\theta_1) + \sigma_2(c - 1/2\theta_2))\right\}
\end{align*}

\begin{align*}
(5.5.\text{i}) \quad & (c_0(c_1 - 1/\theta) - (2c + 1/\theta)^2) \xi \leq c_0(2c_1 - 1/\theta)(c_0 + (2c_1 - 1/\theta)c_\sigma),
\end{align*}

\begin{align*}
(5.5.\text{ii}) \quad & (2c + 1/\theta_1)(2c + 1/\theta_2) \xi \geq p^* + c_0 + (2c_1 - 1/\theta)(\sigma_1(c - 1/2\theta_1) + \\
& \sigma_2(c - 1/2\theta_2)) + 0.05\nu,
\end{align*}

\begin{align*}
(5.5.\text{iii}) \quad & 4c_2 \chi_0^2 \xi_0 \geq 2(p^* + c_0) + 4c_1^2 \chi + 2c_1 \sigma / \theta + (0.44 + 2D^{-1} \text{Log} \chi_0) / G,
\end{align*}

where $\xi_0 = e^{-1} + \frac{1}{Z} \text{Log} \left(\frac{4c_2 \chi_0^2}{X_0 e(1 + 1/2\theta \chi_0)}\right) - \frac{e^{-c}}{Z}, 1 \leq \chi_0 \leq 1.5$.

\begin{align*}
(5.6.\text{i}) \quad & C \geq p^* + c_0 + (4c_2 \chi_0^2 / Z) \text{Log} \left(2e\right) + 2\chi_0(3c_1 \sigma + 1/D)c + 3c_1 \sigma / \theta - 0.48c_0 / G + 0.09,
\end{align*}

\begin{align*}
(5.6.\text{ii}) \quad & C \geq 2(p^* + c_0 + 0.09) + (4c_2 \chi_0^2 / Z) \text{Log} \left(\chi e / \chi_0\right) + 2c_1(2\chi + \\
& \chi_0) + 2c_0 / D + 3c_1 \sigma / \theta + 0.2c_0 / G + (2/DG)(4 \text{Log} \left(e\chi / \chi_0\right) \\
& - \text{Log} \left(19.7c\theta \chi_0\right))^{+},
\end{align*}

[as usual, for a real number $z$ we define $z^+ = \max\{z, 0\}$.]
(5.6.iii) \( C \geq 10c \) and \( CD \geq 200, \)

(5.7) \( 0 < \xi \leq \varepsilon^{-1} + \frac{1}{Z} \log \frac{2cf(1 - 1/2\theta c)^2}{Zc_1/e^2} - \frac{e^{-c}}{Z}, \)

(5.8) \( x_2 = \sqrt{c_0c_1}/c, x = x_1 + x_2, \quad \text{[by (5.2), \( x_2 \geq \sqrt{3} \)]} \)

(5.9) \( x \geq \sqrt{c_0c_1}/c + (1 + \sqrt{c_0\theta})/(2c\theta), \)

(5.10) either \( \alpha_1 \) and \( \alpha_2 \) are multiplicatively independent or

\[
x \geq \sqrt{c_0c_1}/c + 1/(\theta c) + c_1/c.
\]

5.3 Statement of the main result.

THEOREM 5.11. Under the above hypotheses, we have \( |\Lambda| > e^{-CU}. \)

The rest of §5 is devoted to the proof of this inequality. Therefore we assume \( \log |\Lambda| \leq -CU \) and we shall eventually reach a contradiction.

5.4. The parameters

We define \( L_0, L_1, M_1, M_2 \) by

\[
L_0 = [c_0D^3a_1a_2GZ^{-3}], L_1 = [c_1DGZ^{-1}], \\
M_1 = [cD^2Ga_2Z^{-2}], M_2 = [cD^2Ga_1Z^{-2}].
\]

We put

\[
2L_1 + 1 = x_1DGZ^{-1}, \text{ so that } |x_1 - 2c_1| \leq 1/\theta, \\
2M_1 + 1 = y_1D^2Ga_2Z^{-2}, \quad 2M_2 + 1 = y_2D^2Ga_1Z^{-2},
\]

so that \( |y_j - 2c| \leq 1/\theta_j \) for \( j = 1, 2. \)

The following inequalities

(5.12) \( L_0 \geq (c_0 - 1/\theta_0)D^3a_1a_2GZ^{-3} \geq c_0\theta - 1, L_0 \geq 190, \)

\( L_1 \geq (c_1 - 1/\theta)DGZ^{-1} \geq c_1\theta - 1, L_1 \geq 12, \)

\( M_1 \geq (c - 1/\theta_1)D^2Ga_2Z^{-2}, M_2 \geq (c - 1/\theta_2)D^2Ga_1Z^{-2}, \)
and $M_1, M_2 \geq 20, M_1 \geq L_1$ are all consequences of the definition of $Z$ and of (5.1).

By lemma 2.2 and the definition of $a'_i$, we have

\[(*) \quad |\Lambda| \geq 2^{-D} \cdot \exp(-b_1 Da'_1 - b_2 Da'_2) \cdot B^{-1}.
\]

This shows first that the numbers $u + v\beta, |u| \leq 4M_1$ and $|v| \leq 4M_2$ are pairwise distinct (here and in the sequel the letters $u$ and $v$ represent rational integers); otherwise $b_1 < 8M_1$ and $b_2 < 8M_2$, which implies

\[|\Lambda| \geq \exp(-D(1 + 8M_1 a_1 + 8M_2 a_2) - G) \geq \exp(-(16c + 2)U/10D).
\]

and contradicts the assumption $|\Lambda| \leq e^{-GU}$, since $C \geq 10c$ and $CD \geq 200$.

We have $|a_1 b_1 - a_2 b_2| \leq fB \cdot e^{-GU}$, and the hypotheses on $f, C$ and $U$ imply

\[a_1 b_1 \leq (1 + e^{-GU/2}) \cdot a_2 b_2 \leq 1.001 \cdot a_2 b_2.
\]

We also remark that $M_2 \leq b_2/33$ : if not (*) implies the estimate

\[|\Lambda| \geq \exp(-3b_2 Da_2) \leq \exp(-99M_2 a_2 D),
\]

which contradicts $|\Lambda| \leq e^{-GU}$, since $C > 10c, \theta \geq 10$ and $M_2 a_2 D \leq cU/\theta$. This remark is used in the proof of proposition 5.19.

Moreover, we remark also that (*) and (5.0) imply

\[(**) \quad B \geq \max\{(C/2 - 0.1)\theta^2, 49DC\},
\]

so that $\Log B \geq 9.2, G > 12.49, \omega(2B) < 0.09 + \Log \Log 2B$ (see lemma 2.7 of [*] for the definition of $\omega(x)$) and also $U \geq 186D \Log D$.

5.5 The auxiliary function.

We denote by $\{\xi_1, \ldots, \xi_D\}$ a basis of $Q(\alpha_1, \alpha_2)$ over $Q$, where $\xi_d = \alpha_1^{d_1}_1 \alpha_2^{d_2}_2$, $0 \leq d_j < D_j (j = 1, 2)$ and $d_1 + d_2 < D$. This implies the estimate

\[\max\{h(\xi_j); 1 \leq j \leq D\} \leq \min\{(D_1 - 1)a'_1 + (D_2 - 1)a'_2, (D - 1)a'\}.
\]

As in [*], we shall construct an auxiliary function of the form

\[F(z) = \sum_{h=0}^{L_0} \sum_{k=-L_1}^{L_1} p_{h,k} \Delta_h(z) \alpha_1^{kz}, \quad \text{where} \quad p_{h,k} = \sum_{d=1}^{D} p_{h,k,d} \xi_d, p_{h,k,d} \in \mathbb{Z}
\]
and $\Delta_h(z)$ is defined in lemma 2.4 of [*]. For rational integers $u$ and $v$ we put

$$\varphi(u, v) = \sum_{h=0}^{L_0} \sum_{k=-L_1}^{L_1} p_{h,k} \Delta_h(u + v\beta) \alpha_1^k \alpha_2^v.$$

**Proposition 5.14.**— *There exist rational integers $p_{h,k,d}$, not all zero, such that*

$$\varphi(u, v) = 0 \text{ for } -M_1 \leq u \leq M_1, -M_2 \leq v \leq M_2,$$

*with $P := \log(\max|p_{h,k}|)$

$$\leq \eta(L_0(G_0 + 0.016) + ((x_1/4)(\sigma_1y_1 + \sigma_2y_2) + 0.013\nu)U/D)$$

$$\leq (p + 0.013\eta\nu)U/D - \eta(G - G_0 - 0.016)L_0,$$

*where $p = \eta(c_0 + (x_1/4)(\sigma_1y_1 + \sigma_2y_2))$ and $\eta = y_1y_2/(c_0x_1 - y_1y_2)$.

