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RÉSUMÉ. — Le but de cet article est l'étude du prolongement des fonctions analytiques à valeurs multiples. Nous obtenons l'équivalence entre une propriété du genre Liouville et les ensembles pour lesquels on peut prolonger ces fonctions.

ABSTRACT. — The purpose of this note is to study removable singularities for analytic multivalued functions. Moreover, the equivalence between Liouville-type properties and removable singularities results is proved.

Introduction

Let $X$ a complex space. By $F_c(X)$ we denote the hyperspace of non-empty compact subsets of $X$.

As in [8] we say that an upper semi-continuous multivalued function $K : X \to F_c(Y)$, where $X$ and $Y$ are complex spaces, is analytic if for every open subset $W$ of $X$ and every plurisubharmonic function $\psi$ on a neighbourhood of $\Gamma_K \mid W$, the graph of $K$ on $W$, the function

$$\varphi(x) = \sup\{\psi(x, y) \mid y \in K(x)\}$$

is plurisubharmonic on $W$.

Analytic multivalued functions (for short: A.M.V. functions) have been investigated by several authors, in particular by Slodkowski [8, 9] and Ransford [5, 6, 7].

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In [7], Ransford has proved that every A.M.V. function

\[ K : D \rightarrow F_c(V), \]

where \( D = \{ z \in \mathbb{C} | |z| < 1 \} \), \( D^* = D \setminus \{0\} \) and \( V \) is either \( D \) or \( D_{rs} = \{ z \in \mathbb{C} | r < |z| < s \}, 0 < r < s \), can be extended analytically to \( D \).

This note considers a removable-singularity result for A.M.V. functions. Moreover, the equivalence between a Liouville-type property and extendibility of A.M.V. functions is proved.

1. Removable-singularities for analytic multivalued functions

An A.M.V. function \( K : G \rightarrow F_c(Y) \) is said to be locally compact if for every \( x \in X \) there exists a neighbourhood \( U \) of \( x \) such that \( K(U \cap G) \) is relatively compact in \( Y \), where \( G \) is an open subset of \( X \).

**Theorem 1.1.** Let \( G \) be an open set in \( \mathbb{C}^n \), \( S \) a closed subset of \( G \), \( Y \) is a Stein space. Then every A.M.V. function \( K : G \setminus S \rightarrow F_c(Y) \) can be extended analytically to \( G \) if one of the following conditions is satisfied

- a) \( S = H \cap (G \setminus U) \), where \( H \) is an analytic set in \( G \), \( U \) is an open subset of \( G \) such that \( U \) meets every component of \( H \);
- b) \( S \) is a set of zero \((2n - 2)\)-Hausdorff measure in \( G \);
- c) \( S \) is a pluripolar set in \( G \) and \( K \) is locally compact.

We first need the following, which is a generalization of the important result of Wermer [10].

**Lemma 1.2.** Let \( A \) be a uniform algebra with Shilov boundary \( \partial_A^0 \) and \( U \) an open subset of \( \mathbb{C} \). Let \( h : U \rightarrow A \) be a holomorphic map. Then for every \( f \in A \) such that \( \sigma(f) \setminus f(\partial_A^0) \subset U \), where \( \sigma(f) \) is the spectrum of \( f \), the form

\[ K(\lambda) = \{ \tilde{h}(\lambda, w) = \overline{h(\lambda)}(w) \mid w \in \tilde{f}^{-1}(\lambda) \} \]

defines an A.M.V. function on \( \sigma(f) \setminus f(\partial_A^0) \).

