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# A problem of minimization with relaxed energy[(*)]

REJEB HADIJI[(1)] and FENG ZHOU[(2)]

RÉSUMÉ. — On généralise un résultat de F. Bethuel et H. Brezis concernant un problème de minimisation avec l'énergie relaxée. On montre que l'infimum n'est pas atteint et on étudie aussi le comportement de la suite minimisante.

ABSTRACT. — We generalize a result of F. Bethuel and H. Brezis concerning a minimization problem with relaxed energy. We prove that the energy infimum is not achieved. We also study the behaviors of the minimizing sequence.

## 1. Introduction

Let $\Omega$ be the unit ball in $\mathbb{R}^3$ and $S^2$ be the unit sphere. Set

$$H^1(\Omega, S^2) = \left\{u \in H^1(\Omega, \mathbb{R}^3) \mid u(x) \in S^2 \text{ a.e.}\right\}$$

and $R^1(\Omega, S^2) = \left\{u \in H^1(\Omega, S^2) \mid u \text{ is regular except at a finite number of singularities}\right\}$.

Let $f$ be in $R^1(\Omega, S^2)$ which is singular at $\{a_1, \ldots, a_k\}$ with the degree $d_1, \ldots, d_k$ respectively. The degree $d_i$ of $f$ at the point $a_i$ means the topological degree in $\mathbb{Z}$ of $f$ restricted to a sphere centered at $a_i$. By continuity it is independent of the choice of the sphere. For any $\varphi \in H_1(S^2, S^2)$, we define the degree of $\varphi$ by

$$\deg(\varphi) = \frac{1}{4\pi} \int_{S^2} \operatorname{Jac}(\varphi)$$

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which coincides with the topological degree of $\varphi$ when $\varphi$ is regular mapping. If $u \in H^1(\Omega, \mathbb{R}^3)$, $E(u)$ denotes the Dirichlet energy of $u$:

$$E(u) = \int_\Omega \left|\nabla u\right|^2 \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,.$$

We consider the following minimization problem:

$$\alpha = \inf \left\{ E(u - f) \,,\ u \in C^1\left(\overline{\Omega}, S^2\right) \right\} . \tag{1}$$

It is well known that $\alpha > 0$ if $d_i \neq 0$ for some $i \in \{1, \ldots, k\}$. This is due to the fact, proved by R. Schoen and K. Uhlenbeck in [SU1], that $f$ can not be approximated by smooth maps. In this paper, we will prove that for some specific functions $f$, the infimum in (1) can not be achieved and we study some properties of the minimizing sequence of (1). The general case is still an open problem.

Our main result is the following theorem.

THEOREM 1.— *Suppose that $f(x)$ is $\psi\big(x/|x|\big)$ where $\psi$ is a non constant mapping in $H^1(S^2, S^2)$, then the infimum in (1) is not achieved and every minimizing sequence converges strongly in $H^1$ to a mapping of the form $\varphi\big(x/|x|\big)$ such that $\varphi$ is a regular mapping from $S^2$ into $S^2$ with degree zero.*

The problem was originally studied by F. Bethuel and H. Brezis in the case where $f(x) = x/|x|$. For the study of this problem, it is convenient to introduce the relaxed energy associated to (1) as in [BB] or [BCL].

Let $u$ be in $H^1(\Omega, S^2)$, the vector field $D(u)$ is defined as follows [BCL],

$$D(u) = \left(u \cdot u_y \wedge u_z \,,\ u \cdot u_z \wedge u_x \,,\ u \cdot u_x \wedge u_y\right) .$$

Consider

$$L(u) = \frac{1}{4\pi} \sup \int_\Omega D(u) \nabla\xi \,, \quad \xi : \overline{\Omega} \to \mathbb{R}, \,,\ \left\|\nabla\xi\right\|_{L^\infty} \leq 1 \,,\ \xi\big|_{\partial\Omega} = 0 \,. \tag{2}$$

Note that $L(u)$ makes sense for any $u \in H^1(\Omega, S^2)$, since $D(u) \in L^1(\Omega, \mathbb{R}^3)$. In the case where $u$ is in $R^1(\Omega, S^2)$, $L(u)$ coincides with the length of a minimal connection between the singularities " allowing connection to $\partial\Omega$ " (see [BBC]). We know that $L(u)$ is continuous for the strong topology of $H^1$. The relaxed energy associated to (1) is following:

$$F(u) = E(u - f) + 8\pi L(u) \,. \tag{3}$$

For the proof of Theorem 1, we need the following theorem, whose proof uses the same argument as in [BBC].