Moreover

$$P_1 := \log(\Sigma_{h,k,d}|p_{h,k,d}|) \leq P + 0.006\nu U/D + 0.051L_0.$$

and

$$\log(\Sigma_{h,k}|p_{h,k}|) \leq P_2 := P_1 + (D - 1)a'' \leq P_1 + 0.01\nu U.$$

**Remark.**— By (5.2), we have $0.7 \leq \eta \leq \eta^* \leq 1.22$; moreover, $p \leq p^*$.

**Proof of proposition 5.14.**

We have to solve in $\mathbb{Z}$ a linear system of $(2M_1 + 1)(2M_2 + 1)$ equations in the $D(L_0 + 1)(2L_1 + 1)$ unknowns $p_{h,k,d}$. We use lemma 2.1. By definition, we have

$$(2M_1 + 1)(2M_2 + 1)/((L_0 + 1)(2L_1 + 1) - (2M_1 + 1)(2M_2 + 1)) = \eta.$$

With the notations of lemma 2.1, we have

$$i \rightarrow (h, k, d), j \rightarrow (u, v), N_{j,1} = 2L_1|u| + D_1 - 1, N_{j,2} = 2L_1|v| + D_2 - 1,$$

$$P_{ij} = \Delta_h(u + v\beta)b_2^{L_0}\Omega(b_2, L_0)X_1^{L_1}|u| + ku + d_1 X_2^{L_1}|v| + kv + d_2,$$

with $d_1 + d_2 < D$. By lemma 2.4 of [*], using the inequality $\beta a_1 \leq 1.001a_2$ as well as our assumption (5.1) we get

$$L(P_{ij}) \leq 2(X^h/h!)b_2^{o}\Omega(b_2, L_0), X = \max\{1.1|u + v\beta|, L_0/2\} = L_0/2.$$
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Notice that

$$\sum_{h=0}^{L_0} \frac{X^h}{h!} \leq e^X,$$

so that \((\Sigma_i L^2(p_{ij}))^{1/2} \leq 2(D(2L_1 + 1))^{1/2}(\sqrt{e}b_2)^L_0 \Omega(b_2, L_0)\) and

\[ V_{u,v} \leq 2(D(2L_1 + 1))^{1/2}(\sqrt{e}b_2)^L_0 \Omega(b_2, L_0). \]

\cdot \exp\{2L_1|u|h(\alpha_1) + 2L_1|v|h(\alpha_2) + (D - 1)a'\}.

Now we have

$$\sum_{u=-M_1}^{M_1} (2L_1|u|h(\alpha_1) = 2L_1M_1(M_1 + 1)h(\alpha_1).$$

and \(2M_1(M_1 + 1)h(\alpha_1) \leq (1/2)(2M_1 + 1)^2a'_1.\)

Hence, since a similar result holds for the summation over \(v,\)

\[ \Sigma_{u,v} \log V_{u,v} \leq (2M_1 + 1)(2M_2 + 1)\{\log (2(D(2L_1 + 1))^{1/2}(\sqrt{e}b_2)^L_0 \Omega(b_2, L_0)) + a'_1L_1(M_1 + 1/2) + a'_2L_1(M_2 + 1/2) + (D - 1)a'\} \]

An application of lemma 2.1 shows that there is a non trivial solution with

\[ \log (\max |p_{h,k,d}|) \leq \eta L_0 G_0 + L_1(a'_1(M_1 + 1/2) + a'_2(M_2 + 1/2)) + (D - 1)a'' \]

\[ + \log (2(D(2L_1 + 1))^{1/2}), \]

where \(G_0 = 0.59 + \log B + \log \log B \leq G - 0.41.\)

Moreover

\[ \log (\Sigma|p_{h,k,d}|) \leq \log (\max |p_{h,k,d}|) + \log (D(L_0 + 1)(2L_1 + 1)) \]

and

\[ \log (\Sigma|p_{h,k}|) \leq \log (\max |p_{h,k,d}|) + \log (D(L_0 + 1)(2L_1 + 1)) + (D - 1)a'. \]

We have

\[ L_1(a'_1(M_1 + 1/2) + a'_2(M_2 + 1/2)) \leq (x_1/4)(y_1\sigma_1 + y_2\sigma_2)(U/D). \]

By (5.0) and (5.1), we have also

\[ \log (2L_1 + 1) \leq \log ((25/72)L_0) \leq 0.023L_0, \log 2 \leq 0.004L_0, \]

\[ \log (L_0 + 1) \leq 0.028L_0,(D - 1)a' \leq 0.01\nu U/D, \log D \leq 0.006\nu U/D. \]

Now it is easy to get proposition 5.14.
5.6 The extrapolation.

Put $M_{1,0} = [x_0 c D^2 a_2 G Z^{-2} + 0.5]$ and $M_{2,0} = [x_0 c D^2 a_1 G Z^{-2} + 0.5]$. In this section we prove that

$$(#) \quad \varphi(u, v) = 0 \text{ for } -M_{1,0} \leq u \leq M_{1,0}, -M_{2,0} \leq v \leq M_{2,0}.$$ 

By construction, this is true for $-M_1 \leq u \leq M_1$ and $-M_2 \leq v \leq M_2$. We proceed almost exactly like in [*], and we give only the details which are different.

Define $N = M_{1,0} + M_{2,0} - M_1 - M_2$ and $X_n$ by

$$M_1^{(n)} = x_n c a_2 D^2 G Z^{-2}, 1 \leq n \leq N,$$

so that $1 \leq X_n \leq X_0$.

We prove, by induction on $n, (0 \leq n \leq N)$, that

$$(P)_n \quad \varphi(u, v) = 0 \text{ for } |u| \leq M_1^{(n)} \text{ and } |v| \leq M_2^{(n)}.$$ 

As already seen, this is true for $n = 0$, while $(P)_N$ is nothing else than $(#)$.

We consider the set

$$\Gamma_{n-1} = \{z_1, \ldots, z_m\} = \{u + v \beta; |u| \leq M_1^{(n-1)}, |v| \leq M_2^{(n-1)}\},$$

where $m = (2M_1^{(n-1)} + 1)(2M_2^{(n-1)} + 1)$, and a point $z_0 \in \Gamma_n, z_0 \notin \Gamma_{n-1}$.

Since $\beta a_1 \leq a_2(1 + e^{-CU/2})$ we get $M_1^{(n-1)} + 1 \geq \beta M_2^{(n-1)} - e^{-CU/3}$.

Define $R_1 = M_1^{(n)} + M_2^{(n)} \beta, R = m/(L_1 l_1)$.

**Proposition 5.19.** — We have

$$|F(z_0)| \leq E_1 + E_2$$

where

$$\log E_1 \leq -m \log (R/e R_1) + 1 + \log (\Sigma_{h,k} |p_{h,k}| + L_0 (1 + \log (1/2 + R/L_0)),$$

so that

$$\log E_1 \leq P_2 + L_0 G' + 1.5 + L_0 \log (m/(2M_1 + 1)(2M_2 + 1)) - m \xi Z$$

and

$$\log E_2 \leq \log (\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}) + m \log (e R_1/(M_1^{(n-1)} + 1)) + 2M_2^{(n-1)} \log (e B/2M_2^{(n-1)}) + 0.027 M_2^{(n-1)},$$
so that

\[
\log E_2 \leq \log \max_{\gamma \in \Gamma_{n-1}} |F(\gamma)| + m \log 2e + 2M_2^{(n-1)} \log \left(\frac{eb_2}{2M_2^{(n-1)}}\right) + 0.03M_2^{(n-1)}.
\]

(5.21)

**Proof.** Thanks to the interpolation formula, we have

\[
\log E_1 \leq -m \log \left(\frac{R}{R_1}\right) + \log \left(\frac{R}{(R - R_1)}\right) + \log |F|_{R_1};
\]

this implies the first upper bound for \( \log E_1 \). The second one follows like in \([*]\), except that now \n
\[
R_1/(M_1^{(n-1)} + 1) \leq 2 + e^{-CU/3}.
\]

(5.25)