**Proof.** This is basically Slodkowski’s argument [8]. It is enough to show that \( K(\lambda) \) satisfies condition (ii) of [8, theorem 3], i.e. for every
polynomial $p(\lambda)$ and for every $a, b \in \mathbb{C}$ the function $\lambda \mapsto \max |f_\lambda(K(\lambda))|$, where $f_\lambda(z) = (z - \lambda a - b)^{-1} \exp(p(\lambda))$, has local maximum property in $G = \{\lambda \in \sigma(f) \setminus \hat{f}(\partial^0_A) \mid a\lambda + b \not\in K(\lambda)\}$. Let $D$ be a disc such that $\overline{cD} \subset G$. Put $N = \hat{f}^{-1}(D) \subset M_A$, where $M_A$ is maximal ideal space of $A$, and let $B$ denote the uniform closure of $A|_{\overline{cN}}$ on $\overline{cN}$ and the form $k = (h(y) - af - b)^{-1} \exp(p(f))$, where $a, b \in \mathbb{C}$ and $p$ is a polynomial, defines an element of $B$. Denote

$$f_\lambda(z) = (z - \lambda a - b)^{-1} \exp(p(\lambda)).$$

For $\lambda_0 \in D$, we have

$$\max f_{\lambda_0}(K(\lambda_0)) = \max |\hat{k}\hat{f}^{-1}(\lambda_0)|$$

$$\leq \max |\hat{k}|_{\overline{cN}} \text{ (by Rossi's local maximum principle)}$$

$$\leq \max \left\{ \max |\hat{k}(\hat{f}^{-1}(\lambda_0))| \mid \lambda \in \partial D \right\}$$

$$= \max \left\{ \max |f_\lambda(K(\lambda))| \mid \lambda \in \partial D \right\}.$$

Thus the function $\lambda \mapsto \max |f_\lambda(K(\lambda))|$ has the local maximum property.

The lemma is proved. □

**Lemma 1.3** (Slodkowski's theorem [9]). — Let $G$ be a bounded planar domain and $K : G \rightarrow F_c(\mathbb{C}^k)$ be an A.M.V. function such that $\sup_{x \in G} \max_{y \in G} |K(x)| < \infty$. Then there exists a uniform algebra $A$ and functions $f, g_1, \ldots, g_k \in A$ such that

i) $\hat{f}(M_A) \setminus \hat{f}(\partial^0_A) = G$, where $\hat{f}$ denotes the Gelfand transformation of $f$, $M_A$ and $\partial^0_A$ are the maximal ideal space and the Shilov boundary respectively of $A$.

ii) $\hat{g}(\hat{f}^{-1}(x)) = K(x)$ for every $x \in G$, where $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_k)$.

**Lemma 1.4.** — Let $K : G \rightarrow F_c(Y)$ be an upper semi-continuous multivalued function, where $G$ is an open subset of $\mathbb{C}^n$ and $Y$ an analytic set in $\mathbb{C}^k$. If $K : F \rightarrow F_c(\mathbb{C}^k)$ is analytic, then $K : G \rightarrow F_c(Y)$ is also analytic.

**Proof.** — We can assume that $n = 1$. Given $\varphi$ a plurisubharmonic function on a neighborhood $W$ of $\Gamma_K|_U$, where $U$ is an open subset of $G$, consider the plurisubharmonic function $\tilde{\varphi}(z, w) = \varphi(z, \tilde{\varphi}(w))$ on
(id x \tilde{g})^{-1}(W), where f, g, A are constructed as in lemma 1.3. By [3] we have
\[ \tilde{\varphi}(z, w) = \lim \max \left\{ c_j^n \log |\tilde{h}_j^n(z, w)| \right\} \]

for all \((z, w) \in (id x \tilde{g})^{-1}(W), where h_j^n are holomorphic maps from U into A.

