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$H^\infty$-extensibility and finite proper holomorphic surjections


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The extension of holomorphic maps from a Riemann domain $D$ over a Stein manifold to its envelope of holomorphy $\hat{D}_\infty$ for the Banach algebra of bounded holomorphic functions $H^\infty(D)$ has been investigated by some authors.

For holomorphic maps with values in finite dimensional complete $C$-spaces, the problem was considered by Sibony [6], Hirschowitz [3], and recently by Nguyen van Khue and Bui Dac Tac [4]. The aim of the present paper is to consider the problem in the infinite dimensional case.

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Let $X$ be a Banach analytic space in the sense of Douady [1]. As in the finite dimensional case, we define the Carathéodory pseudodistance $C_X$ on $X$ by the formula

$$C_X(x, y) = \sup \{|f(x) - f(y)| : |f| \leq 1, f \in H^\infty(X)\}.$$  

We say that $X$ is a $C$-space if $C_X$ is a distance defining the topology of $X$.

Let $(D, p, B)$ and $(D', q, B)$ be Riemann domains over a Banach space $B$. $D'$ is called a $H^\infty$-extension of $D$ if there is a holomorphic map $e : D \to D'$ such that $p = q \cdot e$ and for every bounded holomorphic function $f$ on $D$, there exists a bounded holomorphic function $f'$ on $D'$ such that $f = f' \cdot e$.

A Banach analytic space $X$ is said to be a space having the holomorphic $H^\infty$-extension property (for short, the HEH$^\infty$-property) if for every holomorphic map $g$ from a Riemann domain $D$ over a Banach space into $X$ there exists a holomorphic map $g'$ from $D'$ into $X$ such that $g = g' \cdot e$, where $D'$ is a $H^\infty$-extension of $D$ and $D'$ is a $C$-space. In this case we say also that $g$ can be extended to a holomorphic map $g'$ on $D'$.

The main result of this note is the following.

**Theorem 1.** — Let $\theta$ be a finite proper holomorphic map from a Banach analytic space $X$ onto a Banach analytic space $Y$. Then:

(i) if $Y$ has the HEH$^\infty$-property and $H^\infty(X)$ separates the points of the fibers of $\theta$, then $X$ has the HEH$^\infty$-property;

(ii) if $X$ has the HEH$^\infty$-property and $X$ does not contain a compact analytic set of positive dimension, then every holomorphic map from $D$ into $Y$ can be extended Gateaux holomorphically on $D'$, where $D'$ is a Riemann domain over a Banach space, $D'$ is a $H^\infty$-extension of $D$ and $D'$ is a $C$-space. In this case we say also that $g$ can be extended to a holomorphic map $g'$ on $D'$.

Moreover, the extension is holomorphic outside a hypersurface.

Let $X$ be a Banach analytic space. We say that an upper semi-continuous function $\varphi : X \to [\pm \infty]$ is plurisubharmonic if for every holomorphic map $\sigma : \Delta \to X$ the function $\varphi \circ \sigma$ is subharmonic, where $\Delta$ is the unit disc.

Let $Z$ be a Banach analytic space. By $F_c(Z)$ we denote the hyperspace of non-empty compact subsets of $Z$. An upper semi-continuous multivalued function $K : X \to F_c(Z)$, where $X$ is a Banach analytic space, is called analytic in the sense of Slodkowski [?] if for every plurisubharmonic function
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Let $K : Y \to F_c(X)$ be an analytic multivalued function such that $\text{card } K(y) < \infty$ for all $y \in Y$, where $Y$ is a connected Banach analytic space. Assume that $U$ and $V$ are disjoint open subsets of $X$ such that $K(y) \subseteq U \cup V$ for all $y \in Y$. Then either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$.

**Proof.** Define $\Psi$ on $Y \times (U \cup V)$ by

$$\Psi(y, z) = \begin{cases} 1 & \text{if } z \in U \\ 0 & \text{if } z \in V. \end{cases}$$

Then $\Psi$ is plurisubharmonic on a neighbourhood of the graph of $K$, so $\varphi$ is plurisubharmonic on $Y$, where

$$\varphi(y) = \max\{\Psi(y, z) \mid z \in K(y)\} = \begin{cases} 0 & \text{if } K(y) \cap U = \emptyset \\ 1 & \text{if } K(y) \cap U \neq \emptyset. \end{cases}$$

By the plurisubharmonicity of $\varphi$ and the connectedness of $Y$, it implies that either $K(y) \cap U = \emptyset$ for all $y \in Y$ or $K(y) \cap U \neq \emptyset$ for all $y \in Y$. The lemma is proved. $\square$

**Lemma 2.** Let $K : Y \to F_c(X)$ be an analytic multivalued function such that $\text{card } K(y) < \infty$ for all $y \in Y$. Then

$$V_m = \{y \in Y \mid \text{card } K(y) < m\}$$

is closed in $Y$ for every $m \geq 1$.

