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Introduction

Let $X$ a complex space. By $F_c(X)$ we denote the hyperspace of non-empty compact subsets of $X$.

As in [8] we say that an upper semi-continuous multivalued function $K : X \to F_c(Y)$, where $X$ and $Y$ are complex spaces, is analytic if for every open subset $W$ of $X$ and every plurisubharmonic function $\psi$ on a neighbourhood of $\Gamma_K|_W$, the graph of $K$ on $W$, the function

$$\varphi(x) = \sup\{\psi(x, y) \mid y \in K(x)\}$$

is plurisubharmonic on $W$.

Analytic multivalued functions (for short: A.M.V. functions) have been investigated by several authors, in particular by Slodkowski [8, 9] and Ransford [5, 6, 7].

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In [7], Ransford has proved that every A.M.V. function

$$K : D \rightarrow F_c(V),$$

where $D = \{z \in \mathbb{C} \mid |z| < 1\}$, $D^* = D \setminus \{0\}$ and $V$ is either $D$ or $D_{rs} = \{z \in \mathbb{C} \mid r < |z| < s\}$, $0 < r < s$, can be extended analytically to $D$.

This note considers a removable-singularity result for A.M.V. functions. Moreover, the equivalence between a Liouville-type property and extendibility of A.M.V. functions is proved.

1. Removable-singularities for analytic multivalued functions

An A.M.V. function $K : G \rightarrow F_c(Y)$ is said to be locally compact if for every $x \in X$ there exists a neighbourhood $U$ of $x$ such that $K(U \cap G)$ is relatively compact in $Y$, where $G$ is an open subset of $X$.

**Theorem 1.1.** — Let $G$ be an open set in $\mathbb{C}^n$, $S$ a closed subset of $G$, $Y$ a Stein space. Then every A.M.V. function $K : G \setminus S \rightarrow F_c(Y)$ can be extended analytically to $G$ if one of the following conditions is satisfied

a) $S = H \cap (G \setminus U)$, where $H$ is an analytic set in $G$, $U$ is an open subset of $G$ such that $U$ meets every component of $H$;
b) $S$ is a set of zero $(2n - 2)$-Hausdorff measure in $G$;
c) $S$ is a pluripolar set in $G$ and $K$ is locally compact.

We first need the following, which is a generalization of the important result of Wermer [10].

**Lemma 1.2.** — Let $A$ be a uniform algebra with Shilov boundary $\partial_A^0$ and $U$ an open subset of $\mathbb{C}$. Let $h : U \rightarrow A$ be a holomorphic map. Then for every $f \in A$ such that $\sigma(f) \setminus f(\partial_A^0) \subset U$, where $\sigma(f)$ is the spectrum of $f$, the form

$$K(\lambda) = \{\hat{h}(\lambda, w) = \overline{h(\lambda)}(w) \mid w \in \hat{f}^{-1}(\lambda)\}$$

defines an A.M.V. function on $\sigma(f) \setminus f(\partial_A^0)$.

**Proof.** — This is basically Slodkowski’s argument [8]. It is enough to show that $K(\lambda)$ satisfies condition (ii) of [8, theorem 3], i.e. for every
polynomial \( p(\lambda) \) and for every \( a, b \in \mathbb{C} \) the function \( \lambda \rightarrow \max |f_\lambda(K(\lambda))| \), where \( f_\lambda(z) = (z - \lambda a - b)^{-1} \exp(p(\lambda)) \), has local maximum property in \( G = \{ \lambda \in \sigma(f) \setminus \tilde{f}(\partial_0^A) \mid \alpha \in \sigma(f) \not\in K(\lambda) \} \). Let \( D \) be a disc such that \( \text{cl}D \subset G \). Put \( N = \tilde{f}^{-1}(D) \subset M_A \), where \( M_A \) is maximal ideal space of \( A \), and let \( B \) denote the uniform closure of \( A|_{\text{cl}N} \) on \( \text{cl}N \) and the form \( k = (h(y) - af(b) - b)^{-1} \exp(p(f)) \), where \( a, b \in \mathbb{C} \) and \( p \) is a polynomial, defines an element of \( B \). Denote
\[
f_\lambda(z) = (z - \lambda a - b)^{-1} \exp(p(\lambda)).
\]

For \( \lambda \in D, \) we have
\[
\max f_\lambda(K(\lambda)) = \max |\tilde{k}\tilde{f}^{-1}(\lambda)| \\
\leq \max |\tilde{k}|_N \text{(by Rossi's local maximum principle)} \\
\leq \max \{ \max |\tilde{k}(\tilde{f}^{-1}(\lambda))| \mid \lambda \in \partial D \} \\
= \max \{ \max |f_\lambda(K(\lambda))| \mid \lambda \in \partial D \}.
\]
Thus the function \( \lambda \rightarrow \max |f_\lambda(K(\lambda))| \) has the local maximum property.

