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On the structure of certain Weingarten surfaces with boundary a circle(\textasteriskcentered)

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1. Introduction

We study in this paper a certain class of surfaces $M$ in $\mathbb{R}^3$ satisfying a Weingarten relation of the form

$$H = f(H^2 - K)$$

where $H$ is the mean curvature, $K$ is the Gaussian curvature and $f$ is a real smooth function defined on a interval $[-\varepsilon, \infty)$, $\varepsilon > 0$.

Furthermore, we require that $f$ satisfies the inequality

$$4t(f(t))^2 < 1.$$
We call such a function \( f \), elliptic, when it satisfies (2). The reason for this denomination is that equation (1) and inequality (2) give rise to a fully nonlinear elliptic equation. We call \( M \) a special surface when \( M \) satisfies \( H = f(H^2 - K) \) for \( f \) elliptic. They have been studied by Hopf [8], Hartman and Wintner [7], Chern [5] and Bryant [3]. Here, we extend some results for constant mean curvature surfaces obtained in [2] and [6], when \( M \) is topologically a disk. Precisely we prove the following theorems.

**Theorem 1.** — Let \( M \) be a disk type special surface immersed in \( \mathbb{R}^3 \). Assume \( \partial M \) is a circle \( S^1 \) of radius 1. Suppose \( f \) is analytic with \( f(0) > 0 \). Then:

- a) \( f(0) \leq 1 \),
- b) if \( f(0) = 1 \), \( M \) is a halfsphere.

**Theorem 2.** — Let \( M \) be a disk type special surface embedded in \( \mathbb{R}^3 \). Assume \( \partial M \) is a circle \( S^1 \) of radius 1 contained in the horizontal plane \( \mathcal{H} = \{ z = 0 \} \). Suppose \( f > 0 \), \( f(0) > 0 \) and \( M \) cuts transversely \( \mathcal{H} \) along \( \partial M \). Then \( M \) is a spherical cap.

We remark that the ellipticity condition (2) on \( M \) allow us to apply maximum principle (for special surfaces) and Alexandrov reflection principle techniques as it was applied in [6] and [10], for constant mean curvature surfaces (see Hopf’s book [8] for further details). Furthermore, we notice that R. Bryant constructed a global quadratic form \( Q \) on a surface \( M \) satisfying (1) such that the zeros of \( Q \) are the umbilical points of \( M \) (see [3]). These facts emphasize the analogy between special surfaces and constant mean curvature surfaces. Now we state and prove the maximum principle for special Weingarten surfaces in \( \mathbb{R}^3 \) satisfying (1) and (2) in the form we shall need: if \( M_1, M_2 \) are tangent at \( p \), \( M_1 \), on one side of \( M_2 \) near \( p \), both \( M_1, M_2 \) satisfying (1) and (2) with respect to the same normal \( N \) at \( p \) then \( M_1 = M_2 \) near \( p \). By a standard argument \( M_1 = M_2 \) everywhere.

**1.1 Interior maximum principle**

Suppose \( M_1, M_2 \) are \( C^2 \) surfaces in \( \mathbb{R}^3 \) which are given as graphs of \( C^2 \) functions \( u, v : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R} \).

Suppose the tangent planes of both \( M_1, M_2 \) agree at a point \((x, y, z)\); i.e. \( T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2 \) for \( z = u(x, y) = v(x, y), \ (x, y) \in \Omega \).
Let $H(N_1)$ and $H(N_2)$ be the mean curvature functions of $u$ and $v$ with respect to unit normals $N_1$ and $N_2$ that agree at $(x, y, z)$. Let $K_i$, be the Gaussian curvature of $M_i$, $i = 1, 2$.

Suppose $M_i$ satisfy

$$H(N_i) = f(H_i^2 - K_i), \quad i = 1, 2,$$

for $f$ satisfying (2).

If $u \leq v$ near $(x, y)$ then $M_1 = M_2$ near $(x, y, z)$, i.e. $u = v$ in a neighbourhood of $(x, y)$.