**Proof.** Given a sequence $\{y_n\}$ in $V_m$, $y_n \to y^*$, choose disjoint neighbourhoods $U_i$ of $x_i$, $i = 1, \ldots, \ell$, where $\{x_1, \ldots, x_{\ell}\} = K(y^*)$. Take a neighbourhood $D$ of $y^*$ such that

$$K(D) \subseteq \bigcup_{i=1}^{\ell} U_i.$$
Then by lemma 1, $K(y) \cap U \neq \emptyset$ for all $i = 1, \ldots, \ell$ and for all $y \in D$. Hence $m > \text{card } K(y_n) \geq 1$ for sufficiently large $n$. This implies that $y^* \in V_m$. The lemma is proved. □

**Lemma 3.** — Let $\theta : X \to Y$ be a finite proper holomorphic surjection, where $X$ and $Y$ are Banach analytic spaces. Then the multivalued function

$$K : Y \to F_c(X)$$

given by

$$K(y) = \theta^{-1}(y)$$

is analytic.

**Proof**

(i) Consider first the case where $Y = \Delta$, the unit disc in $\mathbb{C}$.

Since $\theta$ is proper, $K$ is upper semi-continuous. Let $\Psi$ be a plurisubharmonic function on a neighbourhood of $\Gamma K \mid G$, where $G$ is an open subset of $\Delta$. Since $\theta$ is a branch covering map [2], there exists a discrete sequence $A$ in $\Delta$ such that

$$\theta : X \setminus \theta^{-1}(A) \to \Delta \setminus A$$

is an unbranched covering map of order $m < \infty$. Let $y_0 \in \Delta \setminus A$ and

$$\theta^{-1}(y_0) = \{x_1, \ldots, x_m\}.$$

Take a neighbourhood $W$ of $y_0$ such that

$$\theta^{-1}(W) = U_1 \cup \cdots \cup U_m,$$

where $U_j$ are disjoint, $x_j \in U_j$ and $\theta : W \cong U_j$, $j = 1, \ldots, m$. Then the function

$$\varphi(y) = \max_j \max \{\Psi(y, x) \mid z \in \theta^{-1}(y) \cap U_j\}$$

is subharmonic on $W \cap G$. Since $\varphi$ is locally bounded on $G$, it follows that $\varphi$ is subharmonic on $G$. 

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(ii) Consider now the general case where \( Y \) is a Banach analytic space.

Let \( \varphi \) be as in (i). Obviously \( \varphi \) is upper semi-continuous because of the upper semi-continuity of \( K \) and \( \Psi \). It remains to check that \( \varphi \circ h \) is subharmonic on \( \Delta \) for every holomorphic map \( h : \Delta \rightarrow X \). Consider the commutative diagram

\[
\begin{array}{ccc}
\tilde{\Delta} & \xrightarrow{\tilde{h}} & X \\
\downarrow \theta & & \downarrow \theta \\
\Delta & \xrightarrow{h} & Y
\end{array}
\]

where \( \tilde{\Delta} = \Delta \times_Y X \). By (i) and by the relation

\[
\varphi \circ h(\lambda) = \max \left\{ \Psi(h(\lambda), z) \mid \theta(z) = h(\lambda) \right\}
\]

it follows that \( \varphi \circ h \) is subharmonic on \( \Delta \). The lemma is proved. \( \square \)

Let \( X \) and \( D \) be Banach analytic spaces. A finite proper holomorphic surjection \( \pi : X \rightarrow D \) is called a branch covering map if it satisfies the following:

(i) there is a closed subset \( A \) of \( D \) which is a removable for bounded holomorphic germs on \( D \setminus A \);

(ii) \( \pi : X \setminus \pi^{-1}(A) \rightarrow D \setminus A \) is a local biholomorphism and \( \card \pi^{-1}(z) \) is constant on every connected component of \( D \setminus A \).

**Lemma 4.** Let \( \theta \) be a finite proper holomorphic map from a Banach analytic space \( X \) onto an open set \( D \) in a Banach space \( B \). Then \( \theta \) is a branch covering map.

