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A Viterbo-Hofer-Zehnder Type Result for Hamiltonian Inclusions

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RESUMÉ. — On obtient un résultat de type de Viterbo-Hofer-Zehnder pour les inclusions hamiltoniennes. Soit $H : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ une fonction locale lipschitzienne et $c \in \mathbb{R}$. Supposons que $\Sigma := \{x \in \mathbb{R}^{2N} \mid H(x) = c\}$ soit un ensemble partiel compact et non vide de $\mathbb{R}^{2N}$ et $0 \notin \partial H(x)$ pour $x \in \Sigma$. Donc, pour aucun $\delta > 0$ l'inclusion hamiltonienne $\dot{x} \in J\partial H(x)$ a une solution conservatrice et périodique $x(t)$ de façon que $H(x(t)) \equiv c' \in (c - \delta, c + \delta)$ pour tout $t$.

ABSTRACT. — We obtain a Viterbo-Hofer-Zehnder type result for Hamiltonian inclusions. Let $H : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ be a locally Lipschitz function and $c \in \mathbb{R}$. Suppose that $\Sigma := \{x \in \mathbb{R}^{2N} \mid H(x) = c\}$ is a nonempty compact subset of $\mathbb{R}^{2N}$ and $0 \notin \partial H(x)$ for $x \in \Sigma$. Then for any $\delta > 0$ the Hamiltonian inclusion $\dot{x} \in J\partial H(x)$ has a conservative periodic solution $x(t)$ such that $H(x(t)) \equiv c' \in (c - \delta, c + \delta)$ for all $t$.

1. Introduction and Main Result

Hofer and Zehnder [1] extended the result of Viterbo [2]. The aim of the present paper is to extend the result of [1] to the case of Hamiltonian inclusions.

Let $H : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ be locally Lipschitz continuous, which is written as $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$. Consider the Hamiltonian inclusion.

$$\dot{x} \in J\partial H(x) \quad (1)$$

where $\partial H$ is Clarke's generalized gradient of $H$ and $J$ is the standard $2N \times 2N$ symplectic matrix (see [3]). By a solution of (1) we mean an

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absolutely continuous function $x(t)$ satisfying (1) for almost all $t$. It is well-known that, if $H$ is regular, then any solution of (1) is conservative, i.e. $H(x(t)) \equiv \text{constant}$. However, in general, if $H$ is not regular, then a solution of (1) need not be conservative.

Our main result is the following

**Theorem 1.** — Let $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$ and $c \in \mathbb{R}$. Suppose that $\Sigma_c = H^{-1}(c)$ is a nonempty compact subset of $\mathbb{R}^{2N}$ and

$$0 \notin \partial H(x) \quad \text{for} \quad x \in \Sigma_c.$$  \hspace{1cm} (2)

Then for any bounded neighborhood $\Omega$ of $\Sigma_c$, there are positive constants $\beta$ and $d$ such that for any $\delta > 0$, (1) has a $T = T(\delta)$-periodic conservative solution $x(t)$ in $\Omega$ such that $H(x(t)) \equiv c' \in (c - \delta, c + \delta)$ and

$$\beta \leq \frac{1}{2} \int_0^T (-J \dot{x}, x) \, dt \leq d. \hspace{1cm} (3)$$

The following results obtained by the author [4] will be used in the proof of theorem 1.

**Proposition 1 ([4]).** — Let $\Omega$ be an open subset of $\mathbb{R}^{k}$ and $H \in C^{1-0}(\Omega, \mathbb{R})$. Then for any continuous function $\epsilon : \Omega \to (0, +\infty)$ there is a $C^\infty$-function $g : \Omega \to \mathbb{R}$ such that

i) $|g(x) - H(x)| \leq \epsilon(x)$ for $x \in \Omega$,

ii) $\forall x \in \Omega, \exists y \in \Omega$ and $\xi \in \partial H(y)$ such that $|x - y| \leq \epsilon(x)$ and $|g'(x) - \xi| \leq \epsilon(x)$.

A $C^1$-function $g : \Omega \to \mathbb{R}$ satisfying the condition i) and ii) in proposition 1 is called an $\epsilon(x)$-admissible approximation for $H$ on $\Omega$. In particular, when $\epsilon(x) \equiv \epsilon$, $g$ is called an $\epsilon$-admissible approximation for $H$ on $\Omega$.