The lemma is proved. \( \square \)

**LEMMA 1.3 (Slodkowski's theorem [9]).** — Let \( G \) be a bounded planar domain and \( K : G \rightarrow F_c(\mathbb{C}^k) \) be an A.M.V. function such that \( \sup \max_{x \in G} |K(x)| < \infty \). Then there exists a uniform algebra \( A \) and functions \( f, g_1, \ldots, g_k \in A \) such that
\[
i) \quad \tilde{f}(M_A) \setminus \tilde{f}(\partial_0^A) = G, \text{ where } \tilde{f} \text{ denotes the Gelfand transformation of } f, M_A \text{ and } \partial_0^A \text{ are the maximal ideal space and the Shilov boundary respectively of } A.
\]
\[
ii) \quad \tilde{g}(\tilde{f}^{-1}(x)) = K(x) \text{ for every } x \in G, \text{ where } \tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_k).
\]

**LEMMA 1.4.** — Let \( K : G \rightarrow F_c(Y) \) be an upper semi-continuous multivalued function, where \( G \) is an open subset of \( \mathbb{C}^n \) and \( Y \) an analytic set in \( \mathbb{C}^k \). If \( K : F \rightarrow F_c(\mathbb{C}^k) \) is analytic, then \( K : G \rightarrow F_c(Y) \) is also analytic.

**Proof.** — We can assume that \( n = 1 \). Given \( \varphi \) a plurisubharmonic function on a neighborhood \( W \) of \( \Gamma_K|_U \), where \( U \) is an open subset of \( G \), consider the plurisubharmonic function \( \tilde{\varphi}(z, w) = \varphi(z, \tilde{g}(w)) \) on
For all $(z, w) \in (\text{id} \times \tilde{g})^{-1}(W)$, where $h_j^n$ are holomorphic maps from $U$ into $A$.

Since $(\text{id} \times \tilde{g})$ is continuous and $W$ is open, it implies that
\[
\frac{\partial(\text{id} \times \tilde{g})^{-1}(W) \cup (\text{id} \times \tilde{g})^{-1}(W)}{(\text{id} \times \tilde{g})^{-1}(\partial W)} \subset (\text{id} \times \tilde{g})^{-1}(\partial W).
\]

By lemma 1.2, the multivalued function
\[
L(z) = \{\tilde{h}_j^n(z, w) \mid w \in \hat{f}^{-1}(z)\}
\]
is analytic on $\sigma(f) \setminus \hat{f}(\partial A)$. On the other hand $\hat{f}^{-1}(\partial G) \supset \partial A$, by Rossi's local maximum principle we have
\[
\max|\tilde{h}_j^n(z, w)|_{\partial(\text{id} \times \tilde{g})^{-1}(W)} = \max|\tilde{h}_j^n(z, w)|_{(\text{id} \times \tilde{g})^{-1}(\partial W)}.
\]

Since for every sequence of upper semi-continuous function $\psi_n$, $\psi = \lim \psi_n$ point-wise, $\lim \max(\psi_n|_F) = \max(\psi|_F)$ on every compact subset $F$ [8], and since $(\text{id} \times \tilde{g})^{-1}(\partial W) \subset (\text{id} \times \tilde{g})^{-1}(W)$, it follows that the function $\gamma$ given by
\[
\gamma(z) = \max\{\varphi(z, y) \mid y \in K(z) = \tilde{g}\hat{f}^{-1}(z)\}
\]
is plurisubharmonic on $U$. Hence the multivalued function $K : G \to F_c(Y)$ is analytic.

**Proof of theorem 1.1**

Without loss of generality we may assume that $Y$ is an analytic set in $\mathbb{C}^k$. Then the function
\[
\theta(x) = \sup\{||y|| \mid y \in K(x)\}
\]
is plurisubharmonic on $G_0 = G \setminus S$, where $S$ satisfies one of the conditions a) or b) or c) of the theorem. By [4], $\theta$ can be extended to a plurisubharmonic function on $C$. This implies that for every $x_0 \in S$ there exists a
neighbourhood $U$ of $x_0$ such that $K(U \cap G_0)$ is relatively compact. Define a upper semi-continuous extension of $K$ by

$$
\hat{K}(x) = \begin{cases} 
K(x) & \text{for } x \in G_0 \\
y \in Y \mid \exists \{(x_n, y_n)\} \subset \Gamma_K, (x_n, y_n) \to (x, y) & \text{for } x \in S.
\end{cases}
$$

