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A Picard method without Lipschitz continuity for some ordinary differential equations[*]

FRANCISCO BERNIS[1] et MAN KAM KWONG[2]

RÉSUMÉ. — On présente un théorème d'existence et d'unicité pour des problèmes à valeurs initiales pour l'équation $y^{(m)} = f(x,y)$, où $f(x,y)$ peut ne pas satisfaire la condition de Lipschitz usuelle, et même f peut être discontinue en y. Dans la preuve on utilise une modification de la méthode de Picard. Finalement, un résultat de non-existence montre que les hypothèses du théorème sont optimales.

ABSTRACT. — An existence and uniqueness theorem is presented for initial value problems involving the equation $y^{(m)} = f(x,y)$, where $f(x,y)$ may not satisfy the usual Lipschitz condition and even may be noncontinuous in y. A modified Picard method is used for the proof. Finally, a nonexistence result shows that the hypotheses of the theorem are sharp.

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KEY-WORDS : Existence, uniqueness, nonexistence, Picard method, ordinary differential equation

1. Introduction and statement of results

We present the main ideas of this note by means of the following illustrative example of initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = x^\alpha y^p & x > 0 \\ y(0) = y'(0) = 0 & y''(0) = 1. \end{cases}$$

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Consider first the case $\alpha \geq 0$ and $0 < p < 1$. Since $y(0) = 0$, the usual Lipschitz condition does not hold and no general uniqueness theorems seem to be applicable (see more details below). Then we found a uniqueness proof if $p > -(\alpha+1)/2$. Hence, negative values of p and α are also allowed. It turns out that under the same hypotheses (Theorem 1.1 below) we also obtain an existence proof. The main idea is to observe that a solution must behave as $x^2/2$ as $x \to 0$ and exploit the fact that the equation is of higher order. Actually, the solution can be obtained by appropriate Picard iterations, since we perform the proof by means of the contraction mapping theorem.

We also give a nonexistence result (Theorem 3.1) which shows that the hypotheses of Theorem 1.1 are sharp. For the above example Theorem 3.1 states that no solution exists if $p \leq -(\alpha+1)/2$, i.e. either we have existence and uniqueness or we have nonexistence.

All the above remains true if we replace the equation $y''' = x^\alpha y^p$ by the equation $y''' = -x^\alpha y^p$.

We proceed to the full statement of our result on existence and uniqueness. In the sequel:

I) α, p and b_i are real numbers;

II) m, i and j are integers; and

III) δ, M_0 and M are positive constants.

All these quantities are given data, while the positive number ε will be chosen along the proof and will depend on these data.

We consider the initial value problem

$$y^{(m)} \equiv \frac{\mathrm{d}^m y}{\mathrm{d}x^m} = f(x,y)\,, \quad x > 0 \tag{1.1}$$

$$y^{(i)}(0) = b_i\,, \quad 0 \leq i \leq m-1 \tag{1.2}$$

$$y \in C^{m-1}[0\,,\varepsilon] \cap C^m(0\,,\varepsilon]\,. \tag{1.3}$$

We introduce the following hypotheses on the real-valued function f

$$f(x,y) \quad \text{is continuous in } (0,\delta) \times (0,\delta) \tag{1.4}$$

$$\bigl|f(x,y)\bigr| \leq M_0 x^\alpha y^p\,, \quad 0 < x,\, y < \delta \tag{1.5}$$

$$\big|f(x,y)-f(x,\bar{y})\big| \le M\frac{x^\alpha|y-\bar{y}|}{\big(\min\{y,\bar{y}\}\big)^{1-p}}, \quad 0<x,\, y,\, \bar{y}<\delta \text{ if } p<1 \tag{1.6}$$

$$\big|f(x,y)-f(x,\bar{y})\big| \le Mx^\alpha\big(\max\{y,\bar{y}\}\big)^{p-1}|y-\bar{y}|, \quad 0<x,\, y,\, \bar{y}<\delta \text{ if } p\ge 1\,. \tag{1.7}$$

Notice that a sufficient condition for (1.6)-(1.7) is

$$\left|\frac{\partial f(x,y)}{\partial y}\right| \le Mx^\alpha y^{p-1}, \quad 0<x,\, y<\delta\,. \tag{1.8}$$

Simple examples of functions satisfying (1.4)-(1.7) are

$$f(x,y) = \pm x^\alpha y^p\,.$$

A more complicated example is

$$f(x,y) = x^\alpha \sin\frac{1}{x^\beta}\, y^{p+q}\sin\frac{1}{y^q}, \quad \beta>0\,,\ q>0\,.$$

This example shows that functions satisfying (1.4)-(1.7) may change sign infinitely many times both near $x=0$ and near $y=0$.

The following theorem will be proved in Section 2.

THEOREM 1.1.— *Let $m\ge 2$, $1\le j\le m-1$,*

$$b_i = 0 \text{ if } 0\le i\le j-1 \quad \text{and} \quad b_j>0\,. \tag{1.9}$$

Assume that f satisfies (1.4)-(1.7) and that

$$p > -\frac{\alpha+1}{j}\,. \tag{1.10}$$

Then there exists $\varepsilon>0$ such that Problem (1.1)-(1.3) has a unique solution.