The interpolation formula implies also

\[
\log E_2 \leq \log \left(\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}\right) - (4M_2^{(n-1)} + 2) \log (M_1^{(n-1)})
\]

\[
+ 5.12(M_2^{(n-1)} + 1)^3b_2^{-2} + \log m + (m - 1) \log R_1 + \log \left(\frac{2R^4}{(R^4 - R_1^4)}\right),
\]

where \( \log \left(\frac{2R^4}{(R^4 - R_1^4)}\right) < 2.04, \)

\[
m \leq (41/20)^2M_1^{(n)}M_2^{(n-1)} = (41/20)^2(M_1^{(n)}/R_1)R_1M_2^{(n-1)}
\]

\[
< 4.3R_1M_2^{(n-1)},
\]

\[
5.12(M_2^{(n-1)} + 1)^3b_2^{-2} \leq 6(M_2^{(n-1)})^3b_2^{-2} < 0.01M_2^{(n-1)},
\]

\[
(4M_2^{(n-1)} + 2) \log (M_1^{(n-1)}) \geq \log (2\pi/e),
\]

\[
\log ((2M_2^{(n-1)})!)) \geq 2M_2^{(n-1)} \log (2M_2^{(n-1)}/e) + 0.5 \log (4\pi M_2^{(n-1)}),
\]

so that

\[
\log E_2 \leq \log \left(\max\{|F(\gamma)|; \gamma \in \Gamma_{n-1}\}\right) - m \log ((M_1^{(n-1)} + 1)/eR_1)
\]

\[
+ 2M_2^{(n-1)} \log (eB/2M_2^{(n-1)}) + 0.5 \log (9M_2^{(n-1)}) + 0.01M_2^{(n-1)}.
\]

This implies the first upper bound of \( \log E_2 \). The second one follows from (5.25). This completes the proof of proposition 5.19.
PROPOSITION 5.29. — Put $\lambda = \max\{1/2, |\gamma|/L_0\}$. For $\gamma = u + v\beta \in \Gamma_N$, we have

$$|F(\gamma) - \varphi(u,v)| \leq E_3$$

where

$$\log E_3 \leq -CU + P_2 + \lambda L_0 + \log (L_1 M_{2,0}) + L_1 M_{1,0} D\alpha_1' + 3D\alpha_2 + (L_1 M_{2,0} + 1)D\alpha_2' + 1 \leq (-C + 2a_0 \chi_0 \sigma + c_1 \sigma / \theta + 0.04)U + P_2 + 0.55L_0.$$ 

Proof. — We first show that for $-L_1 \leq k \leq L_1$ and $-M_{2,0} \leq v \leq M_{2,0}$, we have (5.30) $|\alpha_1^{kv} - \alpha_2^{kv}| \leq \exp\{-CU + 3D\alpha_2 + \log (L_1 M_{2,0}) + (L_1 M_{2,0} + 1)D\alpha_2' + L_1 M_{2,0} D\alpha_2/CU\}$.

Indeed,

$$|\alpha_1^{kv} - \alpha_2^{kv}| \leq |\alpha_1^{\beta} - \alpha_2^{\beta}| \cdot L_1 M_{2,0} \cdot \max\{1, |\alpha_1^{\beta} \alpha_2^{\beta}|^{-1}\}.$$ 

$$\cdot \exp((L_1 M_{2,0}^* - 1) \cdot \log (\max\{|\alpha_1^{\beta}|, |\alpha_2^{\beta}|, |\alpha_1^{\beta}|^{-1}, |\alpha_2^{\beta}|^{-1}\}))$$

We have $|\log |\alpha_2^{\beta}|| \leq D\alpha_2'$ and $|\alpha_1^{\beta}| \leq |\alpha_2^{\beta}| + e^{-CU/2}$ so that $\log |\alpha_1^{\beta}| \leq D\alpha_2' + e^{-CU/2}$. Clearly, $|\alpha_1^{\beta}| \geq \exp(-D\alpha_2') - \exp(-CU + 2D\alpha_2 + e^{-CU})$ and this leads to

$$\log |\alpha_1^{\beta}| \geq -D\alpha_2' - D\alpha_2/(CU).$$

These lower bounds give

$$\log (|\alpha_1^{\beta} \alpha_2^{\beta}|^{-1}) \leq 2D\alpha_2' + D\alpha_2/CU,$$

$$\log (\max\{|\alpha_1^{\beta}|, |\alpha_2^{\beta}|, |\alpha_1^{\beta}|^{-1}, |\alpha_2^{\beta}|^{-1}\}) \leq D\alpha_2' + D\alpha_2/CU,$$

and (5.30) follows since $|\alpha_1^{\beta} - \alpha_2^{\beta}| \leq \exp\{-CU + 3D\alpha_2\}$.

We have $|\gamma| \leq 2.03 a_0 cD^2GZ^{-2}a_2$, thus $\lambda = 0.5$ by (5.1). Moreover,

$$\log (L_1, M_{2,0}) \leq \log (L_0 c_1/(c_0 - 1/\theta)) + \log (L_0 c_1/(c_0 - 1/\theta)) \leq 0.04L_0.$$ 

The conclusion follows like in [*].

PROPOSITION 5.34. — For $\gamma = u + v\beta \in \Gamma_N$, either $\varphi(u,v)$ is equal to zero or

$$|\varphi(u,v)| \geq E_4$$
Proof. — From Liouville inequality, we deduce

$$-\log |\varphi(u,v)| \leq (D - 1)P_1 + (D - 1)\lambda L_0 + (1 + \mu)DL_0 \log B$$

$$+ 2DL_1(a_1'M_1^{(n)} + a_2'M_2^{(n)}) + D(D - 1)a'$$

$$\leq (D - 1)P_1 + 0.01\nu U + DL_0G_0 + 2DL_1(a_1'M_1^{(n)} + a_2'M_2^{(n)}) - L_0/2,$$

and the result follows.

**Proposition 5.35.** — Assume that in (5.21) we have

$$\max_{\gamma \in F_{n-1}} |F(\gamma)| \leq E_3$$

then

$$E_1 + E_2 + E_3 < E_4.$$

Proof. — We use (5.5) to check

(5.36)$$E_1 < E_4/3,$$

and then we shall use (5.6) (i) to check

(5.37)$$\max\{E_2, E_3\} < E_4/3.$$
We have also $3 \leq 0.016L_0$ and
\[
P_1 \leq (p + 0.013\eta \nu + 0.006\nu)U/D - \eta(G - G_0 - 0.1)L_0
\leq (p + 0.022\nu)U/D - \eta(G - G_0 - 0.1)L_0
\]
Therefore (5.36) is a consequence of
\[
H := y_1y_2\xi - (p + c_0 + (x_1/2)(y_1\sigma_1 + y_2\sigma_2) + 0.05\nu) \geq 0,
\]
[recall that $p = \eta(c_0 + (x_1/4)(y_1\sigma_1 + y_2\sigma_2))$ and $\eta = y_1y_2/(c_0x_1 - y_1y_2)$].
We see that $\partial H/\partial x_1 \geq 0$, so that the worst value for $x_1$ is $2c_1 - 1/\theta:
\partial H/\partial x_1 > ((y_1y_2)^2/2 - 1)(y_1\sigma_1 + y_2\sigma_2)/2 \geq 0$ (since $y_1, y_2 > 2$).
Whereas
\[
\frac{\partial H}{\partial y_1} = y_2\xi - \frac{y_2c_0x_1(c_0 + x_1(y_1\sigma_1 + y_2\sigma_2)/4)}{(c_0x_1 - y_1y_2)^2} - x_1\sigma_1(\eta + 2)/4.
\]
As easily verified, $\partial^2 H/\partial y_1^2 \leq 0$ and $\partial(\partial^{-1}\partial H/\partial y_1)/\partial y_2 \leq 0$, so that $\partial H/\partial y_1 \leq 0$ when
\[(c_0x_1 - y_1^2)^2\xi \leq c_0x_1(c_0 + x_1y_1)/2,\] where $y = 2c - 1/\theta$.
This proves that (5.5.i) implies $\partial H/\partial y_1 \leq 0$ (and also $\partial H/\partial y_2 \leq 0$). Now, by condition (5.5.ii) we have $H \geq 0$.