Since \((id x \tilde{g})\) is continuous and \(W\) is open, it implies that
\[ (id x \tilde{g})^{-1}(W) \supset \varnothing \Rightarrow \partial (id x \tilde{g})^{-1}(W) \supset \varnothing \]
\[ \partial (id x \tilde{g})^{-1}(W) \cup (id x \tilde{g})^{-1}(W) \subset (id x \tilde{g})^{-1}(W) \cup (id x \tilde{g})^{-1}(\partial W) \Rightarrow \]
\[ \partial (id x \tilde{g})^{-1}(W) \subset (id x \tilde{g})^{-1}(\partial W). \]

By lemma 1.2, the multivalued function
\[ L(z) = \{ \tilde{h}_j^n(z, w) \mid w \in \tilde{f}^{-1}(z) \} \]
is analytic on \(\sigma(f) \setminus \tilde{f}(\partial A)\). On the other hand \(\tilde{f}^{-1}(\partial G) \subset \partial A\), by Rossi's local maximum principle we have
\[ \max |\tilde{h}_j^n(z, w)|_{\partial (id x \tilde{g})^{-1}(W)} = \max |\tilde{h}_j^n(z, w)|_{(id x \tilde{g})^{-1}(\partial W)}. \]

Since for every sequence of upper semi-continuous function \(\psi_n, \psi = \lim \psi_n\) point-wise, \(\lim \max (\psi_n|_F) = \max (\psi|_F)\) on every compact subset \(F\) [8], and since \((id x \tilde{g})^{-1}(\partial W) \supset (id x \tilde{g})^{-1}(W)\), it follows that the function \(\gamma\) given by
\[ \gamma(z) = \max \{ \varphi(z, y) \mid y \in K(z) = \tilde{g}\tilde{f}^{-1}(z) \} \]
\[ = \max \{ \tilde{\varphi}(z, y) \mid w \in \tilde{f}^{-1}(z) \} \]
is plurisubharmonic on \(U\). Hence the multivalued function \(K : G \to F_c(Y)\) is analytic.

Proof of theorem 1.1

Without loss of generality we may assume that \(Y\) is an analytic set in \(C^k\). Then the function
\[ \theta(x) = \sup \{|y| \mid y \in K(x)\} \]
is plurisubharmonic on \(G_0 = G \setminus S\), where \(S\) satisfies one of the conditions a) or b) or c) of the theorem. By [4], \(\theta\) can be extended to a plurisubharmonic function on \(C\). This implies that for every \(x_0 \in S\) there exists a
neighbourhood $U$ of $x_0$ such that $K(U \cap G_0)$ is relatively compact. Define a upper semi-continuous extension of $K$ by

$$\hat{K}(x) = \begin{cases} \frac{K(x)}{y \in Y \mid \exists \{(x_n, y_n)\} \subset \Gamma_K, (x_n, y_n) \to (x, y)} & \text{for } x \in G_0 \\ \frac{\gamma \{aA\}}{8(L \cap \{G' \cap S\})} & \text{for } x \in S. \end{cases}$$

We prove that $\hat{K}$ is analytic at every $x_0 \in S$. Let $G'$ be an open ball around $x_0$, $G' \subset G$. It suffices to show that $\hat{K} |_{L \cap G'}$ is analytic for every complex line $L$ in $\mathbb{C}^n$. Using the Slodkowski theorem we can find a uniform algebra $A$ and $f, g_1, \ldots, g_k \in A$ such that

i) $\hat{g} f^{-1}(x) = \hat{K}(x)$ for all $x \in L \cap (G' \setminus S)$;

ii) $f(\partial_A^0) = \partial (L \cap (G' \setminus S))$.

We have to prove that $f(\partial_A^0) \cap (L \setminus G') = \emptyset$.

Suppose the contrary. Then there exists a complex line $L$ in $\mathbb{C}^n$ such that $f(\partial_A^0) \cap (L \cap G') \neq \emptyset$. Since $\hat{K}$ is analytic on $G' \setminus S$, it follows that $\hat{f}(\partial_A^0) \cap (L \cap (G' \setminus S)) = \emptyset$. Hence there exists $w_0 \in \partial_A^0$ such that $\hat{f}(w_0) = x_0$. Since $G'$ is open and set of peak points of $A$ is dense in $\partial_A^0$, we may assume that $w_0$ is a peak point. Hence there exists $h \in A$ such that $|\hat{h}(w_0)| = 1$ and $|\hat{h}(w)| < 1$ for $w \in M_A \setminus \{w_0\}$.