This result seems to be different from the existence and uniqueness criteria given in the literature; cf. Hartman [H], and Agarwal and Lakshmikantham [AL]. A group of uniqueness theorems (those of Kamke, Nagumo, Osgood, etc.) improve the usual Lipschitz condition, but they do not allow a behaviour as y^p near $y=0$ if $p<1$, while in Theorem 1.1 p may be less than 1 and even negative. Furthermore, Theorem 1.1 is not a Wend type theorem (see [AL]), since it is not based on monotonicity hypotheses on f and even f may change sign infinitely many times. It is also clear that our hypotheses have nothing to do with the dissipative hypotheses of Peano type theorems. The fact that (1.1) is a higher order differential equation is an important aspect, as in the uniqueness result of [BM, Theorem 2.1].

In a different spirit, [KK1], [KK2] and references therein obtain uniqueness results for first order problems such us

$$y' = p(x)y^p + q(x)\,, \quad y(0) = 0$$

with $0 < p < 1$. In these problems the zero solution is excluded by the term $q(x)$ of the equation.

We come back to Theorem 1.1 to add some comments. Although f may not have constant sign, the solution must be positive near $x = 0$ because of (1.9). Hence $f(x, y)$ may not be defined for $y < 0$. Notice also that the data $b_{j+1}, \ldots, b_{m-1}$ may be negative. The condition $b_j > 0$ is essential: if all the b_i are zero and $p < 1$ it is well-known that uniqueness does not hold. For example,

$$y'' = y^p\,, \quad y(0) = y'(0) = 0 \quad \text{with } 0 < p < 1$$

has power solutions in addition to the zero solution.

If $j = 0$ the methods of this paper do not give any new result. This is why $m = 1$ and $j = 0$ are excluded in Theorem 1.1.

Finally, let us remark that entirely similar theorems hold when $x < 0$ and/or $b_j < 0$.

2. Proof of Theorem 1.1

In this section we assume the hypotheses of Theorem 1.1 and $\varepsilon < \delta$.

The integral equation associated to the initial value problem (1.1)-(1.3) is

$$y(x) = P(x) + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f(t, y(t))\, \mathrm{d}t \tag{2.1}$$

where

$$P(x) = \sum_{i=j}^{m-1} \frac{1}{i!} b_i x^i\,. \tag{2.2}$$

For simplicity of notation we set

$$b = \frac{1}{j!} b_j \tag{2.3}$$

and write $P(x)$ in the form

$$P(x) = bx^j + x^j Q(x) \tag{2.4}$$

where $Q(x)$ is a polynomial which satisfies $Q(0) = 0$.

Now we consider the new unknown function

$$z(x) = \frac{y(x)}{x^j}. \tag{2.5}$$

By (1.3) and (1.9) the function $z(x)$ is continuous up to $x = 0$ if we set $z(0) = b$. Hence our initial value problem is reduced to find a function $z \in C[0, \varepsilon]$ such that $z(0) = b$ and z satisfies the integral equation

$$z(x) = (Tz)(x) \tag{2.6}$$

where T is defined by

$$(Tz)(x) = b + Q(x) + \frac{1}{(m-1)!} \int_0^x \frac{(x-t)^{m-1}}{x^j} f\big(t, t^j z(t)\big)\, dt. \tag{2.7}$$

Notice that the function $\left|f\big(t, y(t)\big)\right| = \left|f\big(t, t^j z(t)\big)\right|$ is integrable in $(0, \varepsilon)$ for ε small enough, because by (1.5)

$$\left|f\big(t, t^j z(t)\big)\right| \le M_0 t^{\alpha + pj}\, z(t)^p \tag{2.8}$$

and $\alpha + 1 + pj > 0$ by (1.10). This *assures that the integral equation problem for z is indeed equivalent to Problem (1.1)-(1.3).*

Next we consider the metric space

$$E = C\left([0, \varepsilon]; \left[\frac{b}{2}, \frac{3b}{2}\right]\right)$$

with the usual distance:

$$d(z, \bar{z}) = \sup_{0<x<\varepsilon} \left|z(x) - \bar{z}(x)\right|.$$

For simplicity we assume that $\varepsilon < 1$. Therefore

$$\frac{(x-t)^{m-1}}{x^j} \le 1 \quad \text{if } 0 < t < x \le \varepsilon\,. \tag{2.9}$$

Recall that $\varepsilon < \delta$. We also assume that

$$\varepsilon^j \frac{3b}{2} < \delta$$

in order to assure that $x^j z(x) < \delta$ for all $x \in [\,0\,,\,\varepsilon\,]$. Hence (1.5)-(1.7) can be applied for all $x \in [\,0\,,\,\varepsilon\,]$ if we set $y = x^j z(x)$ and $\bar{y} = x^j \bar{z}(x)$ with $z, \bar{z} \in E$. In particular, (2.8) holds for all $t \in [\,0\,,\,\varepsilon\,]$ if $z \in E$.