THEOREM 2.— *$F$ satisfies:*

*(a) $F$ is l.s.c. for the weak topology of $H^1$, then*

$$\inf\{F(u) \mid u \in H^1(\Omega, S^2)\}$$

*is achieved;*

*(b)* $\inf\{E(u-f) \mid u \in C^1(\Omega, S^2)\} = \min\{F(u) \mid u \in H^1(\Omega, S^2)\}$*;*

*(c) for any $u \in H^1(\Omega, S^2)$, we have $F(u) = \inf\{\liminf E(u_n - f)\}$, where infimum is taken over all sequences $(u_n)$ of $C^1(\Omega, S^2)$ which converges weakly to $u$ in $H^1(\Omega, S^2)$. (The existence of a such sequence is given by a result of [B].)*

We use also the following Lemma 1 which plays a crucial role in the proof of Theorem 1. This lemma is called "construction of a dipole function". It is due to T. Rivière [R]. Here we give a finer estimate.

LEMMA 1.— *Let $u$ and $f$ be two regular mappings in $H^1(\Omega, S^2)$ and $u$ is not constant. Let $x_0$ be any point in $\Omega$ such that $\nabla u(x_0) \neq 0$, then $\forall\ \rho > 0$ there exists $v \in H^1(\Omega, S^2)$ such that:*

*(i) $v = u$ outside $B_\rho(x_0)$,*

*(ii) $v$ is Lipschitz except at two points singularities $p$ and $n$ in $B_\rho(x_0)$ of degree $\pm 1$,*

*(ii) $E(v - f) < E(u - f) + 8\pi|p - n|$.*

The outline of this paper is as follows: section 2 is proof of Lemma 1 and section 3 is proof of Theorem 1.

## 2. Proof of Lemma 1

The dipole construction was originally introduced by H. Brezis, J.-M. Coron and H. Lieb in [BCL] for the calculation of minimum of the Dirichlet energy among the maps in $H^1(\Omega, S^2)$ which have fixed isolated singularities. It has been extensively used to study the minimization problem with relaxed energy. In [R], T. Rivière generalized the construction of a dipole function. For the convenience of readers, we will recall his construction and we will take the same notation. The proof of Lemma 1 consist of several steps.

## 2.1 Construction of a dipole

We may as well assume that $\nabla u(0) \neq 0$. We can always choose a basis $(i,\, j,\, k)$ such that (see [BC]):

$$u_x(0) \neq 0, \quad u_x(0) \cdot u_y = (0)\,. \tag{4}$$

Let $\delta$ be small enough and $p = (0,\, 0,\, \delta)$ and $n = (0,\, 0,\, -\delta)$. We denote by $C_\delta$ the cylinder centered at 0, of axis $0z$, of radius $2\delta^2$ and of length $2(\delta + \delta^2)$. We divide $C_\delta$ into three smaller cylinders, $C_\delta = c_\delta \cup c_\delta^p \cup c_\delta^n$ where:

$$\begin{aligned} c_\delta &= \left\{(x,y,z) \in C_\delta \mid -\delta + \delta^2 \leq z \leq \delta - \delta^2\right\} \\ c_\delta^p &= \left\{(x,y,z) \in C_\delta \mid z \geq \delta - \delta^2\right\} \\ c_\delta^n &= \left\{(x,y,z) \in C_\delta \mid z \leq -\delta + \delta^2\right\}. \end{aligned}$$

We denote also by $\pi^+$ (resp. $\pi^-$) the radial projection centered at $p$ (resp. $n$) onto the boundary of the cylinder $c_\delta^p$ (resp. $c_\delta^n$). Let $a = u_x(0)$ and $b = u_y(0)$.

