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A Viterbo-Hofer-Zehnder Type Result for Hamiltonian Inclusions

XIANLING FAN[1]

RÉSUMÉ. — On obtient un résultat de type de Viterbo-Hofer-Zehnder pour les inclusions hamiltoniennes. Soit $H : \mathbb{R}^{2N} \to \mathbb{R}$ une fonction locale lipschitzienne et $c \in \mathbb{R}$. Supposons que $\Sigma := \{x \in \mathbb{R}^{2N} \mid H(x) = c\}$ soit un ensemble partiel compact et non vide de $\mathbb{R}^{2N}$ et $0 \notin \partial H(x)$ pour $x \in \Sigma$. Donc, pour aucun $\delta > 0$ l'inclusion hamiltonienne $\dot{x} \in J\partial H(x)$ a une solution conservatrice et périodique $x(t)$ de façon que $H(x(t)) \equiv c' \in (c - \delta\, ,\, c + \delta)$ pour tout t.

ABSTRACT. — We obtain a Viterbo-Hofer-Zehnder type result for Hamiltonian inclusions. Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be a locally Lipschitz function and $c \in \mathbb{R}$. Suppose that $\Sigma := \{x \in \mathbb{R}^{2N} \mid H(x) = c\}$ is a nonempty compact subset of $\mathbb{R}^{2N}$ and $0 \notin \partial H(x)$ for $x \in \Sigma$. Then for any $\delta > 0$ the Hamiltonian inclusion $\dot{x} \in J\partial H(x)$ has a conservative periodic solution $x(t)$ such that $H(x(t)) \equiv c' \in (c - \delta\, ,\, c + \delta)$ for all t.

1. Introduction and Main Result

Hofer and Zehnder [1] extended the result of Viterbo [2]. The aim of the present paper is to extend the result of [1] to the case of Hamiltonian inclusions.

Let $H : \mathbb{R}^{2N} \to \mathbb{R}$ be locally Lipschitz continuous, which is written as $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$. Consider the Hamiltonian inclusion.

$$\dot{x} \in J\partial H(x) \tag{1}$$

where ∂H is Clarke's generalized gradient of H and J is the standard $2N \times 2N$ symplectic matrix (see [3]). By a solution of (1) we mean an

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absolutely continuous function $x(t)$ satisfying (1) for almost all t. It is well-known that, if H is regular, then any solution of (1) is conservative, i.e. $H(x(t)) \equiv$ constant. However, in general, if H is not regular, then a solution of (1) need not be conservative.

Our main result is the following

THEOREM 1.— *Let $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$ and $c \in \mathbb{R}$. Suppose that $\Sigma_c = H^{-1}(c)$ is a nonempty compact subset of $\mathbb{R}^{2N}$ and*

$$0 \notin \partial H(x) \quad \text{for} \quad x \in \Sigma_c . \tag{2}$$

Then for any bounded neighborhood Ω of Σ_c, there are positive constants β and d such that for any $\delta > 0$, (1) has a $T = T(\delta)$-periodic conservative solution $x(t)$ in Ω such that $H(x(t)) \equiv c' \in (c - \delta, c + \delta)$ and

$$\beta \leq \frac{1}{2} \int_0^T \langle -J\dot{x}, x\rangle \, \mathrm{d}t \leq d . \tag{3}$$

The following results obtained by the author [4] will be used in the proof of theorem 1.

PROPOSITION 1 ([4]).— *Let Ω be an open subset of $\mathbb{R}^k$ and $H \in C^{1-0}(\Omega, \mathbb{R})$. Then for any continuous function $\epsilon : \Omega \to (0, +\infty)$ there is a C^∞-function $g : \Omega \to \mathbb{R}$ such that*

i) $|g(x) - H(x)| \leq \epsilon(x)$ *for* $x \in \Omega$,

ii) $\forall\, x \in \Omega$, $\exists\, y \in \Omega$ *and* $\xi \in \partial H(y)$ *such that* $|x - y| \leq \epsilon(x)$ *and* $|g'(x) - \xi| \leq \epsilon(x)$.