We now prove (5.37). It is sufficient to check
\[
DP_1 + (c_0 + 6cc_1x_0\sigma + 3c_1\sigma/\theta + 0.04)U + m\log(2e) + D(D - 1)a' + (D - 1)a'' + 2M_{2,0}\log(3B/2M_{2,0}) - DL_0(G - G_0) + 0.05L_0 < CU.
\]
We have
\[
M_{2,0}\log(3B/2M_{2,0}) \leq 2M_{2,0}(G - 1.9) - 2M_{2,0}\log 2M_{2,0}
\leq 2x_0cU/D - 10M_{2,0}.
\]
And now $m$ is bounded by
\[
m = (2M_{1,0} + 1)(2M_{2,0} - 1) \leq (2M_{1,0} - 1)(2M_{2,0} - 1) + 4M_{2,0}
\leq 4x_0^2c^2U/Z + 4M_{2,0}.
\]
This shows that (5.37) is a consequence of (5.6.i).
We have proved (#).
5.7 End of the proof.

We shall prove that the non-zero polynomial $\Delta_{h,k}p_{h,k}\Delta_h(X)Y^L + k$ vanishes at the points

$$(u/2 + \beta v/2, \alpha_1^{u/2} \alpha_2^{v/2}), (u, v) \in \mathbb{Z} \times \mathbb{Z}, |u| \leq M_1^*, |v| \leq M_2^*,$$

where $M_1^* = [\chi cD^2 a_2 GZ^{-2} + 0.5]$ and $M_2^* = [\chi cD^2 a_1 GZ^{-2} + 0.5].$

According to proposition 4.1 of [*] (zero estimate) and the obvious analog of proposition 5.43 of [*], we will obtain a contradiction; and this will prove theorem 5.11.

Consider a point $\gamma = (u + v\beta)/2$, with $|u| \leq M_1^*, |v| \leq M_2^*.$

We suppose that $\gamma$ does not belong to $\Gamma_N$ (otherwise $\varphi(u/2, v/2) = 0$ by the preceding section) and we apply the interpolation formula for $\gamma$ and the points of $\Gamma_N$ with

$$R_1 = (M_1^* + \beta M_2^*)/2, R = m^*/L_1 h_1,$$

where $m^* = \text{Card}(\Gamma_N) = (2M_{1,0} + 1)(2M_{2,0} + 1).$

We get

$$|F(\gamma)| \leq E_1^* + E_2^*,$$

where

$$\log E_1^* \leq -m^* \log (R/eR_1) + 1 + \log(\Sigma_{h,k} |p_{h,k}|) + L_0 (1 + \log (1/2 + R/L_0)),$$

and

$$\log E_2^* \leq \log (\max\{|F(\gamma); \gamma \in \Gamma_N\}) + m^* \log (eR_1/(M_{1,0} + 1))$$

$$+ 2M_{2,0} \log (eB/2M_{2,0}) + 0.027M_{2,0}.$$

Now,

$$\frac{R}{L_0} \leq \frac{(2c\chi_0 + 1/2\theta)^2(e^{G'-1} - 0.5)}{\rho(c_0 - 1/\theta_0)(c_1 - 1/\theta)} \leq \chi_0^2(c^{G'-1} - 0.5)$$

so that

$$\log \left(\frac{R}{L_0} + \frac{1}{2}\right) \leq G' - 1 + 2\log \chi_0,$$

$$\log (R/eR_1) \geq \xi_0 Z$$

and

$$R_1 \leq (\chi/\chi_0)(M_{1,0} + 1)(1 + e^{-CU/3}).$$
This leads to

\[ \log E_1^* \leq -m^* \xi_0 Z + P_2 + (G' + 2 \log \chi_0) L_0 + 1 \]

and

\[
\log E_2^* \leq \log E_3 + m^* \log (e\chi/\chi_0) + 2\chi_0 cU/D + 2M_{2,0} \log (eB/2M_{2,0}) \\
\leq (-C + 2cc_1\chi_0 \sigma + 2c_1 \sigma / \theta + 0.05) U + m^* \log (e\chi/\chi_0) \\
+ 2\chi_0 cU/D - 2M_{2,0} \log (4M_{2,0}) + P_1 + 0.55L_0.
\]

Notice that \( m^* \leq 4\chi_0^2 c^2 U + 4M_{1,0} + 4M_{2,0} + 1 \). Without loss of generality, we may suppose that \( M_{1,0} \leq M_{2,0} \), and then we have

\[
m^* \log (e\chi/\chi_0) + 2M_{2,0} \log (eB/2M_{2,0}) \\
\leq (4\chi_0^2 c^2 U + 8M_{2,0} + 1) \log (e\chi/\chi_0) + \\
2M_{2,0} (G - \log \log 2B - \log (2M_{2,0})) \\
\leq 4\chi_0^2 c^2 U \log (e\chi/\chi_0) + 8M_{2,0} \log (e\chi/\chi_0) + \\
2M_{2,0} (G - \log (19.7M_{2,0})) + 2 \\
\leq 4\chi_0^2 c^2 U \log (e\chi/\chi_0) + 2\chi_0 U/D + \\
2M_{2,0} (4 \log (e\chi/\chi_0) - \log (19.7M_{2,0})) + 2 \\
\leq 4\chi_0^2 c^2 U \log (e\chi/\chi_0) + 2\chi_0 U/D + \\
2(U/DG)(4 \log (e\chi/\chi_0) - \log (19.7c\chi_0))^+ + 2.
\]

(we have used the notation \( x^+ = \max\{x, 0\} \) for real numbers), this gives

\[
\log E_2^* \leq (-C + 2cc_1\chi_0 \sigma + c_1 \sigma / \theta + 4\chi_0^2 c^2 \log (e\chi/\chi_0) + 2\chi_0 c/D + 0.09) U \\
+ P_1 + 0.55L_0 + 2(U/DG)(4 \log (e\chi/\chi_0) - \log (17c\theta\chi_0))^+.
\]

Moreover, \( |\varphi(u/2, v/2) - F(\gamma)| \leq E_3^* \) with

\[
\log E_3^* \leq -CU + P_2 + \lambda^* L_0 + \log (L_1 M_{2^*}^*) + L_1 M_{2^*} D(a_1^*/2) \\
+(L_1 M_{2^*}^* + 1) D(a_1^*/2) + 4 Da_2 + 1,
\]

where \( \lambda^* = \max\{0.5, |\gamma|/L_0\} = 0.5 \) (by (5.1) and \( \chi \leq e \)), thus

\[
\log E_3^* \leq (-C + p/D + \chi cc_1 \sigma + \chi c_1 \sigma / \theta + 0.09 + c_0/2DG) U.
\]

Finally, if \( \varphi(u/2, v/2) \neq 0 \) then \( |\varphi(u/2, v/2)| \geq E_4^* \), where

\[
-\log E_4^* \leq (2D - 1) \log (\Sigma_{k,k'} p_{k,k'}) + 2DL_1(a_1^* M_{1^*}^* + a_2^* M_{2^*}) \\
+ 2DL_0(\lambda^* + \log 2 + \log B + \log \log 2B + 0.09) + 2D(D - 1)a' \\
\leq (2D - 1) P_1 + (2c_0 + 4cc_1 \sigma \chi + 2c_1 \sigma / \theta + 0.02) U + 2DL_0(G - G_0 + \log 2).
\]
Now it is easy to verify that condition (5.5.iii) implies $E_1^* < E_4^*/3$ and that condition (5.6.ii) implies $\max\{E_2^*, E_3^*\} < E_4^*/3$. This proves that

$$\varphi(u/2, v/2) = 0.$$  

This completes the proof of theorem 5.11.

6. Numerical examples

We use the notations and hypotheses of §5.1 and 5.2, and we produce suitable values for the constant $C$, so that the assumptions (5.1) to (5.10) have been checked. Therefore the conclusion

$$|\Lambda| > \exp(-CU)$$

of theorem 5.11 holds.

Here we choose $f = 2\varepsilon$, and $\varepsilon = \sigma = 1$ (the worst values for $\sigma$ and $\varepsilon$). we proceed essentially as in [*]: we fix $\theta > 10$ and $Z \geq 1$, then we choose $c$ and $c_1$, and for those values we search a suitable $c_0$ (if it exists), for this we have to solve a quadratic equation and then to verify the conditions (5.1). From (5.6), with the value of $X$ given by (5.9) and (5.10), we deduce a suitable value for $C$.

The results are given in fig. 1 and 2 in the case of multiplicatively independent numbers. These results improve those of [*], for example in figure 1 for $Z = 1$ and $\theta = 14$, we got $C = 530$ and now we have $C = 258$.