For $\delta$ sufficiently small and $z \in [\,-\delta + \delta^2\,,\, \delta - \delta^2\,]$, we consider the two following regular unit vector fields:

$$I(z) = \frac{u_x(0,\, 0,\, z)}{\left\|u_x(0,\, 0,\, z)\right\|}\,, \tag{5}$$

and

$$K(z) = u(0,\, 0,\, z)\,. \tag{6}$$

Since $u$ takes its values into the unit sphere, it is clear that $I(z)$ and $K(z)$ are orthogonal. If $b \neq 0$, $J(z)$ is choosed such that $\big(I(z),\, J(z),\, K(z)\big)$ be an orthonormal basis having the same orientation as $\big(a,\, b,\, u(0)\big)$. If $b = 0$, $J(z)$ is chose such that $\big(I(z), J(z), K(z)\big)$ is direct orthonormal basis.

The dipole function denoted by $u^\delta$ is defined as follows: $\forall\, (x,y,z) \in \Omega \setminus C_\delta$, $u^\delta(x,y,z) = u(x,y,z)$, $\forall\, (x,y,z) \in c_\delta^p$ (resp. $c_\delta^n$), $u^\delta$ is the composition of the radial projection $\pi^+$ (resp. $\pi^-$) and the value of $u$ on $\partial c_\delta^p$ (resp. $\partial c_\delta^n$) and for any $(x,y,z) \in c_\delta$, let $u^\delta(x,y,z)$ be the map constructed by H. Brezis and J.-M. Coron in [BC], that is: let $(r,\theta)$ be the polar coordinates of $(x,y)$, if $r < \delta^2$:

$$u^\delta(x,y,z) = \frac{2\lambda}{\lambda^2 + r^2}\big(xI(z) + yJ(z) - \lambda K(z)\big) + K(z)\,, \tag{7}$$

where $\lambda = c\delta^4$ ($c$ will be fixed later) and if $\delta^2 \leq r \leq 2\delta^2$,

$$\begin{aligned} u^\delta(x,y,z) = (A_1 r + B_1)I(z) + (A_2 r + B_2)J(z) + \\ + \sqrt{1 - \big(A_1 r + B_1\big)^2 - \big(A_2 r + B_2\big)^2}\, K(z)\,, \end{aligned} \tag{8}$$

where $A_i$ and $B_i$ depend only on $\theta$ and $\delta$ and are determined in such a way as to make $u^\delta$ continuous on $\Omega$. More precisely:

$$\begin{cases} 2\delta^2 A_i + B_i = u^i(2\delta^2\cos\theta\,,\ 2\delta^2\sin\theta\,,\ z) \quad \text{for } i = 1,\ 2\ ; \\ \delta^2 A_1 + B_1 = \dfrac{2\lambda\delta^2}{\lambda^2 + \delta^4}\cos\theta\ ; \\ \delta^2 A_2 + B_2 = \dfrac{2\lambda\delta^2}{\lambda^2 + \delta^4}\sin\theta\,, \end{cases} \tag{9}$$

where $u^i$ is the $i$-th coordinate of $u$ in $\big(I(z),\ J(z),\ K(z)\big)$.

## 2.2. Asymptotic analysis of $u^\delta$ as $\delta$ tends to zero

We recall here some estimates obtained in [R]. Set

$$\begin{aligned} c_\delta(i) &= \big\{(x,y,z) \in c_\delta \mid r < \delta^2\big\}\,, \\ c_\delta(e) &= \big\{(x,y,z) \in c_\delta \mid \delta^2 \leq r \leq 2\delta^2\big\}\,. \end{aligned}$$

By (9), we have:

$$\begin{aligned} 2\delta^2 A_i + B_i = u^i(0,\ 0,\ z) + 2\delta^2 u^i_x(0,\ 0,\ z)\cos\theta \\ + 2\delta^2 u^i_y(0,\ 0,\ z)\sin\theta + \mathrm{o}(\delta^4)\,, \end{aligned}$$

and $u_i(0,\ 0,\ z) = 0$, for $i = 1,\ 2$.