A C^1-function $g : \Omega \to \mathbb{R}$ satisfying the condition i) and ii) in proposition 1 is called an $\epsilon(x)$-admissible approximation for H on Ω. In particular, when $\epsilon(x) \equiv \epsilon$, g is called an ϵ-admissible approximation for H on Ω.

PROPOSITION 2 ([4]).— *Let Ω be an open subset of $\mathbb{R}^{2N}$, $H \in C^{1-0}(\Omega, \mathbb{R})$ and $\epsilon_n \to 0$ $(n \to \infty)$ with $\epsilon_n > 0$. Suppose that for each n, $H_n \in C^1(\Omega, \mathbb{R})$ is an ϵ_n-admissible approximation for H on Ω and x_n is a T_n-periodic solution of the Hamiltonian system*

$$\dot{x} = J H_n'(x) . \tag{4}$$

If

i) $\{T_n \mid n = 1, 2, \ldots\}$ *is bounded,*

ii) $\{x_n(t) \mid t \in \mathbb{R}\,,\ n = 1, 2, \ldots\}$ *is contained in a compact subset of* Ω, *then* $\{x_n\}$ *has a subsequence* $\{x_{n_K}\}$ *which converges uniformly to a* T*-periodic solution* x *of (1) with* $T = \lim T_{n_K}$ *and*

$$H\big(x(t)\big) \equiv c = \lim H_{n_K}\big(x_{n_K}(t)\big)\,.$$

In section 2 we give the proof of theorem 1. In section 3 we extend the a priori bound criterion of Benci-Hofer-Rabinowitz [5] to the case of Hamiltonian inclusions.

2. Proof of theorem 1

Without loss of generality we may assume that $c = 1$ and Σ_1 is connected.

Let Ω, a bounded neighborhood of Σ_1, be given. By the upper semicontinuity of H, the compactness of Σ_1 and the condition (2), we may choose a bounded neighborhood V of Σ_1 such that $\overline{V} \subset \Omega$ and $0 \notin \partial H(x)$ for $x \in V$. Then there are positive constants m and M such that $m < |\xi| < M$ for $\xi \in \partial H(V)$. Using the pseudo-gradient flow (see [6]) we can construct a Lipschitz homeomorphism $\psi : (-s, s) \times \Sigma_1 \to V$ such that

$$H\big(\psi(t, x)\big) = 1 + t \quad \text{for} \quad (t, x) \in (-s, s) \times \Sigma_1\,.$$

Set

$$U = \psi\big((-s, s) \times \Sigma_1\big)\,, \quad D = \text{diam}\, U\,,\ \Sigma_c = \big(H\big|_U\big)^{-1}(c)\,.$$

We fix positive numbers r, b, such that

$$D < r < 2D\,, \quad \frac{3}{2}\,\pi r^2 < b < 2\pi r^2\,.$$

Take a sequence $\epsilon_n \to 0$ such that $0 < \epsilon_n < \min\{s/3\,, m/3\}$ for all n. By proposition 1, for each n, there is an ϵ_n-admissible approximation H_n for H on U and $H_n \in C^\infty(U, \mathbb{R})$. Then we have

$$\begin{cases} \big|H_n(x) - H(x)\big| \le \dfrac{s}{3} & \text{for } x \in U \text{ and all } n, \\ \dfrac{2}{3}\,m < \big|H_n'(x)\big| < M + \dfrac{m}{3} & \text{for } x \in U \text{ and all } n, \end{cases}$$

For each n let ψ_n be the flow in U generated by

$$\dot{x} = -\frac{H_n'(x)}{\left|H_n'(x)\right|^2}, \quad x(0) \in U .$$

Set $\Sigma_{1,n} = H_n^{-1}(1)$. It is easy to see that $\psi_n\left(\left[-{}^s/_2, {}^s/_2\right] \times \Sigma_{1,n}\right) \subset U$ and

$$H_n\big(\psi_n(t,x)\big) = 1 + t \quad \text{for} \quad (t,x) \in \left[-\frac{s}{2}, \frac{s}{2}\right] \times \Sigma_{1,n} .$$