**Proof.** Without loss of generality we may assume that \( D \) is convex. For each \( n \geq 1 \) put

\[
F_n = \{ y \in D \mid \card \theta^{-1}(y) < n \}.
\]

By lemma 2 and lemma 3, \( F_n \) is closed in \( D \). Applying the Baire theorem to \( D = \bigcup_{1}^{\infty} F_n \), we can find \( n_0 \) such that \( \text{Int} F_n \neq \emptyset \). Put

\[
m = \max \{ \card \theta^{-1}(y) \mid y \in \text{Int} F_{n_0} \}.
\]
Since $\theta : \theta^{-1}(E \cap D) \to E \cap D$ is a branch covering map for every finite dimensional subspace $E$ of $B$ [2], by the connectedness of $D \cap E$ for all subspace $E$ of $B$, $\dim E < \infty$, we have

$$\sup\{\text{card } \theta^{-1}(y) \mid y \in D\} = \sup\{\text{card } \theta^{-1}(y) \mid y \in D \cap E, \ E \subset B, \ \dim E < \infty\} = m.$$  

Put

$$Z = \{y \in D \mid \text{card } \theta^{-1}(y) < m\}.$$  

Then $Z$ is closed in $D$, and from the finiteness and properness of $\theta$ it follows that

$$\theta : X \setminus \theta^{-1}(Z) \to D \setminus Z$$

is an unbranched covering map. It remains to show that $Z$ is removable for bounded holomorphic germs. Let $h$ be a bounded holomorphic function on $U \setminus Z$, where $U$ is an open subset of $D$. Then for every finite dimensional space $E$ of $B$ such that

$$\sup\{\text{card } \theta^{-1}(y) \mid y \in E \cap D\} = m,$$

$h|_{U \setminus Z}$ can be extended holomorphically on $U$. From the relation

$$D = \bigcup\left\{E \cap D \mid E \subset B, \ \dim E < \infty, \ \sup\{\text{card } \theta^{-1}(y) \mid y \in D \cap E\} = m\right\},$$

it follows that $h$ can be extended to a bounded Gateaux-holomorphic function $\hat{h}$ on $U$. By the boundedness of $\hat{h}$, we deduce that $\hat{h}$ is holomorphic on $U$. The lemma is proved. □

**Lemma 5.** Let $\theta : X \to D$, where $D$ is a $C$-manifold, be a branch covering map. Denote by $\text{SH}^{\infty}(X)$ and $\text{SH}^{\infty}(D)$ the spectra of Banach algebras $H^{\infty}(X)$ and $H^{\infty}(D)$, respectively. Let $\theta : \text{SH}^{\infty}(X) \to \text{SH}^{\infty}(D)$ be the map induced by $\theta$. Then

$$\hat{\theta} : \hat{\theta}^{-1}(D) \to D$$

is also a branch covering map.
Proof. — Obviously \( \tilde{\theta} : \tilde{\theta}^{-1}(D) \to D \) is finite, proper and surjective, since \( H^\infty(X) \) is an integral extension of finite degree of \( H^\infty(D) \). By lemma 4, it suffices to prove that \( \tilde{\theta}^{-1}(D) \) is a Banach analytic space. Let \( B(0, r) \) (resp. \( B^*(0, r) \)) denote the open ball in \( H^\infty(X) \) (resp. \( (H^\infty(X))^* \)) centred at 0 with radius \( r > 0 \). Consider the holomorphic map

\[
F : (D \setminus Z) \times B^*(0, 2) \to H^\infty(B(0, 2))
\]
given by

\[
F(z, w)(h) = w(h)^m + \sigma_{m-1}(h \circ p_1(z), \ldots, h \circ p_m(z))w(h)^{m-1} + \\
\cdots + \sigma_0(h \circ p_1(z), \ldots, h \circ p_m(z)),
\]

where \( z \) is the branch locus of \( \theta \), \( m \) the order of \( \theta \) and \( \sigma_0, \ldots, \sigma_{m-1} \) are elementary symmetric polynomials in \( m \) variables and

\[
\theta^{-1}(z) = (p_1(z), \ldots, p_m(z)) \quad \text{for} \; z \in D \setminus Z.
\]

Since \( \sigma_0, \ldots, \sigma_{m-1} \) are bounded holomorphic functions on \( D \setminus Z \), it follows that \( F \) is holomorphic on \( D \times B^*(0, 2) \). We have

\[
F^{-1}(0) = \{(z, w) \mid \tilde{\theta}(w) = z\} \cong \tilde{\theta}^{-1}(D).
\]

Hence \( \tilde{\theta} : \tilde{\theta}^{-1}(D) \to D \) is a branch covering map. The lemma is proved. \( \square \)

Lemma 6. — Every Banach space has the HEH\(^\infty\)-property.