**Proposition 2 ([4]).** — Let $\Omega$ be an open subset of $\mathbb{R}^{2N}$, $H \in C^{1-0}(\Omega, \mathbb{R})$ and $\epsilon_n \to 0$ ($n \to \infty$) with $\epsilon_n > 0$. Suppose that for each $n$, $H_n \in C^1(\Omega, \mathbb{R})$ is an $\epsilon_n$-admissible approximation for $H$ on $\Omega$ and $x_n$ is a $T_n$-periodic solution of the Hamiltonian system

$$\dot{x} = JH'_n(x). \hspace{1cm} (4)$$
If

i) \( \{ T_n \mid n = 1, 2, \ldots \} \) is bounded,

ii) \( \{ x_n(t) \mid t \in \mathbb{R}, n = 1, 2, \ldots \} \) is contained in a compact subset of \( \Omega \),

then \( \{ x_n \} \) has a subsequence \( \{ x_{n_k} \} \) which converges uniformly to a

T-periodic solution \( x \) of (1) with \( T = \lim T_{n_k} \) and

\[
H(x(t)) \equiv c = \lim H_{n_k}(x_{n_k}(t)).
\]

In section 2 we give the proof of theorem 1. In section 3 we extend
the a priori bound criterion of Benci-Hofer-Rabinowitz [5] to the case of
Hamiltonian inclusions.

2. Proof of theorem 1

Without loss of generality we may assume that \( c = 1 \) and \( \Sigma_1 \) is connected.

Let \( \Omega \), a bounded neighborhood of \( \Sigma_1 \), be given. By the upper semi-
continuity of \( H \), the compactness of \( \Sigma_1 \) and the condition (2), we may
choose a bounded neighborhood \( V \) of \( \Sigma_1 \) such that \( V \subset \Omega \) and \( 0 \not\in \partial H(x) \) for
\( x \in V \). Then there are positive constants \( m \) and \( M \) such that \( m < |\xi| < M \)
for \( \xi \in \partial H(V) \). Using the pseudo-gradient flow (see [6]) we can construct a
Lipschitz homeomorphism \( \psi : (-s, s) \times \Sigma_1 \to V \) such that

\[
H(\psi(t, x)) = 1 + t \quad \text{for} \quad (t, x) \in (-s, s) \times \Sigma_1.
\]

Set

\[
U = \psi((-s, s) \times \Sigma_1), \quad D = \text{diam } U, \quad \Sigma_c = (H|_U)^{-1}(c).
\]

We fix positive numbers \( r, b \), such that

\[
D < r < 2D, \quad \frac{3}{2} \pi r^2 < b < 2\pi r^2.
\]

Take a sequence \( \epsilon_n \to 0 \) such that \( 0 < \epsilon_n < \min \{ s/3, m/3 \} \) for all \( n \). By
proposition 1, for each \( n \), there is an \( \epsilon_n \)-admissible approximation \( H_n \) for
\( H \) on \( U \) and \( H_n \in C^\infty(U, \mathbb{R}) \). Then we have

\[
\begin{align*}
\left| H_n(x) - H(x) \right| & \leq \frac{s}{3} \quad \text{for } x \in U \text{ and all } n, \\
\frac{2}{3} m & < \left| H'_n(x) \right| < M + \frac{m}{3} \quad \text{for } x \in U \text{ and all } n,
\end{align*}
\]
For each $n$ let $\psi_n$ be the flow in $U$ generated by

$$\dot{x} = -\frac{H'_n(x)}{|H'_n(x)|^2}, \quad x(0) \in U.$$ 

Set $\Sigma_{1,n} = H_n^{-1}(1)$. It is easy to see that $\psi_n \left( \left[ -\frac{s}{2}, \frac{s}{2} \right] \times \Sigma_{1,n} \right) \subset U$ and

$$H_n(\psi_n(t,x)) = 1 + t \quad \text{for} \quad (t,x) \in \left[ -\frac{s}{2}, \frac{s}{2} \right] \times \Sigma_{1,n}.$$