LEMMA 1. — *For each n, $\Sigma_{1,n}$ is a connected compact hypersurface in U.*

Proof. — It suffices to prove the connectedness of $\Sigma_{1,n}$. For fixed n let $x_1, x_2 \in \Sigma_{1,n}$. Then there are $-t_1 < 0$ and $-t_2 < 0$ such that

$$\psi_n(-t_1, x_1) = y_1 \in \Sigma_{1+s/2} \quad \text{and} \quad \psi_n(-t_2, x_2) = y_2 \in \Sigma_{1+s/2} .$$

Note that $\Sigma_{1+s/2}$ is connected since $\Sigma_{1+s/2}$ is homeomorphic to Σ_1. Let p be a path in $\Sigma_{1+s/2}$ joining y_1 to y_2. It is easy to see that along the descent flow lines of ψ_n, p can be deformed to a path in $\Sigma_{1,n}$ joining x_1 to x_2. So $\Sigma_{1,n}$ is connected and the proof of lemma 1 is complete.

Set $U_n = \psi_n\left(\left(-{}^s/_2, {}^s/_2\right) \times \Sigma_{1,n}\right)$. Then $\psi_n : \left(-{}^s/_2, {}^s/_2\right) \times \Sigma_{1,n} \to U_n \subset U$ is a diffeomorphism. We denote by A_n and B_n the unbounded and bounded component of $\mathbb{R}^{2N} \setminus U_n$ respectively and by B the bounded component of $\mathbb{R}^{2N} \setminus U$. We may assume that $0 \in B$, then $0 \in B_n$ since $B \subset B_n$ for all n.

Let $\delta > 0$ be given. We may assume $\delta < {}^s/_2$.

Following [1], we pick a C^∞-function $f : \left(-{}^s/_2, {}^s/_2\right) \to \mathbb{R}$ satisfying

$$f\big|_{(-s/2,\,-\delta]} = 0, \quad f\big|_{[\delta,\,s/2)} = b \quad \text{and} \quad f'(t) > 0 \text{ for } -\delta < t < \delta .$$

Choose a C^∞-function $g : (0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} g(t) = b & \text{for } t \le r, \\ g(t) = \dfrac{3}{2}\pi t^2 & \text{for } t \text{ large}, \\ g(t) \ge \dfrac{3}{2}\pi t^2 & \text{for } t > r, \\ 0 < g'(t) \le 3\pi t & \text{for } t > r. \end{cases}$$

For each n define a C^∞-function $G_n : \mathbb{R}^{2N} \to \mathbb{R}$ by

$$G_n(x) = \begin{cases} 0 & \text{if } x \in B_n \\ f(t) & \text{if } x \in \psi_n(t \times \Sigma_{1,n}),\ -\delta \le t \le \delta \\ b & \text{if } x \in A_n \text{ and } |x| \le r \\ g(|x|) & \text{if } |x| > r. \end{cases}$$

Then, by [1], for each n the Hamiltonian system

$$\dot{x} = J G_n'(x) \tag{5}$$

has a 1-periodic solution x_n in U_n such that

$$H_n\big(x_n(t)\big) = c_n \in (1+\delta\,,\,1-\delta) \quad \text{for all } t$$

and

$$\beta \le \frac{1}{2}\int_0^1 \langle -J\dot{x}_n\,,\,x_n\rangle\,\mathrm{d}t \le d\,,$$

where β and $d = 16\,\pi D^2$ are positive constants independent of n and δ.

By the definition of G_n we have

$$G_n(x) = f\big(H_n(x)-1\big) \quad \text{and} \quad G_n'(x) = f'\big(H_n(x)-1\big)H_n'(x)$$

for $x \in \big(H_n\big|_{U_n}\big)^{-1}\big((1-\delta\,,\,1+\delta)\big)$.