We prove that $\hat{K}$ is analytic at every $x_0 \in S$. Let $G'$ be an open ball around $x_0$, $G' \subset G$. It suffices to show that $\hat{K}|_{L \cap G'}$ is analytic for every complex line $L$ in $\mathbb{C}^n$. Using the Slodkowski theorem we can find a uniform algebra $A$ and $f, g_1, \ldots, g_k \in A$ such that

i) $\hat{g}_f^{-1}(x) = \hat{K}(x)$ for all $x \in L \cap (G' \setminus S)$;

ii) $f(\partial_A^0) = \partial(L \cap (G' \setminus S))$.

We have to prove that $f(\partial_A^0) \cap (L \setminus G') = \emptyset$.

Suppose the contrary. Then there exists a complex line $L$ in $\mathbb{C}^n$ such that $f(\partial_A^0) \cap (L \cap G') \neq \emptyset$. Since $\hat{K}$ is analytic on $G' \setminus S$, it follows that $\hat{f}(\partial_A^0) \cap (L \cap (G' \setminus S)) = \emptyset$. Hence there exists $w_0 \in \partial_A^0$ such that $\hat{f}(w_0) = x_0$. Since $G'$ is open and set of peak points of $A$ is dense in $\partial_A^0$, we may assume that $w_0$ is a peak point. Hence there exists $h \in A$ such that $|\hat{h}(w_0)| = 1$ and $|\hat{h}(w)| < 1$ for $w \in M_A \setminus \{w_0\}$.

Consider the plurisubharmonic function

$$
\varphi(x) = \log \max |\hat{h}f^{-1}(x)| \quad \text{on} \quad G' \setminus S.
$$

Then $\varphi$ is plurisubharmonic on $G' \cap L$. Since

$$
\log \max |\hat{h}f^{-1}(x)| \leq 0 = \log \max |\hat{h}f^{-1}(x_0)|
$$

for every $x \in G'$, it follows that $\varphi = \text{constant}$, which is impossible.

Thus $f(\partial_A^0) \cap (G' \cap L) = \emptyset$.

Theorem 1.1 is proved. $\square$

2. Liouville-type property for analytic multivalued functions

In the section we study the relation between a Liouville-type property and removable singularities of A.M.V. functions with values in convex domains.
THEOREM 2.1. — Let $D$ be a convex domain in $\mathbb{C}^n$. Then the following conditions are equivalent:

a) for every A.M.V. function $K : \mathbb{C} \to F_c(D)$, the multivalued function $	ilde{K} : \mathbb{C} \to F_c(D)$ given by $	ilde{K}(x) = K(x)$, where $K(x)$ is polynomial convex hull of $K(x)$, is constant;

b) every A.M.V. function $K : \Delta^* \to F_c(D)$ can be extended analytically on $\Delta$, where $\Delta$ is the unit disc, $\Delta^* = \Delta \setminus \{0\}$;

c) every A.M.V. function $L : \Delta \setminus S \to F_c(D)$ can be extended analytically on $\Delta$, where $S$ is a polar set in $\Delta$.

To prove the theorem we shall use the hyperboliticity of convex domains. In [1] Bath proved that a convex domain $D$ is hyperbolic if and only if $D$ does not contain complex lines (i.e. every holomorphic map $h : \mathbb{C} \to D$ is constant).

Proof of theorem 2.1

Consider the condition:

$$D \text{ is hyperbolic} \quad (1)$$

We shall prove that a) $\iff$ (1) $\implies$ c) $\implies$ b) $\implies$ (1).

We first write

$$D = \bigcap_{\alpha \in I} \{ \text{Re } x_\alpha^* < \varepsilon_\alpha \},$$

where $\{x_\alpha^*\}$ are linear forms on $\mathbb{C}^n$. Without loss of generality we may assume that $0 \in D$. Then $\varepsilon_\alpha > 0$ for all $\alpha$.