Moreover, we find:

$$\begin{cases} u^1_x(0,\ 0,\ z) = |a| + \mathrm{O}(\delta)\,, \\ u^1_y(0,\ 0,\ z) = \mathrm{O}(\delta)\,, \\ u^2_x(0,\ 0,\ z) = 0\,, \\ u^2_y(0,\ 0,\ z) = |b| + \mathrm{O}(\delta)\,. \end{cases} \tag{10}$$

Therefore

$$\begin{cases} A_1 = 2\big(|a| - c\big)\cos\theta + \mathrm{O}(\delta)\,, \\ B_1 = 2\delta^2\big(2c - |a|\big)\cos\theta + \mathrm{O}(\delta^3)\,, \\ A_2 = 2\big(|b| - c\big)\sin\theta + \mathrm{O}(\delta)\,, \\ B_2 = 2\delta^2\big(2c - |a|\big)\sin\theta + \mathrm{O}(\delta^3)\,. \end{cases} \tag{11}$$

We obtain exactly in the same way that:

$$\begin{cases} A_{1\theta} = -2\big(|a| - c\big)\sin\theta + \mathrm{O}(\delta)\,, \\ B_{1\theta} = -2\delta^2\big(2c - |a|\big)\sin\theta + \mathrm{O}(\delta^3)\,, \\ A_{2\theta} = 2\big(|b| - c\big)\cos\theta + \mathrm{O}(\delta)\,, \\ B_{2\theta} = 2\delta^2\big(2c - |b|\big)\cos\theta + \mathrm{O}(\delta^3)\,. \end{cases} \tag{12}$$

For $(x, y, z) \in c_\delta(e)$ we have:

$$u^{\delta,1} = \mathrm{O}(\delta^2)\,, \quad u^{\delta,2} = \mathrm{O}(\delta^2) \quad \text{and} \quad u^{\delta,3} = 1 + \mathrm{O}(\delta^2)\,, \tag{13}$$

$$u_z^{\delta,1} = \mathrm{O}(\delta^2)\,, \quad u_z^{\delta,2} = \mathrm{O}(\delta^2) \quad \text{and} \quad u_z^{\delta,3} = \mathrm{O}(\delta^4)\,, \tag{14}$$

$$u_r^{\delta,3} = \mathrm{O}(\delta^2)\,, \quad \frac{1}{r}\, u_\theta^{\delta,3} = \mathrm{O}(\delta^2)\,. \tag{15}$$

### 2.3 Proof of Lemma 1 completed

In [R], T. Rivière has obtained the following estimate:

$$\begin{aligned} E(u^\delta) \le E(u) &+ 16\,\pi\delta + \\ &+ 8\pi\delta^5 \Big[4c^2 - \big(|a|^2 + |b|^2 + 8c^2 - 4c|a| - 4c|b|\big)\ln 2\Big] + \\ &+ \mathrm{O}\big(-\delta^6 \ln(\delta)\big)\,. \end{aligned} \tag{16}$$

We claim that:

$$\int_\Omega \nabla u^\delta \nabla f = \int_\Omega \nabla u \nabla f + \mathrm{O}(\delta^6)\,. \tag{17}$$

Once the estimate (17) is proved, the assertion (iii) of Lemma 1 follows by choosing $c = \frac{1}{2}\max\big(|a|\,,\ |b|\big)$ as in [BC].

*Proof of (17).* — Let $f^i$ be the $i$-th coordinates of $f$ in $\big(I(z),\ J(z),\ K(z)\big)$, then we have:

$$\int_{C_\delta} \nabla u^\delta \nabla f = 8\pi\delta^5 \Big(|a| f_x^1(0) + |b| f_y^2(0) + K'(0)\cdot f_z(0)\Big) + \mathrm{O}(\delta^6)\,, \tag{18}$$

where $(\,\cdot\,)$ denotes the scalar product in $\mathbb{R}^3$.