1.2 Boundary maximum principle

Suppose $M_1$, $M_2$ as in the statement of the interior maximum principle with $C^2$ boundaries $B_1$, $B_2$ given by restrictions of $u$ and $v$ to part of the boundary $\partial \Omega$.

Suppose $T_{(x,y,z)}M_1 = T_{(x,y,z)}M_2$ and $T_{(x,y,z)}B_1 = T_{(x,y,z)}B_2$ for $z = u(x,y) = v(x,y)$, with $(x, y, z)$ in the interior of both $B_1$ and $B_2$.

Suppose $M_1$, $M_2$ satisfy (1) and (2) with respect the same normal $N$ at $(x, y, z)$.

If $u \leq v$ near $(x, y)$ then $M_1 = M_2$ near $(x, y, z)$, i.e. $u = v$ in a neighbourhood of $(x, y)$.

2. Proof of the interior and boundary maximum principle

Clearly, by applying a rigid motion of $\mathbb{R}^3$ which does not change the geometry of the statements, we may suppose the tangent planes of both $M_1$, $M_2$ at $(x, y, z)$ are the horizontal $xy$ plane $P = \{z = 0\}$, and the unit normals $N_1$, $N_2$ at $(x, y, z)$ are equal to $N = (0, 0, 1)$.

First, we fix some notations. We denote

$$p_1 = \frac{\partial u}{\partial x}, \quad q_1 = \frac{\partial u}{\partial y},$$

$$p_2 = \frac{\partial v}{\partial x}, \quad q_2 = \frac{\partial v}{\partial y},$$

$$r_1 = \frac{\partial^2 u}{\partial x^2}, \quad \tau_1 = \frac{\partial^2 u}{\partial y^2}, \quad s_1 = \frac{\partial^2 u}{\partial x \partial y},$$

$$r_2 = \frac{\partial^2 v}{\partial x^2}, \quad \tau_2 = \frac{\partial^2 v}{\partial y^2}, \quad s_2 = \frac{\partial^2 v}{\partial x \partial y}. $$
With this convention the normals $N_1$ and $N_2$ are given by

$$N_i = \frac{1}{(1 + p_i^2 + q_i^2)^{1/2}} (-p_i, -q_i, 1), \quad i = 1, 2.$$ 

The mean curvature $H_i$ and the Gaussian curvature $K_i$ are given by

$$2H_i = \frac{1}{(1 + p_i^2 + q_i^2)^{3/2}} \left( (1 + p_i^2) \tau_i - 2p_i q_i s_i + (1 + q_i^2) r_i \right)$$
$$K_i = \frac{1}{(1 + p_i^2 + q_i^2)^2} (r_i \tau_i - s_i^2)$$

for $i = 1, 2$.

We may write equation (1) for $M_1$ and $M_2$ in the following way

$$F(p_i, q_i, r_i, s_i, \tau_i) = H_i - f(H_i^2 - K_i) = 0 \quad (3)$$

for $i = 1, 2$, where $F$ is a $C^1$ function in the $p, q, r, s, \tau$ variables. We fix $(x, y) \in \Omega$ and we define for $t \in [0, 1]$:

$$\alpha(t) = F(tp_1 + (1 - t)p_2, tq_1 + (1 - t)q_2, tr_1 + (1 - t)r_2, ts_1 + 1(1 - t)s_2, t\tau_1 + (1 - t)\tau_2). \quad (4)$$

Let $w = u - v$.

By applying the mean value theorem, using equation (3) and differentiating equation (4) we are led to the linearized operator on $\Omega$ defined by

$$Lw := \frac{\partial F}{\partial r}(\xi) \frac{\partial^2 w}{\partial x^2} + \frac{\partial F}{\partial s}(\xi) \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial F}{\partial \tau}(\xi) \frac{\partial^2 w}{\partial y^2} +$$
$$+ \frac{\partial F}{\partial p}(\xi) \frac{\partial w}{\partial x} + \frac{\partial F}{\partial q}(\xi) \frac{\partial w}{\partial y} = 0 \quad (5)$$