In fig. 1 we fix $Z = 1, \theta$ varies, and we display the optimal value of $C$ together with the corresponding choices of $c, c_1$ and $c_0$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>258</td>
<td>257</td>
<td>255</td>
<td>254</td>
<td>253</td>
<td>252</td>
<td>251</td>
</tr>
<tr>
<td>$c_0$</td>
<td>28.25</td>
<td>28.51</td>
<td>28.04</td>
<td>28.33</td>
<td>27.98</td>
<td>27.98</td>
<td>28.12</td>
</tr>
<tr>
<td>$c_1$</td>
<td>1.45</td>
<td>1.44</td>
<td>1.44</td>
<td>1.42</td>
<td>1.42</td>
<td>1.41</td>
<td>1.41</td>
</tr>
<tr>
<td>$c$</td>
<td>3</td>
<td>2.99</td>
<td>2.99</td>
<td>2.98</td>
<td>2.98</td>
<td>2.97</td>
<td>2.97</td>
</tr>
</tbody>
</table>

figure 1 : multiplicatively independent numbers, $Z = 1$.

In fig. 2, both $Z$ and $\theta$ vary and we display the optimal value of $CZ^{-3}$. At the end of each row we display the range for $(c, c_1)$ corresponding to the given row. For instance, at the end of the first row in fig. 2 the indication

$$2.93 \leq c \leq 3.01, 1.4 \leq c_1 \leq 1.45$$
means that for $Z = 1$ and for the given values of $\theta$ (with $10 \leq \theta \leq 100$) we always choose $c$ and $c_1$ in these intervals.

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
Z & \theta & 12 & 13 & 14 & 15 & 20 & 30 & 50 & 100 \\
\hline
1 & 263 & 261 & 259 & 257 & 251 & 245 & 243 & 236 & 1.4; 1.45 \\
1.1 & 234 & 232 & 231 & 229 & 225 & 220 & 215 & 212 & 1.51; 1.59 \\
1.2 & 209 & 208 & 207 & 206 & 202 & 198 & 194 & 191 & 1.62; 1.67 \\
1.3 & 188 & 187 & 186 & 185 & 182 & 178 & 175 & 173 & 1.72; 1.76 \\
1.4 & 170 & 169 & 168 & 167 & 164 & 161 & 159 & 156 & 1.8; 1.88 \\
1.5 & 153 & 152 & 152 & 151 & 149 & 146 & 144 & 142 & 1.91; 1.97 \\
2 & 96 & 96 & 95 & 95 & 94 & 92 & 91 & 91 & 2.3; 2.36 \\
3 & 43 & 42.8 & 42.7 & 42.6 & 42.2 & 41.7 & 41.4 & 41 & 2.87; 2.93 \\
5 & 12.6 & 12.5 & 12.5 & 12.5 & 12.4 & 12.3 & 12.2 & 12.1 & 3.49; 3.55 \\
8 & 3.4 & 3.4 & 3.4 & 3.4 & 3.3 & 3.3 & 3.3 & 3.3 & 3.82; 3.83 \\
\hline
\end{array}
$$

figure 2 : multiplicatively independent numbers, values of $C/Z^3$.

Figures 3 and 4 correspond respectively to figures 1 and 2 for multiplicatively dependent numbers.

$$
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\theta & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\hline
C & 270 & 268 & 267 & 266 & 265 & 264 & 263 \\
\hline
c_1 & 1.41 & 1.39 & 1.39 & 1.38 & 1.38 & 1.38 & 1.37 \\
c & 2.99 & 2.98 & 2.98 & 2.97 & 2.97 & 2.97 & 2.96 \\
\hline
\end{array}
$$

figure 3 : dependent numbers, $Z = 1$.

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
Z & 1 & 1.1 & 1.2 & 1.3 & 1.4 & 1.5 & 2 & 3 & 5 & 8 \\
\hline
C/Z^3 & 273 & 243 & 217 & 194 & 175 & 158 & 98 & 44 & 12.7 & 3.5 \\
c_0 & 29.52 & 35.15 & 40.64 & 46.65 & 53.04 & 59.59 & 88.78 & 136.89 & 186.74 & 209.05 \\
c_1 & 1.41 & 1.51 & 1.62 & 1.72 & 1.8 & 1.88 & 2.28 & 2.84 & 3.46 & 3.84 \\
c & 3 & 3.4 & 3.8 & 4.19 & 4.57 & 4.94 & 6.64 & 9.15 & 11.71 & 12.94 \\
\hline
\end{array}
$$

figure 4 : multiplicatively dependent numbers, values of $C/Z^3$ if $\theta \geq 12$.

7. A consequence of the main result

With the notations and hypotheses of §5.1, we shall deduce from theorem 5.11 :

**Corollary 7.1.** — Take $f = 2e, \theta = 11$ and suppose that $\Lambda$ is not zero then

$$
|\Lambda| > \exp\{-1770\; U\}.
$$
Proof. — We suppose $|\Lambda| < \exp\{-1000 U\}$. This implies $\log B > 10.8$.

By Liouville estimate, if $\log B \leq 10.8$ then

$$|\Lambda| > 2^{-D} \cdot B^{-1} \cdot \exp(-Db_1a_1 - Db_2a_2) > B^{-1} \cdot \exp(-2Da_2 - D),$$

with $2DBa_2 + \log B + D \leq (2e^{10.8}+12)Da_2 < 98100Da_2$, and $Da_2 \leq \theta^{-2}U$.

Thus $G > 14.24$.

We get the result with the constant $C$ by dividing the interval $[1, \infty[$ in small intervals like in [*]. We check condition (5.5) in the worst case, namely with $\varepsilon$ and $\sigma$ replaced by $1$.

The numerical values we obtain are displayed in fig. 5 below. For instance, in the range $1 \leq \Omega < 2$, one can choose $c_1 = 2.26, c = 6.65, (and c_0 = 89.21, a value which is not given in the table), and one gets $C = 1000$.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>15</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>1000</td>
<td>1330</td>
<td>1610</td>
<td>1670</td>
<td>1720</td>
<td>1740</td>
<td>1770</td>
<td>1760</td>
<td>1750</td>
<td>1730</td>
<td></td>
</tr>
<tr>
<td>$c_1$</td>
<td>2.26</td>
<td>2.84</td>
<td>3.28</td>
<td>3.44</td>
<td>3.66</td>
<td>3.68</td>
<td>3.75</td>
<td>3.78; 4</td>
<td>4.03</td>
<td>4.06</td>
<td>4.06</td>
</tr>
</tbody>
</table>

figure 5 : values of $C$ in intervals on $Z, \theta \geq 11$.

8. Proof of corollary 1.1

We assume that the hypotheses of corollary 1.1 are fulfilled, and we shall prove the conclusion by considering several cases. As we may, we assume $a_1b_1 \leq a_2b_2$.

a) Assume $\log B \leq 10.7$. Then we prove the estimate in corollary 1.1 with the constant 268 instead of 270. For this we use lemma 2.2 of [*] :

$$|\Lambda| \geq 2^{-D}b_2^{-1}|b_1 \log \alpha_1 - b_2 \log \alpha_2| \geq \exp\{-D\log 2 - 2DBa_2 - \log B\}.$$

Since $\log B \leq 10.7$ we have $2B + \log 2 + \log B < 268(7.5 + \log B)^2$, hence $2DBa_2 + D\log 2 + \log B \leq 268Da_2(7.5 + \log B)^2$, which proves our claim.

b) From now on we assume $\log B \geq 10.7$. There is no loss of generality to assume that $b_1$ and $b_2$ are relatively prime. We are going to use theorem 5.11 with

$$f \geq 2e \text{ and } G = \log B + \log \log 2B + \max\{1, 0.59 + G'\},$$

where $G' \geq \log (e/2 + 2e/l_1), l_j = \lfloor \log \alpha_j \rfloor \quad (j = 1, 2)$. 

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In [*] we proved that \( l_i \geq \exp(-Da_i) \) for \( i = 1, 2 \) and \( 2l_1 \geq \exp(-Da_2) \).

c) Assume \( l_1 \geq 1/22.3 \). In this case we take \( Z = 1 \),

\[
G' = 1 + \log \left( 0.5 + 4/3l_1 \right) < 4.41,
\]

and then we may choose \( G = 5 + \log B + \log \log 2B \).

We prove the inequality of corollary 1.1. We use the estimates of §6 with admissible choices of \( \theta \).

Put \( F = (5 + \log B + \log \log 2B)/(7.5 + \log B) \).