Consider the plurisubharmonic function

$$\varphi(x) = \log \max |\hat{h} f^{-1}(x)| \ \text{on} \ \ G' \setminus S.$$ 

Then $\varphi$ is plurisubharmonic on $G' \cap L$. Since

$$\log \max |\hat{h} f^{-1}(x)| \leq 0 = \log \max |\hat{h} f^{-1}(x_0)|$$

for every $x \in G'$, it follows that $\varphi$ = constant, which is impossible.

Thus $f(\partial_A^0) \cap (G' \cap L) = \emptyset$.

Theorem 1.1 is proved. $\square$

2. Liouville-type property for analytic multivalued functions

In the section we study the relation between a Liouville-type property and removable singularities of A.M.V. functions with values in convex domains.
THEOREM 2.1. — Let $D$ be a convex domain in $\mathbb{C}^n$. Then the following conditions are equivalent

a) for every A.M.V. function $K : \mathbb{C} \to F_c(D)$, the multivalued function $\hat{K} : \mathbb{C} \to F_c(D)$ given by $\hat{K}(x) = \overline{K(x)}$, where $\overline{K(x)}$ is polynomial convex hull of $K(x)$, is constant;

b) every A.M.V. function $K : \Delta^* \to F_c(D)$ can be extended analytically on $\Delta$, where $\Delta$ is the unit disc, $\Delta^* = \Delta \setminus \{0\}$;

c) every A.M.V. function $L : \Delta \setminus S \to F_c(D)$ can be extended analytically on $\Delta$, where $S$ is a polar set in $\Delta$.

To prove the theorem we shall use the hyperbolicity of convex domains. In [1] Bath proved that a convex domain $D$ is hyperbolic if and only if $D$ does not contain complex lines (i.e. every holomorphic map $h : \mathbb{C} \to D$ is constant).

Proof of theorem 2.1

Consider the condition:

$$D \text{ is hyperbolic}$$ \hspace{1cm} (1)

We shall prove that a) $\iff$ (1) $\Rightarrow$ c) $\Rightarrow$ b) $\Rightarrow$ (1).

We first write

$$D = \bigcap_{\alpha \in I} \{ \text{Re} x_\alpha^* < \varepsilon_\alpha \},$$

where $\{x_\alpha^*\}$ are linear forms on $\mathbb{C}^n$. Without loss of generality we may assume that $0 \in D$. Then $\varepsilon_\alpha > 0$ for all $\alpha$.

Let $\{x_{\alpha_1}^*, \ldots, x_{\alpha_p}^*\}$ be a maximal linearly independent system of $\{x_\alpha^*\}$. Take $\theta_\alpha : H_\alpha \to \Delta$, where $H_\alpha = \{z \in \mathbb{C} : \text{Re} z < \varepsilon_\alpha\}$, is a biholomorphism. Define a holomorphic map

$$\gamma : D_1 \to \Delta^p, \text{ where } D_1 = \bigcap_{j=1}^p \{ \text{Re} x_{\alpha_j}^* \},$$

by

$$\gamma(x) = \left( \theta_{\alpha_1}(x_{\alpha_1}^*(x)), \ldots, \theta_{\alpha_p}(x_{\alpha_p}^*(x)) \right).$$

Obviously, $\gamma$ is a biholomorphism if and only if $\bigcap_{j=1}^p \text{Ker} x_{\alpha_j}^* = \{0\}$ or, equivalently, $D_1$ does not contain $C$. 

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a) \implies (1) Because every holomorphic map $h : \mathbb{C} \to D$ is an A.M.V. function and $h(z) = h(z)$, from a) we have $h = \text{const}$, thus $D$ is hyperbolic.