where

$$\xi = (p, q, r, s, \tau)$$

$$p = cp_1 + (1 - c)p_2, \quad q = cq_1 + (1 - c)q_2$$

$$r = cr_1 + (1 - c)r_2, \quad s = cs_1 + (1 - c)s_2, \quad \tau = c\tau_1 + (1 - c)\tau_2$$
for $0 < c(x, y) < 1$. Notice that the principal part of $L$ is given by the symmetric matrix

$$A = A(p, q, r, s, \tau) = \begin{bmatrix}
\frac{\partial F}{\partial r} & \frac{1}{2} \frac{\partial F}{\partial s} \\
\frac{1}{2} \frac{\partial F}{\partial s} & \frac{\partial F}{\partial \tau}
\end{bmatrix}$$

Computations show that if $p = q = 0$ then trace $A = 1$ and

$$\det A = \frac{1}{4} \left(1 - 4t(f'(t))^2\right),$$

where

$$t = \frac{\left(1 + p^2\right)r - 2pq + (1 + q^2)s}{2(1 + p^2 + q^2)^{3/2}} - \frac{1}{(1 + p^2 + q^2)^2}(r^2 - s^2). \tag{6}$$

Now, consider in formula (6)

$$p = cp_1 + (1 - c)p_2, \quad q = cq_1 + (1 - c)q_2$$

$$r = cr_1 + (1 - c)r_2, \quad s = cs_1 + (1 - c)s_2, \quad \tau = c\tau_1 + (1 - c)\tau_2$$

where $p_i, q_i, r_i, s_i$ and $\tau_i$ are varying in a neighbourhood of $(x, y)$ and $c$ is varying in the interval $[0, 1]$. We see easily that the non negative quantity $t = t(p, q, r, s, \tau)$ is bounded from above. Hence $1 - 4t(f'(t))^2 \geq \mu > 0$ in this neighbourhood ($c$ is varying between 0 and 1), for some positive real number $\mu$. As $p_i = q_i = 0$ at $(x, y), i = 1, 2$, by continuity we have that in a neighbourhood $V$ of $(x, y)$ the matrix $A(\xi)$ is positive definite. Furthermore, there is a positive real number $\lambda_0$ such that

$$\frac{\partial F}{\partial r}(\xi)\eta_1^2 + \frac{\partial F}{\partial s}(\xi)\eta_1\eta_2 + \frac{\partial F}{\partial \tau}(\xi)\eta_2^2 \geq \lambda_0(\eta_1^2 + \eta_2^2)$$

for any $(x, y)$ in $V$ and any real numbers $\eta_1, \eta_2$. Consequently, $L$ is a linear second order uniformly elliptic operator with bounded coefficients in a neighbourhood of $(x, y)$. The same conclusion holds if $(x, y)$ is a boundary point as in the hypothesis of the boundary maximum principle statement.

Finally we have in a neighbourhood of $(x, y)$

$$Lw = 0, \quad w \leq 0, \quad w(x, y) = 0.$$
If \((x, y)\) is an interior point then \(w = u - v = 0\) in a neighbourhood of \((x, y)\), by applying the interior maximum principle of Hopf.

If \((x, y)\) is a boundary point lying in the interior of a \(C^2\) portion contained in \(\Omega\), then \(w\) attains again a local maximum at \((x, y)\) with \((\partial w / \partial \nu)(x, y) = 0\), where \(\nu\) is the exterior unit normal to \(\Omega\) at \((x, y)\). This implies by using the boundary maximum principle of Hopf that \(w = 0\) in a neighbourhood of \((x, y)\), as desired. We conclude the proof of the maximum principle for special Weingarten surfaces in \(\mathbb{R}^3\).

We remark that the maximum principle above leads to an Alexandrov theorem for special Weingarten surfaces. That is, a closed embedded special Weingarten surface \(M\) given by equation (1) with respect to a unit global normal \(N\), for \(f\) elliptic, is a sphere. Hence, \(f(0) \neq 0\) and \(M\) is a sphere of radius \(R = 1/|f(0)|\).