Set $z_n(t) = x_n\big(f'(c_n-1)t\big)$. Then z_n is a T_n-periodic solution in U_n of the Hamiltonian system

$$\dot{z} = J H_n'(z) \tag{6}$$

with $T_n = f'(c_n-1)$ and

$$\beta \le \frac{1}{2}\int_0^{T_n} \langle -J\dot{z}_n\,,\,z_n\rangle\,\mathrm{d}t \le d\,. \tag{7}$$

From the fact that $|c_n - 1| < \delta$ and f' is bounded on $(-\delta\,,\,\delta)$ it follows that $\{T_n \mid n = 1, 2, \ldots\}$ is bounded. Noting that

$$U_n \subset \left\{x \in U \;\middle|\; 1-\frac{5}{6}s \le H(x) \le 1+\frac{5}{6}s\right\} \subset U\,,$$

from proposition 2 it follows that $\{z_n\}$ has a subsequence $\{z_{n_K}\}$ which converges uniformly to a conservative T-periodic solution z of (1) such that

$$T = \lim T_{n_K}\,, \quad H\big(z(t)\big) = \bar{c} = \lim c_{n_K} \in [\,1-\delta\,,\,1+\delta\,] \text{ and } z(t) \in U\,,\ \forall\, t\,.$$

(3) follows from (7). The proof of theorem 1 is complete. □

3. A criterion for a priori bounds

For $x \in \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$, set $x = (p, q) = (\pi_1 x, \pi_2 x)$. Note that in general neither of the sets $\partial_p H(x) \times \partial_q H(x)$ and $\partial H(x)$ need be contained in the other, but both of them are contained in $\pi_1 \partial H(x) \times \pi_2 \partial H(x)$ (see [3]). The following theorem is an extension of the result of Benci-Hofer-Rabinowitz [5].

THEOREM 2.— *Under the assumptions of theorem 1, if there is a function $K \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ and constants a, $b \geq 0$ with $a + b > 0$ such that*

$$a\langle \pi_1 x , \pi_1 \xi \rangle + b\langle \pi_2 x , \pi_2 \xi \rangle + \big\langle K'(x) , J\xi \big\rangle > 0 , \quad \forall\, x \in \Sigma_c , \ \xi \in \partial H(x) \tag{8}$$

then (1) has a periodic solution on Σ_c.

Proof.— We use the notations used in the proof of theorem 1 and assume $c = 1$. By the upper semicontinuity of ∂H and the compactness of Σ_c, for $s > 0$ small, there is a constant $\gamma > 0$ such that

$$a\langle \pi_1 x , \pi_1 \xi \rangle + b\langle \pi_2 x , \pi_2 \xi \rangle + \big\langle K'(x) , J\xi \big\rangle > \gamma , \quad \forall\, x \in U , \ \xi \in \partial H(x) \tag{9}$$

where $U = \psi\big((-s, s) \times \Sigma_1\big)$.

Let z be a conservative T-periodic solution of (1) in U. Setting $\xi(t) = -J\dot z(t)$, then $\xi(t) \in \partial H\big(z(t)\big)$ a.e. and

$$A(z) := \frac{1}{2} \int_0^T \langle -J\dot z , z \rangle \, \mathrm{d}t = \int_0^T \langle \pi_1 z , \pi_1 \xi \rangle \, \mathrm{d}t = \int_0^T \langle \pi_2 z , \pi_2 \xi \rangle \, \mathrm{d}t .$$

Noting that

$$\int_0^T \langle K'(z) , J\xi \rangle \, \mathrm{d}t = \int_0^T \langle K'(z) , \dot z \rangle \, \mathrm{d}t = 0 ,$$

integrating for (9) over $[\,0 , T\,]$ gives

$$(a + b) A(z) \geq \gamma T . \tag{10}$$

We now take a sequence $\delta_n \to 0$ with $0 < \delta_n < s/2$. By theorem 1, for each n, (1) has a conservative T_n-periodic solution z_n in U such that $A(z_n) \leq d$ and $|H(z_n(t)) - 1| < \delta_n$. From (10) it follows that $\{T_n \mid n = 1, 2, 3, \dots\}$ is bounded. It is easy to see that $\{z_n\}$ has a subsequence which converges uniformly to a conservative T-periodic solution z of (1) and $z(t) \in \Sigma_1$, $\forall\, t$.