To prove our claim we consider the following seven cases:

1. \( 10.7 \leq \log B < 14.3 \), then \( \theta \geq 18, C = 264, F < 1.01 \) and \( F^2C < 270 \),
2. \( 14.3 \leq \log B < 16.25 \), then \( \theta \geq 22, C = 262, F < 1.014 \) and \( F^2C < 270 \),
3. \( 16.25 \leq \log B < 19.1 \), then \( \theta \geq 24, C = 260, F < 1.0183 \) and \( F^2C < 270 \),
4. \( 19.1 \leq \log B < 20.95 \), then \( \theta \geq 27, C = 259, F < 1.021 \) and \( F^2C < 270 \),
5. \( 20.95 \leq \log B < 24.77 \), then \( \theta \geq 29, C = 258, F < 1.0229 \) and \( F^2C < 270 \),
6. \( 24.77 \leq \log B < 30.6 \), then \( \theta \geq 33, C = 257, F < 1.0248 \) and \( F^2C < 270 \),
7. \( 30.6 \leq \log B \), then \( \theta \geq 39, C = 256, F < 1.026 \) and \( F^2C < 270 \).

d) Now on we assume \( l_2 \leq l_1 < 1/22.3 \). With the present notations

\[
l_1 \geq \exp(-Da_i), \text{ for } i = 1, 2, \]

so that we have \( Da_1 > 3.1 \). Looking again at Liouville estimate, we see that if \( \log B \leq 11.97 \) then

\[
|\Lambda| \geq \exp\{-269D^2a_1a_2(7.5 + \log B)^2\}.
\]

Hence we may suppose \( \log B > 11.97 \). Besides one may choose

\[
G' = 2.41 + Z_0, G = 3 + \log B + \log \log 2B + Z_0 \text{ where } Z_0 = \log (1/l_1).
\]

Thus \( G \geq 20.6 \) and

\[
3.1 < Z_0 \leq \min\{Da_1, Da_2, \log (ef)\}.
\]

We take \( Z = \min\{G/11, Z_0\} \). We obviously have

\[
1 \leq Z \leq \min\{DG/11, Da_1, Da_2, \log (ef), \sqrt{DG}/8\}.
\]
Now, from corollary 7.1 we deduce

\[ |b_1 \log \alpha_1 - b_2 \log \alpha_2| \geq \exp\{-1770D^4a_1a_2G^2Z^{-3}\}. \]

If \( G \geq 11Z_0 \) then \( Z = Z_0 > 3.1 \),

\[ G^2/Z^3 < (5 + \Log B + \Log \Log 2B)^2/27 < (1.1^2/27)(7.5 + \Log B)^2, \]

and we have proved our claim.

Whereas if \( G < 11Z_0 \) then by the results of figure 5:

\- if \( Z \leq 3 \) then \( C = 1330 \) and \( 1330G^2/Z^3 < 1330 \cdot 11^3/20.6 \),
\- if \( Z > 3 \) then \( C = 1770 \) and \( 1770G^2/Z^3 < 1770 \cdot 11^3/33 \),

and the result follows in each case.

Now the proof of corollary 1.1 is complete.

9. Examples

a) class number one

J. M. Cherubini and R. V. Wallisser [C.W.], applied an estimate of linear forms in logarithms taken from [M. W. 1] to compute all the imaginary quadratic fields of class number one.

The linear form which is used by these authors is

\[ \Lambda = p\Log (5 + 2\sqrt{6}) - 2q\Log (2 + \sqrt{3}), \quad p, q \in \mathbb{Z}. \]

Put \( \alpha_1 = 5 + 2\sqrt{6}, \alpha_2 = 2 + \sqrt{3}; \) then \( D = 4, l_1 = 2.29243 \ldots \) and \( l_2 = 1.31695 \ldots \) We take \( a_1 = l_1/2, a_2 = l_2/2, f = 1, Z = \Log 2e, \varepsilon = 1, \sigma = 1 \) and \( G = 1 + \Log 4\Delta + \Log \Log 8\Delta. \)

If we choose \( c_0 = 403.51, c_1 = 2.91 \) and \( c = 16.58 \) then the same computation than in §6 gives \( C = 3455.5. \) This leads to \( d > -5.1 \cdot 10^{17}, \) whereas the lower bound of [M. W. 1] gave only \( d > -10^{34}, \) and we got \( d > -2.5 \cdot 10^{19} \) in [*].

b) quotient of two pure powers

Let \( x, y, p, q \) be positive rational integers with \( x^p \neq y^q. \) Let \( X, Y, B \) be positive real numbers satisfying \( X \geq \max\{x, 3\}, Y \geq \max\{y, 3\}, B \geq \max\{p, q\}. \) We prove that

\[ |x^py^{-q} - 1| > \exp\{-1905\Log X \Log Y(8 + \Log B)^2\}. \]
If $x$ and $y$ are multiplicatively dependent the result is obvious (see the end of §5 of [*]). Now we assume that $x$ and $y$ are multiplicatively independent and consider two cases:

i) If $\log B \leq 13.8$, we have $|x^p y^{-q} - 1| \geq y^{-q} \geq \exp\{-B\log Y\}$, and the assumption $\log B \leq 13.8$ implies $B < 2072(8 + \log B)^2$. Now we have

$$\log X \geq \log 3 \quad \text{and} \quad 2072/\log 3 < 1887.$$  

Therefore, we get the conclusion.

ii) If $\log B > 12.33$ we use theorem 5.11 with $Z = \varepsilon = f = D = 1$. We choose now $G = 1.0021(8 + \log B), \theta = 21.86, c_0 = 212.77, c_1 = 1.99, c = 10.02$ and find $C = 1896$ and the result follows easily.

c) ray class field

In this section we present a work of J. Cougnard and V. Fleckinger which uses some linear form in two logarithms. They consider the ray class field $K$, extension of $k = \mathbb{Q}(\sqrt{-19})$, associated to $\mathcal{P}_7$, the principal ideal of $k$ generated by $\left(1 + \frac{1 + \sqrt{-19}}{2}\right)$. They show that the ring of integers $O_K$ of the field $K$ does not have any basis over $O_k$ composed of the powers of some element of $O_K$.

They reduce this problem, via the study of the integers points of some elliptic curve, to the computation of the integers for which a certain linear form in two logarithms is very small. Namely, they want to find all the rational integers $b_1$ and $b_2$ such that

$$|b_1 \log (\alpha_1) - b_2 \log (\alpha_2)| \leq \exp(-0.0367B),$$

where $B = \max\{|b_1|, |b_2|\}$ and $\alpha_1$ is a real root of the polynomial

$$P = x^9 - 16x^8 - 38x^7 - 179x^6 + 41x^5 - 237x^4 + 307x^3 - 120x^2 + 19x - 1,$$

and $\alpha_2$ a root of the reciprocal polynomial with respect to $P$, with

$$l_1 = \log \alpha_1 = 2.92112\ldots, l_2 = \log \alpha_2 = 2.24999\ldots.$$

Using Baker's estimate they prove $B < 10^{212}$. Here, we apply theorem 5.11 to this linear form.
Firstly, using the method of computation of the measure of a polynomial described in [CMP], we see that \( M(P) < 344.56 \), so that

\[
h(\alpha_1) = h(\alpha_2) < 5.843/9.\]

We put \( D = 9, f = 5.843, a_1 = f l_1/D, a_2 = f l_2/D, \sigma = 0.852, Z = 1.97 \) (it is easy to verify that this value of \( Z \) satisfies the inequalities \( Z \leq \min \{ D a_1, D a_2, \log (e, f) \} \)). We have \( G' < 0.2D \) so that we can take \( G = 1 + \log B + \log \log 2B \).

We suppose \( \log B \geq 23.17 \), then \( G \geq 27 \) and \( \theta > 120 \). Applying theorem 5.11 we find the constant \( C < 1235 \) (for \( c_0 = 140.56, c_1 = 2.58, c = 9.11 \)). This gives

\[
0.0367B \leq 1235 \cdot D^4 \cdot a_1 \cdot a_2 \cdot (1 + \log B + \log \log 2B)^2/Z^3, 
\]
so that \( \log B < 23.23 \), and \( B < 1.3 \cdot 10^{10} \).

10. The case of a root of unity

We consider here the special case when one of the numbers \( \alpha_i \) is a root of unity. We choose the following notations \( \zeta \) is a root of unity, \( \zeta = e^{i\pi/m} \), and \( \alpha \) is a non zero algebraic number.

We put \( D = [Q(\zeta, \alpha) : Q] \). We choose \( \log \zeta = i\pi/m \), and \( \log \alpha \) is any non-zero determination of the logarithm of \( \alpha, l = |\log \alpha| \). As before \( \beta = b_1/b_2 \) is a rational number, \( b_1, b_2 \in \mathbb{Z}, 0 < b_1, b_2, (b_1, b_2) = 1 \) such that

\[
\Lambda = i\beta\pi/m - \log \alpha
\]
does not vanish. We put \( B = \max \{ b_1, b_2 \}, a' = h(\alpha) \) [notice that \( h(\zeta) = 0 \)].