Indeed,

$$\begin{aligned} \int_{C_\delta} \nabla u^\delta \nabla f &= |C_\delta| \nabla u^\delta \nabla f(0) + \mathrm{O}(\delta^6) \\ &= 8\pi(\delta + \delta^2)\delta^4 \big(u_x \cdot f_x(0) + u_y \cdot f_y(0) + u_z \cdot f_z(0)\big) + \mathrm{O}(\delta^6)\,. \end{aligned}$$

On the other hand, since $(I, J, K)$ is an orthonormal basis, combining (5) and (6) we have:

$$u_x \cdot f_x(0) = |a| I(0) \cdot f_x(0) = |a| f_x^1(0)$$

and

$$u_y \cdot f_y(0) = |b| f_y^2(0) .$$

Thus (18) follows. Secondly, to estimate the left hand side of (17), we claim that:

$$\int_{c_\delta} \nabla u^\delta \nabla f = 8\pi\delta^5 \big( |a| f_x^1(0) + |b| f_y^2(0) + K'(0) \cdot f_z(0) \big) + \mathrm{O}(\delta^6) , \qquad (19)$$

$$\int_{c_\delta^p} \nabla u^\delta \nabla f = \mathrm{O}(\delta^6) , \qquad (20)$$

$$\int_{c_\delta^n} \nabla u^\delta \nabla f = \mathrm{O}(\delta^6) . \qquad (21)$$

Verification of (19): for the simplicity, we denote by $A = \lambda^2 + r^2$. We write:

$$\int_{C_\delta} \nabla u^\delta \nabla f = \int_{c_\delta(i)} \nabla u^\delta \nabla f + \int_{c_\delta(e)} \nabla u^\delta \nabla f = \mathrm{I} + \mathrm{II} ,$$

so

$$\mathrm{I} = \int_{c_\delta(i)} (u_x^\delta f_x + u_y^\delta f_y + u_z^\delta f_z) . \qquad (22)$$

But in the cylinder $c_\delta(i)$, by (7), we have:

$$u_x^\delta f_x = \frac{2\lambda}{A} \left( \left( 1 - \frac{2x^2}{A} \right) f_x^1 - \frac{2xy}{A} f_x^2 + \frac{2\lambda x}{A} f_x^3 \right) . \qquad (23)$$

Direct computation shows that:

$$\int_{c_\delta(i)} u_x^\delta f_x = 4\pi c f_x^1(0) \delta^5 + \mathrm{O}(\delta^6) ;$$

we get also:

$$\int_{c_\delta(i)} u_y^\delta f_y = 4\pi c f_y^2(0) \delta^5 + \mathrm{O}(\delta^6) .$$

By (7), we have:

$$u_z^\delta = \frac{2\lambda x}{A}\, I'(z) + \frac{2\lambda y}{A}\, J'(z) + \frac{r^2-\lambda^2}{A}\, K'(z)$$

which implies that:

$$\int_{c_\delta(i)} u_z^\delta f_z = 2\pi\delta^5\big(K'(0)\cdot f'(0)\big) + \mathrm{O}(\delta^6)\,.$$

Therefore,

$$I = 4\pi\delta^5\left(cf_x^1(0) + cf_x^2(0) + \frac{1}{2}\, K'(0)\cdot f'(0)\right) + \mathrm{O}(\delta^6)\,. \tag{24}$$

In the exterior domain $c_\delta(e)$, we estimate II as above. First, we have:

$$\int_{c_\delta(e)} u_x^\delta f_x = \sum_{i=1}^{3}\int_{c_\delta(e)} u_x^{\delta,i} f_x^i\,.$$

By (15), it is clear that:

$$\int_{c_\delta(e)} u_x^{\delta,3} f_x^3 = \int_{c_\delta(e)}\left(u_r^{\delta,3}\cos\theta - \frac{1}{r}\, u_\theta^{\delta,3}\sin\theta\right) f_x^3 = \mathrm{O}(\delta^6)\,.$$