### 3. Proof of Theorem 1

We consider \(M\) an immersed smooth special surface in \(\mathbb{R}^3\) and \(N\) an unit normal vector field. We denote by \(\langle \cdot, \cdot \rangle\) the inner product in \(\mathbb{R}^3\) and by \(\nabla\) the standard covariant derivative in \(\mathbb{R}^3\). The mean curvature vector \(\overrightarrow{H}\) of \(M\) at \(p\) is given by

\[
\overrightarrow{H}(p) = \frac{\lambda_1(p) + \lambda_2(p)}{2} N(p)
\]

where \(\lambda_1(p), \lambda_2(p)\) are the principal curvatures of \(M\) at \(p\) (respecting to \(N\)).

#### 3.1 Proof of assertion a)

Suppose first that there is an umbilical boundary point \(p \in \partial M\). Denote by \(v\) a unit tangent field along \(\partial M = S^1\). Then,

\[
f(0) = H(p) = \langle \nabla_v v, N \rangle_p \leq 1.
\]  

(3.1)

Suppose now there are no umbilical points on the boundary. Notice that the set \(U\) of umbilical points of \(M\) is finite. Otherwise \(M\) is a spherical cap and \(f(0) \leq 1\). This follows from the proof of theorem 3.2 of H. Hopf’s book [8, p. 142], and from the fact that \(M\) is compact.

Let \(\lambda_1, \lambda_2 : M \setminus U \to \mathbb{R}\) be the principal curvature functions with \(\lambda_1 < \lambda_2\) on \(M \setminus U\). Let us prove first that ellipticity condition yields

\[
\lambda_2 > f(0) \quad \text{on} \quad M \setminus U.
\]  

(3.2)
Indeed,  
\[ \lambda_2 = H + \sqrt{H^2 - K} = f(H^2 - K) + \sqrt{H^2 - K} \]
and the ellipticity condition  
\[ 4t(f'(t))^2 < 1 \]
assures  
\[ g(t) = f(t) + \sqrt{t} \]
is a monotonic increasing function for \( t \geq 0 \).

Denote by \( \mathcal{F}_2 \) the principal line distribution on \( M \setminus U \) associated to the principal curvature \( \lambda_2 \). Clearly, there is a point \( p \in \partial M \) where \( \mathcal{F}_2 \) is tangent to \( \partial M \) at \( p \), i.e. \( T_p \partial M = \mathcal{F}_2(p) \). If not we would obtain a line foliation of \( M \) transverse to \( \partial M \) and finite number (possibly none) of singularities of negative indices (see [8]); this is impossible since \( M \) has disk topological type. Choose then \( p \in \partial M \) such that \( T_p \partial M = \mathcal{F}_2(p) \).

Clearly  
\[ \lambda_2(p) = \langle \nabla_v v, N \rangle_p \leq 1 \quad (3.3) \]
by inequalities (3.1), (3.2), (3.3)  
\[ f(0) \leq 1. \]
This proves assertion a). \( \square \)

3.2 Proof of assertion b)

Notice first that there is an extension for \( M \) beyond \( \partial M \) satisfying \( H = f(H^2 - K) \), \( f \) elliptic and analytic. This is so, because of the boundary regularity for the underlying analytic elliptic partial differential equation ([4], [11]). If \( f(0) = 1 \) we will show that there are infinitely many umbilical points in \( \partial M \). The resulting non-discreteness of \( U \) will so imply \( M \) is totally umbilical [8].

Suppose by absurd \( \partial M \) has finitely many umbilical points. Observe that the foliation \( \mathcal{F}_2 \) defined on \( M \setminus U \) is transverse to \( \partial M \setminus U \). To prove this, suppose \( p \in \partial M \setminus U \) is such that \( \mathcal{F}_2(p) \) is tangent to \( \partial M \setminus U \). By equations (3.2) and (3.3), we derive a contradiction because \( f(0) < \lambda_2(p) \leq 1 \).

Suppose now, there are no umbilical points on the boundary \( \partial M \). This means (by what we have just proved) that \( \mathcal{F}_2 \) is transverse to \( \partial M \). In this
case $\mathcal{F}_2$ may be seen as a foliation of $M$ with finite number of singularities with negative index [8]. This is a contradiction since by our hypothesis $M$ is a topological disk.