We denote by \( a, G, G', Z, \theta, f, \rho \) positive real numbers which satisfy the following relations :

\[
1 \leq f \leq 2e^{D(a'+1)}, \theta \geq 10, a = f D^{-1}, a \geq 1/D, f \geq m/\pi,
\]
\[
G' = 1 + \log (0.5 + \rho m/\pi), G_0 = 0.59 + \log B + \log \log 2B,
\]
\[
G = G_0 + \max \{ 0.41, G'/D \},
\]
\[
1 \leq Z \leq \min \{ DG/\theta, f \theta/m, Da, \log (e f), \sqrt{D} G/10 \}.
\]

With the present notations, we have \( \theta_1 = \theta f l/Z, \theta_2 = \theta f \pi/m Z \).
We define $\mu, \varepsilon$ as before and put now $\sigma = h(\alpha)/2a$ (so that $0 \leq \sigma \leq 1/2$), and

$$U = D^3 f(\pi/m) a G^2 Z^{-3}.$$ 

Now suppose that $c_0, c_1, c, \mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, C, \eta^*, \mu, \rho, p^*, \xi, \xi_0$ are positive real numbers which satisfy the hypotheses (5.1) to (5.9), and replace (5.10) by

$$(5.10)' \quad \mathcal{X}c \geq \sqrt{c_0 c_1} + \frac{1}{2} \cdot \max \left\{ \frac{c_1 Z_m}{f \pi} + \frac{1}{\theta_2}, \sqrt{c_0/\theta} + \frac{1}{\theta} \right\}.$$ 

**Theorem 10.1.** Under the hypotheses of this paragraph, we have

$$|\Lambda| > \exp(-C'D^3 f a G^2 Z^{-3}),$$

where $C' = C \pi/m$.

Proof of theorem 10.1.

We suppose that $|\Lambda| < e^{-C U}$, and we show that this leads to a contradiction. We first remark that this implies $|\alpha| = 1$. Indeed, if $|\alpha| \neq 1$ then by Liouville inequality,

$$|\Lambda| \geq |\text{Re}(\Lambda)| = |\text{Re}(\log \alpha)| = |\text{Log} |\alpha|| \geq 2^{-D} \exp\{-h(|\alpha|)\},$$

and $h(|\alpha|) \leq h(\alpha)$, which contradicts $|\Lambda| < e^{-C U}$. Thus $D \geq 2$. Remark also that now we have $a'' = 0$.

An easy proof (given at the end of §5 of [*]) shows that $\alpha$ is not a root of unity.

Then the proof of theorem 10.1 follows exactly that of theorem 5.11, except at the very end, when we apply the zero-estimate.

Recall that $\mathcal{X}_2 = \sqrt{c_0 c_1}/c$ and $\mathcal{X}_1 = \mathcal{X} - \mathcal{X}_2$. We define the integers $U_2$ and $V_2$ by

$$\mathcal{X}_2 c D^2 a G Z^{-2} - 0.5 < U_2 \leq \mathcal{X}_2 c D^2 a G Z^{-2} - 0.5,$$

$$\mathcal{X}_2 c D^2 a_0 G Z^{-2} - 0.5 < V_2 \leq \mathcal{X}_2 c D^2 a_0 G Z^{-2} - 0.5,$$

where $a_0 = f \pi/m$,

and $U_1, V_1$ by

$$U_1 = M_1^* - U_2, V_1 = M_2^* - V_2$$

[recall that $M_1^* = [\mathcal{X} c D^2 a Z^{-2}] + 0.5$ and $M_2^* = [\mathcal{X} c D^2 a_0 Z^{-2}] + 0.5$.]
Linear forms in two logarithms and Schneider's method (III)

Since $\alpha$ is not a root of unity, we have

$$ \text{Card } \{ \zeta^u \alpha^v; |u| \leq L_1, |v| \leq L_1 \} = 2m(2V_1 + 1) $$

and $$(2V_1 + 1)^2 - 2(\lambda_2 cD^2 a_0 GZ^{-2} + 0.5) + 1 > (2\lambda_1 c - 1/\theta_2)D^2 a_0 GZ^{-2};$$
thus the condition

$$ \text{Card } \{ \zeta^u \alpha^v; |u| \leq L_1, |v| \leq L_1 \} > 2L_1 $$

is implied by (5.10)'.

To conclude, we have to verify that $(2U_1 + 1)(2V_1 + 1) > L_0$. We have

$$(2U_1 + 1)(2V_1 + 1)/L_0 > (2\lambda_1 cD^2 a_0 GZ^{-2} - 1)(2\lambda_1 cD^2 aGZ^{-2} - 1)/L_0$$
$$\geq (2\lambda_1 c - 1/\theta)D^4 a_0 aG^2 Z^{-4}/L_0 \geq (2\lambda_1 c - 1/\theta)^2 DG/cZ$$
$$\geq (2\lambda_1 c - 1/\theta)^2 \theta/c;$$

and the conclusion follows again from (5.10)'.

This completes the proof of theorem 10.1.

11. Numerical examples for theorem 10.1

We keep the notations and hypotheses of §10 and we give suitable values for the constant $C'$. The results are given in figures 6, 7 and 8 below.

In figure 6 we fix $m = 1, f$ varies, and we give a value of $C'f$ and the corresponding choices of $c_0, c_1$ and $c$.

In figure 7, we suppose $f \geq 5$ and $m = 1$, and we give the values of the product $5C'$ for $D$ in the range $2 \leq D \leq 10$.

Figure 8 displays some cases with $m \geq 2$ and $f = 5$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C'f$</td>
<td>4448</td>
<td>3644</td>
<td>3246</td>
<td>3008</td>
<td>2854</td>
<td>2750</td>
<td>2675</td>
<td>2586</td>
<td>2521</td>
<td>2469</td>
</tr>
<tr>
<td>$c_0$</td>
<td>152.05</td>
<td>81.76</td>
<td>53.56</td>
<td>39.76</td>
<td>31.45</td>
<td>26.09</td>
<td>22.13</td>
<td>15.79</td>
<td>13.72</td>
<td>9.97</td>
</tr>
<tr>
<td>$c_1$</td>
<td>2.09</td>
<td>1.98</td>
<td>1.9</td>
<td>1.79</td>
<td>1.72</td>
<td>1.65</td>
<td>1.58</td>
<td>1.6</td>
<td>1.45</td>
<td>1.38</td>
</tr>
<tr>
<td>$c$</td>
<td>8.88</td>
<td>6.27</td>
<td>4.95</td>
<td>4.13</td>
<td>3.58</td>
<td>3.18</td>
<td>2.88</td>
<td>2.48</td>
<td>2.16</td>
<td>1.78</td>
</tr>
</tbody>
</table>

figure 6: case of a root of unity and $Z = 1$. 

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**PROPOSITION 12.1.** — Under the hypotheses of theorem 10.1, we have

\[ |\alpha| > \exp(-2555 \cdot (f/2e) \cdot a \cdot (6.5 + \log B)^2) \] for \( f \geq 2e \).

For \( f \geq 2e \) we can take \( C' = 2545 \) (choose \( c_0 = 15, c_1 = 1.53, c = 2.32 \)).

The estimate of Euler function given in the appendix enables us to get \( G'/D \leq 0.76 \) (the worst case is \( D = 2 \) and \( m = 3 \)). Thus we can always take \( G = 1.35 + \log B + \log \log 2B \). It is easy to verify that \( G < 1.00212(6.5 + \log B) \). Hence the result.

13. A corollary of theorem 10.1

**COROLLARY 13.1.** — Let \( \alpha \) be an algebraic number of degree \( D \), \( b \) a positive rational integer and \( \zeta \) a root of unity of order \( m \).

Let \( a = \max\{1/D, h(\alpha), f|\log \alpha|/D\} \), suppose \( f \geq m/\pi \) then

\[ |\alpha^b - \zeta| \geq \exp\{-5000D^3fa(5 + \log b)^2\}. \]
14. An example of a measure of irrationality

Let $\alpha = (3 + i\sqrt{7})/4$, then $2\alpha^2 - 3\alpha + 2 = 0$ and $\alpha = e^{i\theta}$ with

$$\theta = \arccos \left(\frac{3}{4}\right)$$

$$= 0.72273424781341561117837735264133336202521848642\ldots$$

We want to get a lower bound for $\left|\frac{\theta}{\pi} - \frac{p}{q}\right|$, where $p$ and $q$ are rational integers.