Proof. — Let \( D \) be a Riemann domain over a Banach space \( B \) and \( D' \) a \( H^\infty \)-extension of \( D \). Let \( f : D \to E \) be a holomorphic map, where \( E \) is a Banach space.

For each \( x^* \in E^* \), by \( \widehat{x^*f} \) we denote the holomorphic extension of \( x^*f \) on \( D' \). Since \( D' \) is a \( H^\infty \)-extension of \( D \), from the open mapping theorem, it follows that

\[
\|\widehat{x^*f}\| = \|x^*f\| \quad \text{for all} \; x^* \in E^*.
\]

On the other hand, by the uniqueness, \( \widehat{x^*f}(z) \) is a continuous linear function on \( E^* \) for every \( z \in D' \). Thus we can define a bounded map \( \widehat{f} : D' \to E^{**} \) by

\[
(\widehat{f}(z))(x^*) = \widehat{x^*f}(z)
\]

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which is separately holomorphic in variables \( z \in D' \) and \( x^* \in E^* \). From the boundedness of \( \tilde{f}(D') \) we deduce that \( \tilde{f} \) is holomorphic and \( \tilde{f}(D') \subset E \). Obviously \( \tilde{f} \) is a holomorphic extension of \( f \) on \( D' \). The lemma is proved. □

Proof of theorem 1

(i) Let first \( Y \) have the \( \text{HEH}^\infty \)-property. Let \( f : D \to X \) be a holomorphic map, where \( D \) is a Riemann domain over a Banach space \( B \). By hypothesis, there is a holomorphic map \( g : D' \to Y \) which is a holomorphic extension of \( \theta f \) on \( D' \), where \( D' \) is a \( H^\infty \)-extension of \( D \). Consider the commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{f} & X \\
\downarrow \alpha & & \downarrow \theta \\
D' & \xrightarrow{\tilde{g}} & Y
\end{array}
\]

where \( G = D' \times_Y X \), \( \tilde{\theta} \) and \( \tilde{g} \) are the canonical projections, \( \alpha \) and \( e \) are the canonical maps. By lemma 4, \( \tilde{\theta} \) is a branch covering map. Let \( H \) denote the branch locus of \( \tilde{\theta} \). Consider the commutative diagram

\[
\begin{array}{ccc}
G \setminus \tilde{\theta}^{-1}(H) & \xrightarrow{\tilde{\theta}^{-1}} & (D' \setminus H) \\
\downarrow \tilde{\theta} & & \downarrow \delta \\
D' \setminus H & \xrightarrow{\delta} & SH^\infty(D' \setminus H) = SH^\infty(D')
\end{array}
\]

where

\[
\tilde{\theta} : SH^\infty(G \setminus \tilde{\theta}^{-1}(H)) \to SH^\infty(D' \setminus H) \cong SH^\infty(D')
\]

is induced by \( \tilde{\theta} : G \setminus \tilde{\theta}^{-1}(H) \to D' \setminus H \). From lemma 5, it follows that

\[
\tilde{\theta} : \tilde{\theta}^{-1}(D' \setminus H) \to D' \setminus H
\]

is a branch covering map. By lemma 6, \( \left( H^\infty(G \setminus \tilde{\theta}^{-1}(H)) \right)^* \) has the \( \text{HEH}^\infty \)-property.
Since \( D' \setminus H \) is also a \( H^\infty \)-extension of \( D \setminus e^{-1}(H) \), there exists

\[
h : D' \setminus H \rightarrow \left( H^\infty \left( G \setminus \tilde{\theta}^{-1}(H) \right) \right)^*\]

which is a holomorphic extension of

\[
\tilde{id} \alpha : D \setminus e^{-1}(H) \rightarrow \left( H^\infty \left( G \setminus \tilde{\theta}^{-1}(H) \right) \right)^*.
\]