**Lemma 1.** For each $n$, $\Sigma_{1,n}$ is a connected compact hypersurface in $U$.

**Proof.** It suffices to prove the connectedness of $\Sigma_{1,n}$. For fixed $n$ let $x_1, x_2 \in \Sigma_{1,n}$. Then there are $-t_1 < 0$ and $-t_2 < 0$ such that

$$\psi_n(-t_1, x_1) = y_1 \in \Sigma_{1+s/2} \quad \text{and} \quad \psi_n(-t_2, x_2) = y_2 \in \Sigma_{1+s/2}.$$ 

Note that $\Sigma_{1+s/2}$ is connected since $\Sigma_{1+s/2}$ is homeomorphic to $\Sigma_1$. Let $p$ be a path in $\Sigma_{1+s/2}$ joining $y_1$ to $y_2$. It is easy to see that along the descent flow lines of $\psi_n$, $p$ can be deformed to a path in $\Sigma_{1,n}$ joining $x_1$ to $x_2$. So $\Sigma_{1,n}$ is connected and the proof of lemma 1 is complete.

Set $U_n = \psi_n \left( \left( -\frac{s}{2}, \frac{s}{2} \right) \times \Sigma_{1,n} \right)$. Then $\psi_n : \left( -\frac{s}{2}, \frac{s}{2} \right) \times \Sigma_{1,n} \to U_n \subset U$ is a diffeomorphism. We denote by $A_n$ and $B_n$ the unbounded and bounded component of $\mathbb{R}^{2N} \setminus U_n$ respectively and by $B$ the bounded component of $\mathbb{R}^{2N} \setminus U$. We may assume that $0 \in B$, then $0 \in B_n$ since $B \subset B_n$ for all $n$.

Let $\delta > 0$ be given. We may assume $\delta < \frac{s}{2}$.

Following [1], we pick a $C^\infty$-function $f : \left( -\frac{s}{2}, \frac{s}{2} \right) \to \mathbb{R}$ satisfying

$$f\big|_{(-\delta/2, -\delta)} = 0, \quad f\big|_{[\delta, s/2)} = b \quad \text{and} \quad f'(t) > 0 \quad \text{for} \quad -\delta < t < \delta.$$ 

Choose a $C^\infty$-function $g : (0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} 
  g(t) = b & \text{for } t \leq r, \\
  g(t) = \frac{3}{2} \pi t^2 & \text{for } t \text{ large}, \\
  g(t) \geq \frac{3}{2} \pi t^2 & \text{for } t > r, \\
  0 < g'(t) \leq 3\pi t & \text{for } t > r.
\end{cases}$$
For each \( n \) define a \( C^\infty \)-function \( G_n : \mathbb{R}^{2N} \to \mathbb{R} \) by

\[
G_n(x) = \begin{cases} 
0 & \text{if } x \in B_n \\
f(t) & \text{if } x \in \psi_n(t \times \Sigma_{1,n}), -\delta \leq t \leq \delta \\
b & \text{if } x \in A_n \text{ and } |x| \leq r \\
g(|x|) & \text{if } |x| > r.
\end{cases}
\]

Then, by [1], for each \( n \) the Hamiltonian system

\[
\dot{x} = JG'_n(x)
\]

has a 1-periodic solution \( x_n \) in \( U_n \) such that

\[
H_n(x_n(t)) = c_n \in (1 + \delta, 1 - \delta) \quad \text{for all } t
\]

and

\[
\beta \leq \frac{1}{2} \int_0^1 \langle -J\dot{z}_n, z_n \rangle \, dt \leq d,
\]

where \( \beta \) and \( d = 16\pi D^2 \) are positive constants independent of \( n \) and \( \delta \).

By the definition of \( G_n \) we have

\[
G_n(x) = f(H_n(x) - 1) \quad \text{and} \quad G'_n(x) = f'(H_n(x) - 1)H'_n(x)
\]

for \( x \in (H_n|_{U_n})^{-1}((1 - \delta, 1 + \delta)) \).