The proof is complete.

COROLLARY 1.— *Suppose that $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$, $c \in \mathbb{R}$ and $\Sigma_c = H^{-1}(c)$ is compact. If*

$$\langle x, \xi \rangle > 0 \quad \textit{for} \quad x \in \Sigma_c \textit{ and } \xi \in \partial H(x), \tag{11}$$

then (1) has a periodic solution on Σ_c.

Proof.— Note that (11) implies (2). Hence all assumptions of theorem 1 are satisfied. Taking $a = b = 1$ and $K = 0$ gives (8). Corollary 1 follows from theorem 2.

COROLLARY 2.— *Suppose that $H \in C^{1-0}(\mathbb{R}^{2N}, \mathbb{R})$, $c \in \mathbb{R}$ and $\Sigma_c = H^{-1}(c)$ is compact. If*

(p_1) $\langle \pi_1 x, \pi_1 \xi \rangle > 0$ *for $x \in \Sigma_c$ with $\pi_1 x \neq 0$ and $\xi \in \partial H(x)$,*

(p_2) $0 \notin \pi_2 \partial H(x)$ *for $x \in \Sigma_c$ with $\pi_1 x = 0$,*

then (1) has a periodic solution on Σ_c.

Proof.— It is clear that (p_1) and (p_2) imply (2). By the upper semicontinuity of ∂H and the compactness of Σ_c there is a bounded neighborhood U of Σ_c such that (p_1) and (p_2) are also true if Σ_c is replaced by U. Applying the acute angle approximation theorem (see e.g. [7]) for the multivalued map $\pi_2 \partial H : \mathbb{R}^{2N} \to 2^{\mathbb{R}^N}$, it is not difficult to construct a map $W \in C^1(\mathbb{R}^{2N}, \mathbb{R}^N)$ such that

$$\langle W(x), \pi_2 \xi \rangle > 0 \quad \text{for} \quad x \in U \text{ with } \pi_1 x = 0 \text{ and } \xi \in \partial H(x).$$

Set $K(x) = \langle -W(x), \pi_1 x \rangle$ for $x \in \mathbb{R}^{2N}$. Then $K \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ and

$$\langle K'(x), J\xi \rangle = \langle -W'(x) \cdot J\xi, \pi_1 x \rangle + \langle W(x), \pi_2 \xi \rangle$$

for $x \in \mathbb{R}^{2N}$ and $\xi \in \partial H(x)$.

It is easy to see that there are constants σ, $\gamma > 0$ such that

$$\langle W(x), \pi_2\xi\rangle \geq 2\gamma \quad \text{and} \quad \left|\langle W'(x)\cdot J\xi, \pi_1 x\rangle\right| \leq \gamma$$

for $x \in U$ with $|\pi_1 x| \leq \sigma$, and $\xi \in \partial H(x)$. Let

$$M = \sup\left\{\langle K'(x), J\xi\rangle \;\middle|\; x \in U, \xi \in \partial H(x)\right\},$$
$$m = \inf\left\{\langle \pi_1 x, \pi_1\xi\rangle \mid x \in U \text{ with } |\pi_1 x| \geq \sigma, \xi \in \partial H(x)\right\}.$$

Set $a = (M+\gamma)/m$ and $b = 0$. Then for $x \in U$ and $\xi \in \partial H(x)$ we have

$$a\langle \pi_1 x, \pi_1\xi\rangle + \langle K'(x), J\xi\rangle \geq 0 + 2\gamma - \gamma = \gamma - 0 \text{ if } |\pi_1 x| \leq \sigma,$$
$$a\langle \pi_1 x, \pi_1\xi\rangle + \langle K'(x), J\xi\rangle \geq M + \gamma - M = \gamma > 0 \text{ if } |\pi_1 x| \geq \sigma.$$

Thus (8) holds and corollary 2 follows from theorem 2.

Remark.— When $H \in C^1$, (2) and (p_1) imply (p_2) (see [5]), but such conclusion is not true when $H \in C^{1-0}$.

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