Using (11) and (12), we find:

$$\begin{aligned}\int_{c_\delta(e)} u_x^{\delta,2} f_x^2 &= \int_{c_\delta(e)}\left(A_2\cos\theta - \left(A_{2\theta} + \frac{B_{2\delta}}{r}\right)\sin\theta\right) f_x^2\\ &= \int_{c_\delta(e)} \frac{\delta^2\big(|b|-2c\big)}{r}\, f_x^2\sin(2\theta) = \mathrm{O}(\delta^6)\,.\end{aligned}$$

and

$$\begin{aligned}\int_{c_\delta(e)} u_x^{\delta,1} f_x^1 &= \int_{c_\delta(e)}\left(A_1\cos\theta - \left(A_{1\theta} + \frac{B_{1\theta}}{r}\right)\sin\theta\right) f_x^1\\ &= 4\pi\big(2|a|-c\big) f_x^1(0)\delta^5 + \mathrm{O}(\delta^6)\,.\end{aligned}$$

So

$$\int_{c_\delta(e)} u_y^\delta f_y = 4\pi\big(2|b|-c\big) f_y^2(0)\delta^5 + \mathrm{O}(\delta^6)\,.$$

For the last term, combining (13) and (14), we obtain:

$$\begin{aligned}\int_{c_\delta(e)} u_z^\delta f_z &= \int_{c_\delta(e)} u^{\delta,3} K'(z)\cdot f_z + \mathrm{O}(\delta^6)\\ &= 6\pi\delta^5\big(K'(0)\cdot f_z(0)\big) + \mathrm{O}(\delta^6).\end{aligned}$$

Finally,

$$II = 8\pi\delta^5\left(\left(|a| - \frac{c}{2}\right) f_x^1(0) + \left(|b| - \frac{c}{2}\right) f_y^2(0) + \frac{3}{4}K'(0)\cdot f_z(0)\right) + \mathrm{O}(\delta^6). \tag{25}$$

Together with (24), we get:

$$\int_{C_\delta} \nabla u^\delta \nabla f = 8\pi\delta^5\big(|a| f_x^1(0) + |b| f_y^2(0) + K'(0)\cdot f_z(0)\big) + \mathrm{O}(\delta^6), \tag{26}$$

and the claim (19) is proved.

Verification of (20): we divide $c_\delta^p$ into two parts. Let $G$ be $\left(\pi^+\right)^{-1}\big((\partial c_\delta^p)\cap(\partial C_\delta)\big)$, $G$ is a litter cone of vertex $p$. Let $H$ be the complement of $G$ in $c_\delta^p$, thus

$$\int_{c_\delta^p} \nabla u^\delta \nabla f = \int_G \nabla u^\delta \nabla f + \int_H \nabla u^\delta \nabla f.$$

$u^\delta = u$ on $\partial H \setminus (\partial G \cap \partial H)$, since $u$ is regular, we conclude that:

$$\int_H \nabla u_\delta \nabla f = \mathrm{O}(\delta^6), \tag{27}$$

and $\partial G\setminus(\partial G\cap\partial H)$ is the horizontal disk $D_{2\delta^2}$ centered at $(0,\,0,\,\delta-\delta^2)$ and of radius $2\delta^2$. We have $D_{2\delta^2} = D_{\delta^2}\cup(D_{2\delta^2}\setminus D_{\delta^2})$. Set $G_1 = \left(\pi^+\right)^{-1}(D_{\delta^2})$ and $G_2 = G\setminus G_1$. On $D_{2\delta^2}\setminus D_{\delta^2}$, $\left|\nabla_{x,y}u^\delta(x,\,y,\,\delta-\delta^2)\right|$ is uniformly bounded. Thus

$$\int_{G_2} \nabla u^\delta \nabla f = \mathrm{O}(\delta^6). \tag{28}$$

On the other hand, for any $(x,y,z)\in G_1$, we have:

$$\pi^+(x,y,z) = \left(\frac{\delta^2}{\delta - z}\,x\,,\ \frac{\delta^2}{\delta - z}\,y\,,\ \delta - \delta^2\right),$$