From the relation \( \tilde{\theta} h = \delta \), where \( \delta : D' \setminus H \rightarrow SH^\infty(D' \setminus H) \) is the canonical map, we have \( h(D' \setminus H) \subset \tilde{\theta}^{-1}(D' \setminus H) \). Since \( H^\infty(X) \) separates the points of the fibers of \( \tilde{\theta} \), there exists a holomorphic mapping \( \tilde{g} : \tilde{\theta}^{-1}(D' \setminus H) \rightarrow X \) such that \( g \tilde{\theta} = \theta \tilde{g} \). Put

\[
f_1 = \tilde{g} h.
\]

Assume now \( z \in H \). Take two neighbourhoods \( U \) and \( V \) of \( z \) and \( g(z) \), respectively, such that \( g(U) \subset V \) and \( \theta^{-1}(V) \) is an analytic set in a finite union \( W \) of balls in a Banach space. Then \( f_1 : U \setminus H \rightarrow W \) can be extended holomorphically on \( U \). This implies that \( f_1 \) and hence \( f \) can be extended holomorphically on \( D' \).

(ii) Let \( X \) be a space having the \( HEH^\infty \)-property and let \( g : D \rightarrow Y \) be a holomorphic map, where \( D \) is a Riemann domain over a Banach space \( B \). Let \( D' \) be a \( H^\infty \)-extension of \( D \) which is a \( C \)-space. Consider the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\tilde{g}} & X \\
\downarrow \tilde{\theta} & & \downarrow \theta \\
D & \xrightarrow{g} & Y
\end{array}
\]

where \( G = D \times_Y X \), \( \tilde{\theta} \) and \( \tilde{g} \) are the canonical projections.

Obviously \( \tilde{\theta} : SH^\infty(G) \rightarrow SH^\infty(D') \) is finite, proper and surjective, since \( H^\infty(G) \) is an integral extension of finite degree of \( H^\infty(D) \) and every bounded holomorphic function on \( D \) can be extended to a bounded holomorphic function on \( D' \). By lemmas 4 and 5, \( \tilde{\theta} \) and \( \tilde{\theta}^{-1}(D') \) are branch covering maps. Let \( H \) denote the branch locus of \( \tilde{\theta} : \tilde{\theta}^{-1}(D') \rightarrow D' \). Consider the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\tilde{g}} & X \\
\end{array}
\]
where $\delta$ is the canonical map. Since every bounded holomorphic function on $G \setminus \tilde{\theta}^{-1}(e^{-1}(H))$ can be extended to a bounded holomorphic function on $\text{SH}^\infty\left( G \setminus \tilde{\theta}^{-1}(e^{-1}(H)) \right)$ and the topology of $\tilde{\theta}^{-1}(D' \setminus H)$ is defined by bounded holomorphic functions, it follows that $\tilde{\theta}^{-1}(D' \setminus H)$ is a $H^\infty$-extension of $G \setminus \tilde{\theta}^{-1}(e^{-1}(H))$ and it is a $C$-space. By hypothesis, $\tilde{g}$ can be extended to a holomorphic map

$$\tilde{g}_0 : \tilde{\theta}^{-1}(D' \setminus H) \to X.$$ 

It is easy to see that $e \tilde{\theta}^{-1}(x) = \tilde{\theta}^{-1}(e(x))$ for every $x \in D \setminus e^{-1}(H)$. This yields

$$\tilde{g}_0 |_{\tilde{\theta}^{-1}(e(x))} = \text{const} \quad \text{for all } x \in D \setminus e^{-1}(H).$$

Since $\tilde{\theta} : \tilde{\theta}^{-1}(D' \setminus H) \to D' \setminus H$ is a branch covering map, it follows that there exists a holomorphic map $\tilde{g}_0 : D' \setminus H \to Y$ such that $\theta \tilde{g}_0 = \tilde{g} \tilde{\theta}$.

$X$ does not contain a compact set of positive dimension. By the Hironaka singular resolution theorem, for every finite dimensional subspace $E$ of $B$ such that $q^{-1}(E) \not\subset e(H)$,

$$\tilde{g}_0 |_{\tilde{\theta}^{-1}(q^{-1}(E) \setminus H)}$$

can be extended to a holomorphic map $\tilde{g}_E : \tilde{\theta}^{-1}(q^{-1}(E)) \to X$. This yields that $\tilde{g}_0 |_{q^{-1}(E) \setminus H}$ can be extended to a holomorphic map $\tilde{g}_E : q^{-1}(E) \to Y$. Thus $\tilde{g}_0$ and hence $g$ can be extended to a Gateaux holomorphic map $\tilde{g} : D' \to Y$ which is holomorphic on $D' \setminus H$. The theorem is proved. $\Box$

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References