Set \( z_n(t) = x_n(f'(c_n - 1)t) \). Then \( z_n \) is a \( T_n \)-periodic solution in \( U_n \) of the Hamiltonian system

\[
\dot{z} = JH'_n(z)
\]

with \( T_n = f'(c_n - 1) \) and

\[
\beta \leq \frac{1}{2} \int_0^{T_n} \langle -J\dot{z}_n, z_n \rangle \, dt \leq d. \tag{7}
\]

From the fact that \( |c_n - 1| < \delta \) and \( f' \) is bounded on \((-\delta, \delta)\) it follows that \( \{T_n \, | \, n = 1, 2, \ldots\} \) is bounded. Noting that

\[
U_n \subset \left\{ x \in U \left| 1 - \frac{5}{6}s \leq H(x) \leq 1 + \frac{5}{6}s \right. \right\} \subset U,
\]

from proposition 2 it follows that \( \{z_n\} \) has a subsequence \( \{z_{nK}\} \) which converges uniformly to a conservative \( T \)-periodic solution \( z \) of (1) such that

\[
T = \lim T_{nK}, \quad H(z(t)) = \bar{c} = \lim c_{nK} \in [1 - \delta, 1 + \delta] \quad \text{and} \quad z(t) \in U, \quad \forall \ t.
\]

(3) follows from (7). The proof of theorem 1 is complete. \( \square \)
3. A criterion for a priori bounds

For \( x \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N \), set \( x = (p, q) = (\pi_1 x, \pi_2 x) \). Note that in general neither of the sets \( \partial_p H(x) \times \partial_q H(x) \) and \( \partial H(x) \) need be contained in the other, but both of them are contained in \( \pi_1 \partial H(x) \times \pi_2 \partial H(x) \) (see [3]). The following theorem is an extension of the result of Benci-Hofer-Rabinowitz [5].

**Theorem 2.** — Under the assumptions of theorem 1, if there is a function \( K \in C^1(\mathbb{R}^{2N}, \mathbb{R}) \) and constants \( a, b > 0 \) with \( a + b > 0 \) such that

\[
a(\pi_1 x, \pi_1 \xi) + b(\pi_2 x, \pi_2 \xi) + \langle K'(x), J\xi \rangle > 0,
\quad \forall x \in \Sigma_c, \xi \in \partial H(x)
\]

then (1) has a periodic solution on \( \Sigma_c \).

**Proof.** — We use the notations used in the proof of theorem 1 and assume \( c = 1 \). By the upper semicontinuity of \( \partial H \) and the compactness of \( \Sigma_c \), for \( s > 0 \) small, there is a constant \( \gamma > 0 \) such that

\[
a(\pi_1 x, \pi_1 \xi) + b(\pi_2 x, \pi_2 \xi) + \langle K'(x), J\xi \rangle > \gamma,
\quad \forall x \in U, \xi \in \partial H(x)
\]

where \( U = \psi((-s, s) \times \Sigma_1) \).

Let \( z \) be a conservative \( T \)-periodic solution of (1) in \( U \). Setting \( \xi(t) = -Jz(t) \), then \( \xi(t) \in \partial H(z(t)) \) a.e. and

\[
A(z) := \frac{1}{2} \int_0^T \langle -Jz, z \rangle \, dt = \int_0^T \langle \pi_1 z, \pi_1 \xi \rangle \, dt = \int_0^T \langle \pi_2 z, \pi_2 \xi \rangle \, dt.
\]

Noting that

\[
\int_0^T \langle K'(z), J\xi \rangle \, dt = \int_0^T \langle K'(z), \dot{z} \rangle \, dt = 0,
\]

integrating for (9) over \( [0, T] \) gives

\[
(a + b)A(z) \geq \gamma T.
\]

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We now take a sequence \( \delta_n \to 0 \) with \( 0 < \delta_n < s/2 \). By theorem 1, for each \( n \), (1) has a conservative \( T_n \)-periodic solution \( z_n \) in \( U \) such that \( A(z_n) \leq d \) and \( |H(z_n(t)) - 1| < \delta_n \). From (10) it follows that \( \{T_n : n = 1, 2, 3, \ldots\} \) is bounded. It is easy to see that \( \{z_n\} \) has a subsequence which converges uniformly to a conservative \( T \)-periodic solution \( z \) of (1) and \( z(t) \in \Sigma_1, \forall t \).

The proof is complete.