hence

$$\begin{aligned}u^\delta(x,y,z) &= u\big(\pi^+(x,y,z)\big)\\ &= \frac{2\lambda}{\lambda^2 + r'^2}\left(\frac{\delta^2}{\delta - z}\,xI(z_0) + \frac{\delta^2}{\delta - z}\,yJ(z_0) - \lambda K(z_0)\right) + K(z_0),\end{aligned} \tag{29}$$

where $z_0 = \delta - \delta^2$ and $r'^2 = \big(\delta^4/(\delta - z)^2\big)(x^2 + y^2)$. We can also write:

$$u^\delta(x,y,z) = \frac{2\lambda'}{\lambda'^2 + r^2}\big(xI(z_0) + yJ(z_0) - \lambda' K(z_0)\big) + K(z_0)\,, \tag{30}$$

where $\lambda' = c\delta^2(\delta - z)$. Therefore,

$$\begin{aligned}
&\int_{G_1} u_x^\delta f_x = \\
&= \int_{z_0}^{\delta} \mathrm{d}z \int_{r<\delta-z} \frac{2\lambda'}{A'}\left(\left(1 - \frac{2x^2}{A'}\right) I(z_0) - \frac{2xy}{A'}\, J(z_0) + \frac{2\lambda' x}{A'}\, K(z_0)\right) f_x \\
&= \int_{z_0}^{\delta} 2\pi c\delta^2(\delta - z) I(z_0)\cdot f_x(0)\,\mathrm{d}z + \mathrm{O}(\delta^6) = \mathrm{O}(\delta^6)\,,
\end{aligned} \tag{31}$$

with $A' = \lambda'^2 + r^2$, we obtain in the same way that:

$$\int_{G_1} u_y^\delta f_y = \mathrm{O}(\delta^6)\,. \tag{32}$$

Moreover, by (30), we have:

$$u_z^\delta = -2(\delta - z)\frac{r^2 - \lambda'^2}{A'^2}\big(xI(z_0) + yJ(z_0)\big) + \frac{4c\delta^2 r^2 \lambda'^2}{A'^2}\, K(z_0)\,.$$

Thus

$$\int_{G_1} u_z^\delta f_z = \mathrm{O}(\delta^6)\,. \tag{33}$$

Therefore the estimate (20) follows from the combination of (27), (28), (31), (32) and (33). In the same way, we can establish (21). The claim (17) is then proved.

COROLLARY 1.— *Suppose that $u$ is a mapping in $R^1(\Omega, S^2)$ with non zero degree, then $u$ is not a minimizer of (a) in Theorem 2. In particular, we have:*

$$\alpha = \inf\big\{E(u - f) \mid u \in C^1(\overline{\Omega}, S^2)\big\} < 8\pi L(f)\,. \tag{34}$$

*Proof.*— Assuming our assertion fails. Let $\underline{u}$ be a such minimizer. By Lemma 1, there exists a dipole function $v$ in $R^1(\Omega, S^2)$ such that:

$$E(v-f) < E(\underline{u}-f) + 8\pi|p-n|,$$

where $\{p,n\}$ are the new point singularities of $v$.

Moreover, we can place the points $p$ and $n$ such that $L(v) = L(\underline{u}) - |p-n|$. That means

$$E(v-f) + 8\pi L(v) < E(\underline{u}-f) + 8\pi L(\underline{u}),$$

which contradicts the fact that $\underline{u}$ is a minimizer.

## 3. Proof of Theorem 1

For the proof of Theorem 1, we use the following lemma due to [BB].

LEMMA 2.— *Let*

$$\beta = \inf_A \int_{S^2} \left|\nabla_T(\varphi-\psi)\right|^2,$$

*where* $A = \{\varphi \in H^1(S^2,S^2) \mid \deg(\varphi) = 0\}$ *and* $\nabla_T$ *denotes the tangential gradient. Then* $\alpha = \beta$.