Let $\{x_{\alpha_1}^*, \ldots, x_{\alpha_p}^*\}$ be a maximal linearly independent system of $\{x_\alpha^*\}$. Take $\theta_\alpha : H_\alpha \to \Delta$, where $H_\alpha = \{z \in \mathbb{C} : \text{Re } z < \varepsilon_\alpha\}$, is a biholomorphism. Define a holomorphic map

$$\gamma : D_1 \to \Delta^p, \quad \text{where} \quad D_1 = \bigcap_{j=1}^p \{ \text{Re } x_{\alpha_j}^* \}$$

by

$$\gamma(x) = \left( \theta_{\alpha_1}(x_{\alpha_1}^*(x)), \ldots, \theta_{\alpha_p}(x_{\alpha_p}^*(x)) \right).$$

Obviously, $\gamma$ is a biholomorphism if and only if $\bigcap_{j=1}^p \text{Ker } x_{\alpha_j}^* = \{0\}$ or, equivalently, $D_1$ does not contain $C$. 

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a) ⇒ (1) Because every holomorphic map $h : \mathbb{C} \to D$ is an A.M.V. function and $h(z) = \overline{h(z)}$, from a) we have $h = \text{const}$, thus $D$ is hyperbolic.

(1) ⇒ a) Let $K : \mathbb{C} \to F_2(D)$ be an A.M.V. function. Suppose $\overline{K(z_1)} \neq \overline{K(z_2)}$ for two points $z_1, z_2 \in \mathbb{C}$. Take a plurisubharmonic function $\varphi$ on $\Delta^P$ such that

$$\sup\{\varphi(y) \mid y \in \gamma_{\overline{K}}(z_1)\} \neq \sup\{\varphi(y) \mid y \in \gamma_{\overline{K}}(z_2)\}.$$ 

Since $K$ is analytic, the function

$$\overline{\varphi}(z) = \sup\{\varphi(y) \mid y \in \gamma K(z)\}
= \sup\{\varphi(y) \mid y \in \gamma \overline{K}(z)\}
= \sup\{\varphi(y) \mid y \in \gamma \overline{\overline{K}}(z)\}$$

is subharmonic on $\mathbb{C}$. On the other hand, since $\gamma \overline{K}(z) \subset \Delta^P$ for all $z \in \mathbb{C}$, $\overline{\varphi}$ is bounded on $\mathbb{C}$. This is impossible because of the subharmonicity of $\overline{\varphi}$ and of the relation $\overline{\varphi}(z_1) \neq \overline{\varphi}(z_2)$.

(1) ⇒ c) By the hypothesis, $D$ and hence $D_1$ is hyperbolic. By theorem 1.1, $\gamma L$ and hence $L$ can be extended to an A.M.V. function $\widetilde{L} : \Delta \to F_c(D_1)$. It remains to show that $\widetilde{L}(z_0) \subset D$ for every $z_0 \in S$.

Let $\alpha \in I$ and $x_{\alpha}^* L$ be an extension of $x_{\alpha}^* L$ with values in $F_c(H_{\alpha})$.

Assume that $x_{\alpha}^* L(z_0) \neq x_{\alpha}^* L(z_0)$ for $z_0 \in S$. Take a plurisubharmonic function $\varphi$ on $\mathbb{C}$ such that $\varphi_1(z_0) \neq \varphi_2(z_0)$, where

$$\varphi_1(z) = \sup\{\varphi(y) \mid y \in x_{\alpha}^* L(z)\} = \sup\{\varphi(y) \mid y \in x_{\alpha}^* L(z)\}$$

and

$$\varphi_2(z) = \sup\{\varphi(y) \mid y \in x_{\alpha}^* L(z)\} = \sup\{\varphi(y) \mid y \in x_{\alpha}^* L(z)\}$$

for $z \in \mathbb{C}$.

Since $\varphi_1$ and $\varphi_2$ are plurisubharmonic on $\Delta$ and $\varphi_1 = \varphi_2$ on $\Delta \setminus \{z_0\}$ we have $\varphi_1(z_0) = \varphi_2(z_0)$. This is impossible because of the choice of $\varphi$. Thus, $\text{Re} x_{\alpha}^* (z) < \epsilon_{\alpha}$ for all $z \in \overline{L}(z_0)$ and for all $\alpha \in I$. Hence $\overline{L}(z_0) \subset D$.

c) ⇒ b) Obvious.
b) ⇒ (1) By [1], it suffices to show that every holomorphic map $\beta : \mathbb{C} \to D$ is constant. By the hypothesis, $\beta$ can be extended to an A.M.V. function $\tilde{\beta}$ on $\mathbb{C}P^1$. By the normality of $\mathbb{C}P^1$, it follows that $\tilde{\beta}$ is holomorphic on $\mathbb{C}P^1$ [2]. Since $\tilde{\beta} : \mathbb{C}P^1 \to D$ is holomorphic on the compact space $\mathbb{C}P^1$, it implies that $\tilde{\beta}$ and hence $\beta$ is constant.

The theorem is proved. □

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References