**Corollary 1.** Suppose that \( H \in C^{1-0}(\mathbb{R}^2, \mathbb{R}), c \in \mathbb{R} \) and \( \Sigma_c = H^{-1}(c) \) is compact. If

\[
\langle x, \xi \rangle > 0 \quad \text{for} \quad x \in \Sigma_c \text{ and } \xi \in \partial H(x),
\]

then (1) has a periodic solution on \( \Sigma_c \).

**Proof.** Note that (11) implies (2). Hence all assumptions of theorem 1 are satisfied. Taking \( a = b = 1 \) and \( K = 0 \) gives (8). Corollary 1 follows from theorem 2.

**Corollary 2.** Suppose that \( H \in C^{1-0}(\mathbb{R}^2, \mathbb{R}), c \in \mathbb{R} \) and \( \Sigma_c = H^{-1}(c) \) is compact. If

\[
(p_1) \langle \pi_1 x, \pi_1 \xi \rangle > 0 \quad \text{for} \quad x \in \Sigma_c \text{ with } \pi_1 x \neq 0 \text{ and } \xi \in \partial H(x),
\]

\[
(p_2) 0 \notin \pi_2 \partial H(x) \quad \text{for} \quad x \in \Sigma_c \text{ with } \pi_1 x = 0,
\]

then (1) has a periodic solution on \( \Sigma_c \).

**Proof.** It is clear that \( (p_1) \) and \( (p_2) \) imply (2). By the upper semicontinuity of \( \partial H \) and the compactness of \( \Sigma_c \) there is a bounded neighborhood \( U \) of \( \Sigma_c \) such that \( (p_1) \) and \( (p_2) \) are also true if \( \Sigma_c \) is replaced by \( U \). Applying the acute angle approximation theorem (see e.g. [7]) for the multivalued map \( \pi_2 \partial H : \mathbb{R}^2 \to 2^{\mathbb{R}^N} \), it is not difficult to construct a map \( W \in C^1(\mathbb{R}^2, \mathbb{R}^N) \) such that

\[
\langle W(x), \pi_2 \xi \rangle > 0 \quad \text{for} \quad x \in U \text{ with } \pi_1 x = 0 \text{ and } \xi \in \partial H(x).
\]

Set \( K(x) = \langle -W(x), \pi_1 x \rangle \) for \( x \in \mathbb{R}^2 \). Then \( K \in C^1(\mathbb{R}^2, \mathbb{R}) \) and

\[
\langle K'(x), J \xi \rangle = \langle -W'(x) \cdot J \xi, \pi_1 x \rangle + \langle W(x), \pi_2 \xi \rangle
\]

for \( x \in \mathbb{R}^2 \) and \( \xi \in \partial H(x) \).
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It is easy to see that there are constants $\sigma, \gamma > 0$ such that

$$\langle W(x), \pi_2 \xi \rangle \geq 2\gamma \quad \text{and} \quad |\langle W'(x) \cdot J\xi, \pi_1 x \rangle| \leq \gamma$$

for $x \in U$ with $|\pi_1 x| \leq \sigma$, and $\xi \in \partial H(x)$. Let

$$M = \sup \left\{ \langle K'(x), J\xi \rangle \mid x \in U, \xi \in \partial H(x) \right\},$$

$$m = \inf \left\{ \langle \pi_1 x, \pi_1 \xi \rangle \mid x \in U \text{ with } |\pi_1 x| \geq \sigma, \xi \in \partial H(x) \right\}.$$

Set $a = (M + \gamma)/m$ and $b = 0$. Then for $x \in U$ and $\xi \in \partial H(x)$ we have

$$a\langle \pi_1 x, \pi_1 \xi \rangle + \langle K'(x), J\xi \rangle \geq 0 + 2\gamma - \gamma = \gamma - 0 \text{ if } |\pi_1 x| \leq \sigma,$$

$$a\langle \pi_1 x, \pi_1 \xi \rangle + \langle K'(x), J\xi \rangle \geq M + \gamma - M = \gamma > 0 \text{ if } |\pi_1 x| \geq \sigma.$$

Thus (8) holds and corollary 2 follows from theorem 2.

Remark. — When $H \in C^1$, (2) and $(p_1)$ imply $(p_2)$ (see [5]), but such conclusion is not true when $H \in C^{1-0}$.

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