*Proof of Lemma 2.*— Let $\varphi_n$ be a minimizing sequence for $\beta$, i.e.:

$$\deg(\varphi_n) = 0 \quad \text{and} \quad \int_{S^2} \left|\nabla_T(\varphi-\psi)\right|^2 = \beta + o(1),$$

since $C^1(S^2,S^2)$ is dense in $H^1(S^2,S^2)$ (see [SU2]) and the function $\deg(\varphi)$ is continuous for strong topology of $H^1(S^2,S^2)$. We can as well assume that $\varphi_n \in C^1(S^2,S^2)$, $\forall\, n \in \mathbb{N}$.

Set $v_n = \varphi_n(x/|x|)$, then $v_n \in H^1(\Omega,S^2)$ and $v_n \in C^1(\Omega\setminus\{0\},S^2)$ with $\deg(v_n,0) = 0$. Thus we can construct (by [BZ, Lemma 5]) a sequence $\omega_n \in C^1(\overline{\Omega},S^2)$ such that $\|v_n-\omega_n\|_{H^1} \to 0$ as $n\to\infty$. Therefore

$$\begin{aligned}
\alpha &\le \int_\Omega \left|\nabla\left(\omega_n - \varphi\Big(\frac{x}{|x|}\Big)\right)\right|^2 = \\
&= \int_\Omega \left|\nabla\left(v_n - \psi\Big(\frac{x}{|x|}\Big)\right)\right|^2 + o(1) \\
&= \int_{S_2} \left|\nabla_T(\varphi_n-\psi)\right|^2 + o(1) = \beta + o(1).
\end{aligned}$$

Thus $\alpha \leq \beta$. On the other hand, we have:

$$\int_\Omega \left|\nabla\left(v-\psi\Big(\frac{x}{|x|}\Big)\right)\right|^2 \geq \int_\Omega \left|\nabla_T\left(v-\psi\Big(\frac{x}{|x|}\Big)\right)\right|^2 \geq \beta\,,$$

$\forall\ v \in C^1(\overline{\Omega}, S^2)$, which proves Lemma 2.

*Proof of Theorem 1 completed.*— Let $u$ be any minimizer of (4). By Theorem 2, there exists a sequence $(v_n) \in C^1(\overline{\Omega}, S^2)$ such that $v_n \to u$ weakly in $H^1$ and $F(v_n) \to F(u)$. Moreover $v_n$ is also a minimizing sequence for $\alpha$ (see (b), Theorem 2). Thus,

$$\begin{aligned}\alpha + o(1) &= \int_\Omega \left|\nabla\left(v_n-\psi\Big(\frac{x}{|x|}\Big)\right)\right|^2\\ &= \int_\Omega \left|\nabla\left(v_n-\psi\Big(\frac{x}{|x|}\Big)\right)\right|^2 + \int_\Omega \left|\frac{\partial v_n}{\partial r}\right|^2\\ &\geq \beta + \int_\Omega \left|\frac{\partial v_n}{\partial r}\right|^2 .\end{aligned}$$

Using Lemma 2 and passing to the limit, we obtain that $\partial u/\partial r = 0$. Therefore $u$ is the form of $\omega(x/|x|)$ with $\omega \in H^1(S^2, S^2)$. We deduce from Corollary 1 that $\deg(\omega) = 0$. Since $L\big(\omega(x/|x|)\big) = \big|\deg(\omega)\big| = 0$ ([BB, Lemma 3]), $v_n \to \omega(x/|x|)$ strongly in $H^1$. Furthermore $\omega$ realises the infimum for $\beta$. In fact, we have:

$$\beta = \alpha = \int_\Omega \left|\nabla\left(u-\psi\Big(\frac{x}{|x|}\Big)\right)\right|^2 = \int_{S^2} |\nabla(\omega-\psi)|^2 .$$

By a result of [M] (see also [SU1]), $\omega$ is a regular function. Using the fact that $\beta = \alpha$ and $\beta$ is achieved and

$$\int_\Omega |\nabla f|^2 \geq 8\pi L(f)\,,$$

We conclude as in [BB] that $\alpha$ is not achieved.

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