One verifies that the number

$$\lambda := \frac{\theta}{\pi}$$

$$= 0.23005345616261588521378056770514289300991139527071410205\ldots$$

has the following expansion as continued fraction

$$[0; 4, 2, 1, 7, 1, 1, 2, 1, 5, 1, 27, 8, 6, 2, 4, 3, 4, 1, 2, 6, 1, 3, 2, 3, 538, 2, 7, 101, 1, 2, 3, 3, 2, 3, 3, 1, 124, 1, 5, 1, 1, 1, 14, 2, 14, 1, \ldots].$$

This implies

$$\left|\lambda - \frac{p}{q}\right| > \frac{1}{540q^2} \quad \text{for} \quad 0 < q < 5 \cdot 10^{27}.$$ 

We put $\Lambda = ip\pi - \theta iq$ and apply theorem 10.1 with $D = 2, h(\alpha) = (\log 2)/2, m = 1, l_1 = \pi, l_2 = \theta, f = 1/\theta = 1.383\ldots, \sigma = \alpha' = \log 2/2, \epsilon^{-1} = \log (\epsilon f)$, since $G'/2 \leq (1 + \log (0.5 + 1.22/\pi))/2 < 0.45$ we can take

$$G = 1.05 + \log q + \log \log 2q < 1.082\log q,$$

then $G \geq 68.9$ and we get (take $c_0 = 67.26, c_1 = 1.81, c = 6.68$, then $C'f = 4179$).

**Proposition.** The number $\lambda = \arccos (3/4)/\pi$ has the following measure of irrationality

$$\left|\lambda - \frac{p}{q}\right| > \exp(-19600(\log q)^2), \quad \text{for} \quad q \geq 2.$$
Remark: Our measure of irrationality of $\lambda$ is not the best known. Baker proved that there exists a constant $c$ such that $|\lambda - p/q| > q^{-c}$ for $q \geq 2$. But the best known estimate of linear form of logarithms with Baker's method which comes from [B G M M S] gives

$$|\lambda - p/q| > c'q^{-c} \text{ with } c' = \exp(-1.3 \cdot 10^{14}) \text{ and } c = 2.4 \cdot 10^{14},$$

so that our result is better for $\log q < 10^{10}$.

Added in proof. — Our main result, theorem 5.11, is not very good when one of the logarithms is very small. A remedy is to introduce a new parameter $q = a_1b_1/(a_2b_2)$. This leads to several minor changes which concern essentially $\eta_0, \eta$, the conditions (5.6.i) and (5.6.ii): for example, the term $\log(2e)$ in (5.6.i) is replaced by $\log(1 + q)e$.

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[MW2] Mignotte (M.) and Waldschmidt (M.).— Linear forms in two logarithms Schneider’s method (II), Acta Arith., to appear (preprint in Publications de l’IRMA, Strasbourg, 1988, 51 pages). [This paper is denoted [*] throughout the present article.]

[RS] Rosser (J.B.) and Schoenfeld (L.).— Approximate formulas for some functions of prime numbers; Illinois J. Math., v.6, 1962, p. 64-94.
APPENDIX

A lower bound for the Euler function

For the proof of the proposition 12.1, we have used the following estimate

**PROPOSITION A.1.** — For all positive rational integers one has

\[ n < 2.685 \cdot \varphi(n)^{1.161}. \]

Suppose that the decomposition of \( n \) in prime factors is \( n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \) with \( p_1 < \ldots < p_k \), then

\[ \frac{n}{\varphi(n)} = \frac{p_1}{p_1 - 1} \cdots \frac{p_k}{p_k - 1}. \]

Put \( \lambda = 1 - \log 4/\log 5 \), then \( p_i/(p_i - 1) \leq 5/4 \leq p_i^{\lambda} \) for \( k \geq 3 \). Thus,

\[ \frac{n}{\varphi(n)} \leq 3p_3^\lambda \cdots p_k^\lambda < 2.341p_1^\lambda \cdots p_k^\lambda \leq 2.341n^\lambda, \]

so that, \( n < (2.341\varphi(n))^{1/(1-\lambda)} < 2.685\varphi(n)^{1.161}. \)

**Remark.** — It is clear that, for any \( \theta > 1 \), the method used to prove the proposition permits to compute a sharp constant \( c_\theta \) such that \( n \leq c_\theta \cdot \varphi(n)^\theta \) for all \( n \); for example

\[ n \leq 3.046 \cdot \varphi(n)^{1.1}. \]

In fact our proof shows that the maximum value of the quotient \( n/\varphi(n)^\theta \) is reached for the integer \( 2.3.5.\ldots p_j \), where \( p_j \) is the largest prime \( p \) such that \( p/(p - 1) \leq p^{1-1/\theta} \).

Moreover, this method can also be applied to many multiplicative arithmetical functions.

**COROLLARY A2.** — If a number field of degree \( D \) contains a root of unity of order \( k \) then

\[ k < 2.685D^{1.161}. \]

The estimate \( n \ll \varphi(n)^{1+\varepsilon} \), for any fixed \( \varepsilon > 0 \), is not the best possible for \( n \) large : the following result holds.
PROPOSITION A3. — For $D \geq 2$ we define the function

$$\varphi_{-1}(D) = \max \{N \geq 1; \varphi(N) \leq D\}.$$ 

Then, for any $\varepsilon > 0$, there exists an integer $D_0(\varepsilon) \geq 2$ such that, for all $D \geq D_0$, we have

$$\varphi_{-1}(D) \leq (e^C + \varepsilon)D \log \log D,$$

where $C$ is Euler's constant (so that $e^C = 1.78107\ldots$). Moreover,

$$\varphi_{-1}(D) \leq 4D \log \log (D + 7) \text{ for } D \geq 2.$$

Inequality (A.1) is an easy consequence of inequality (3.42) in theorem 15 of [RS], namely

$$\frac{n}{\varphi(n)} < e^C \cdot \log \log n + \frac{2.50637}{\log \log n}.$$ 

Indeed this inequality is more precise than the first estimate of proposition A3 and permits to compute admissible values for $D_0(\varepsilon)$. For example:

$$\varphi_{-1}(D) \leq 2\log \log D \text{ for } D > e^{34.1}.$$

Proof of (A.4) : Consider $D > e^{34.1}$ and $N \geq 1$ such that $\varphi(N) = D$. Denote by $\omega(N)$ the number of different prime factors of $N$.

If $\omega(N) \geq 15$ then $\log \log N \geq 3.71, \log \log D \geq 3.66$ and (A.3) gives

$$N/D \leq e^C \cdot \log \log N + 2.51/3.71.$$

So that $N < 1.97 \cdot D \cdot \log \log N < D^{21/20}$, since $1.97 \log \log N < N^{1/20}$. This implies

$$\log \log N < \log \log D + 0.05,$$

and inequality (A.4) follows easily.

If $\omega(N) < 15$ then $N \leq 7.06 \cdot D$ and (A.4) is also true.

Another special case of (A.1) is

$$\varphi_{-1}(D) \leq 3.24D \log \log D \text{ for } D \geq 48.$$

The proof of (A.5) is similar to that of (A.4) : Consider integers $D \geq 48$ and $N \geq 1$ such that $\varphi(N) = D$. 

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If \( \omega(N) \geq 5 \) then \( N \geq 2 \times 3 \times 5 \times 7 \times 11 = 2310 \) and \( D \geq 480 \). Then (A.3) implies

\[
N/D \leq (e^C + 2.51(\log \log 2310)^{-2}) \cdot \log \log N.
\]

By proposition A.1, we have \( N < 2.7 \times D^{1.161} < D^{1.148} \), since \( D \geq 480 \); so that

\[
\log \log N \leq \log \log D + 0.35,
\]

and inequality (A.4) follows in this case.

If \( \omega(N) < 5 \) then \( N \leq \frac{35}{8} D \) and (A.4) is also true.

Now, we prove (A.2). From (A.5) we have only to consider \( D < 48 \). Let \( D < 48 \) and suppose that \( \varphi^{-1}(D) = N \).

Since \( D < 48 \), we have \( \omega(N) \leq 3 \). Thus \( N \leq \frac{15}{4} D \), so that (A.2) is true for \( D \geq 12 \).

Finally, if \( D \leq 12 \) then a direct study shows that (A.2) is still true.