(1) \implies a) Let $K : \mathbb{C} \to F_c(D)$ be an A.M.V. function. Suppose $\bar{K}(z_1) \neq \bar{K}(z_2)$ for two points $z_1, z_2 \in \mathbb{C}$. Take a plurisubharmonic function $\varphi$ on $\Delta^p$ such that
\[
\sup \{ \varphi(y) \mid y \in \gamma \bar{K}(z_1) \} \neq \sup \{ \varphi(y) \mid y \in \gamma \bar{K}(z_2) \}.
\]
Since $K$ is analytic, the function
\[
\bar{\varphi}(z) = \sup \{ \varphi(y) \mid y \in \gamma K(z) \} = \sup \{ \varphi(y) \mid y \in \gamma \bar{K}(z) \} = \sup \{ \varphi(y) \mid y \in \gamma \bar{K}(z) \}
\]
is subharmonic on $\mathbb{C}$. On the other hand, since $\gamma \bar{K}(z) \subset \Delta^p$ for all $z \in \mathbb{C}$, $\bar{\varphi}$ is bounded on $\mathbb{C}$. This is impossible because of the subharmonicity of $\bar{\varphi}$ and of the relation $\bar{\varphi}(z_1) \neq \bar{\varphi}(z_2)$.

(1) \implies c) By the hypothesis, $D$ and hence $D_1$ is hyperbolic. By theorem 1.1, $\gamma L$ and hence $L$ can be extended to an A.M.V. function $\bar{L} : \Delta \to F_c(D_1)$. It remains to show that $\bar{L}(z_0) \subset D$ for every $z_0 \in S$.

Let $\alpha \in I$ and $\tilde{x}_\alpha^* L$ be an extension of $x_\alpha^* L$ with values in $F_c(H_{\alpha})$.

Assume that $\tilde{x}_\alpha^* L(z_0) \neq \tilde{x}_\alpha^* L(z_0)$ for $z_0 \in S$. Take a plurisubharmonic function $\varphi$ on $\mathbb{C}$ such that $\varphi_1(z_0) \neq \varphi_2(z_0)$, where
\[
\varphi_1(z) = \sup \{ \varphi(y) \mid y \in \tilde{x}_\alpha^* L(z) \} = \sup \{ \varphi(y) \mid y \in \tilde{x}_\alpha^* L(z) \}
\]
and
\[
\varphi_2(z) = \sup \{ \varphi(y) \mid y \in \tilde{x}_\alpha^* L(z) \} = \sup \{ \varphi(y) \mid y \in \tilde{x}_\alpha^* L(z) \}
\]
for $z \in \mathbb{C}$.

Since $\varphi_1$ and $\varphi_2$ are plurisubharmonic on $\Delta$ and $\varphi_1 = \varphi_2$ on $\Delta \setminus \{z_0\}$ we have $\varphi_1(z_0) = \varphi_2(z_0)$. This is impossible because of the choice of $\varphi$. Thus, $\Re x_\alpha^*(z) < \epsilon_\alpha$ for all $z \in \bar{L}(z_0)$ and for all $\alpha \in I$. Hence $\bar{L}(z_0) \subset D$.

c) \implies b) Obvious.
b) ⇒ (1) By [1], it suffices to show that every holomorphic map \( \beta : \mathbb{C} \to D \) is constant. By the hypothesis, \( \beta \) can be extended to an A.M.V. function \( \tilde{\beta} \) on \( \mathbb{C}P^1 \). By the normality of \( \mathbb{C}P^1 \), it follows that \( \tilde{\beta} \) is holomorphic on \( \mathbb{C}P^1 \) [2]. Since \( \tilde{\beta} : \mathbb{C}P^1 \to D \) is holomorphic on the compact space \( \mathbb{C}P^1 \), it implies that \( \tilde{\beta} \) and hence \( \beta \) is constant.

The theorem is proved. □

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References