LEMMA 2.1.— *There exists $\varepsilon > 0$ such that T maps E into E.*

Proof.— Let $z \in E$. Consider the integral term of (2.7), namely

$$(Jz)(x) \equiv \frac{1}{(m-1)!} \int_0^x \frac{(x-t)^{m-1}}{x^j} f\big(t, t^j z(t)\big)\,\mathrm{d}t\,. \tag{2.10}$$

From (2.8) and (2.9) it follows that the integral (2.10) is absolutely convergent and $(Jz)(x)$ is continuous for all $x \in [0, \varepsilon]$. Observing that

$$z(t)^p \le \begin{cases} (3b/2)^p & \text{if } p \ge 0 \\ (b/2)^p & \text{if } p < 0 \end{cases}$$

we obtain that for all $x \in [\,0\,,\,\varepsilon\,]$

$$\big|(Jz)(x)\big| \le C_0 \frac{M_0}{(m-1)!}\, b^p \varepsilon^{\alpha+1+pj}$$

where C_0 is a positive constant depending only on j, p and α. Hence we may choose ε so that

$$\big|(Jz)(x)\big| \le \frac{1}{4}\, b \quad \text{for all } x \in [\,0\,,\,\varepsilon\,]\,. \tag{2.11}$$

Since $Q(0) = 0$ we may also choose ε such that

$$\big|Q(x)\big| \le \frac{1}{4}\, b \quad \text{for all } x \in [\,0\,,\,\varepsilon\,]\,. \tag{2.12}$$

From (2.11), (2.12) and (2.7) we obtain that

$$\big|(Tz)(x) - b\big| \le \frac{1}{2}\, b \quad \text{for all } x \in [\,0\,,\,\varepsilon\,]$$

and hence $Tz \in E$. This completes the proof of Lemma 2.1.

LEMMA 2.2.— *There exists $\varepsilon > 0$ such that T is a contraction mapping from E into E.*

Proof.— Let z, $\bar{z} \in E$. Observe that if $p \ge 1$

$$\Big(\max\big\{t^j z(t)\,,\, t^j \bar{z}(t)\big\}\Big)^{p-1} \le t^{j(p-1)} \left(\frac{3b}{2}\right)^{p-1}$$

while if $p < 1$

$$\Big(\min\big\{t^j z(t)\,,\, t^j \bar{z}(t)\big\}\Big)^{p-1} \le t^{j(p-1)} \left(\frac{b}{2}\right)^{p-1} .$$

Hence, from (1.6) and (1.7) it follows that for all $t \in [\,0\,,\,\varepsilon\,]$,

$$\Big|f\big(t, t^j z(t)\big) - f\big(t, t^j \bar{z}(t)\big)\Big| \le CMb^{p-1} t^{\alpha+pj} \big|z(t) - \bar{z}(t)\big|$$

where C is a positive constant depending only on j, p and α. This and (2.9) imply that

$$d(Tz, T\bar{z}) \le K\varepsilon^{\alpha+1+pj}\, d(z, \bar{z})$$

with

$$K = \frac{C}{\alpha + 1 + pj}\, \frac{M}{(m-1)!}\, b^{p-1} .$$

Recalling that $\alpha + 1 + pj > 0$ by (1.10), this inequality implies Lemma 2.2.

By Picard-Banach fixed point theorem, it follows that the integral equation (2.6)-(2.7) has a unique solution $z \in C[\,0\,,\,\varepsilon\,]$ with $z(0) = b$. This completes the proof of Theorem 1.1.

3. Nonexistence theorem

THEOREM 3.1.— *Let $m \geq 2$, $1 \leq j \leq m-1$. Assume that the numbers b_i satisfy (1.9), while f satisfies (1.4) and*

$$\left|f(x,y)\right| \geq M_1 x^{\alpha} y^{p}, \quad 0 < x, y < \delta$$

where M_1 is a positive constant. Then Problem (1.1)-(1.3) has no solution if

$$p \leq -\frac{\alpha+1}{j}.$$

Example.— Under the above hypotheses on m, j and the b_i, let

$$f(x,y) = \pm x^{\alpha} y^{p}.$$

Then Problem (1.1)-(1.3) has a (local) solution *if and only if* $p > -(\alpha+1)/j$. This solution is unique.

Proof of Theorem 3.1.— Assume for contradiction that there exists a solution y defined in some interval $[0, \varepsilon]$. From (1.3) and (1.9) it follows that near $x = 0$

$$C_1 x^j \leq y(x) \leq C_2 x^j$$

where C_1 and C_2 are positive constants. Using the lower bound for y if $p \geq 0$ and the upper bound if $p < 0$ we obtain that near $x = 0$

$$\left|y^{(m)}(x)\right| \geq M_1 x^{\alpha} y(x)^p \geq C_3 x^{\alpha+pj}, \quad C_3 > 0.$$

Since $\alpha + 1 + pj \leq 0$ this implies that $y^{(m-1)}$ is unbounded, which contradicts (1.3).

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