For the case where $\partial M$ has a non zero finite number of umbilical points, consider a umbilical point $p \in \partial M$, and let $\widetilde{M}$ to be an extension of $M$ beyond the boundary $\partial M$.

We first see that $p$ is a singularity of $\mathcal{F}_2$ with negative index and finite number of separatrices, all of them smooth at $p$. Moreover, there is at least one separatrix going from $p$ to the interior of $M$. In other words there is at least one separatrix such that, its interior tangent vector at $p$, say $u$, satisfies $\langle u, \eta \rangle > 0$, where $\eta$ is the interior co-normal of $M$ at $p$. This is a consequence of a straightforward computation using Bryant holomorphic quadratic form [3] such that, in a neighbourhood of $p$, the foliation is diffeomorphically equivalent to the standard foliation

$$\Im z^n (dz)^2 = 0$$

on the complex $z$-plane.

Observe now that the foliation $\mathcal{F}_2$ on $M \setminus U$ is topologically equivalent to a foliation with finite number of singularities on $M$. Some of them are interior singularities on $M$. Others are in the boundary $\partial M$. Those which are in the boundary have separatrices (at least one) coming tranversally to $\partial M$ (fig. 1). In order to see this situation is topologically impossible, we just recall $M$ is a topological disk and use double construction to obtain a foliation of a topological sphere $S^2$ with finite number of singularities, all of them with negative index.

This concludes the proof of Theorem 1. $\square$

\[ \text{Fig. 1} \]
4. Proof of Theorem 2

Suppose without loss of generality that \( M \) is locally contained in the upper halfspace \( \mathcal{H}^+ = \{ z \geq 0 \} \) in a neighbourhood of \( \partial M \). We also identify \( \partial M \) with the unit circle \( S^1 \) centered at the origin of \( \mathcal{H} \).

We first show that boundary roundness determines the behavior of the mean curvature vector \( \overrightarrow{H} \) along the boundary (in fact, only convexity of \( \partial M \) is required). Precisely we state the follows result.

**Claim 1.** — Let \( p \in \partial M \). Then \( \langle \overrightarrow{H}(p), p \rangle < 0. \)

**Proof of Claim 1**

Suppose first that there is a umbilical point \( p \in \partial M \). Take a unit vector field \( v \) tangent to \( \partial M \). Then umbilicity yields

\[
H(N) = \langle \nabla_v v, N \rangle_p
\]

If \( N = \overrightarrow{H}/|H| \) then the mean curvature \( H \) is positive and \( \langle \nabla_v v, N \rangle = |H| > 0. \) So \( \langle -p, M \rangle > 0 \), as desired, for \( \nabla_v v = -p \) is the acceleration vector of \( S^1 \).

For the case where there is no umbilical points on \( \partial M \) we recall that the foliation \( \mathcal{F}_2 \), parallel to the line field associated to the bigger principal curvature \( \lambda_2 \) defined over \( M \setminus U \), has to be tangent to \( \partial M = S^1 \) in some point \( p \). Let \( p \in \partial M \) be such that \( \mathcal{F}_2(p) \) is tangent to \( \partial M \). Clearly

\[
\lambda_2(p) = \left\langle \nabla_v v, \frac{\overrightarrow{H}}{|H|} \right\rangle_p > 0.
\]

Notice that Claim 1 means the following: the orthogonal projection of the mean curvature vector \( \overrightarrow{H} \) on \( \mathcal{H} \) points into the interior of the planar domain \( D \) contained in \( \mathcal{H} \) bounded by \( \partial M \). We will denote \( D \) by \( \text{int} \, \partial M \).

We now define \( M_1 \subset M \) to be the connected component of \( M \cap \mathcal{H}^+ \) which contains \( \partial M \).

**Claim 2.** — \( M_1 \cap \mathcal{H} \subset \text{int} \, \partial M \).

This follows from Claim 1 and from Alexandrov Reflection Principle techniques used exactly in the same way it was used in the proof of Theorem 1 of [6, p. 337].
Let us denote \( C_{f(0)} \) the vertical cylinder on \( \mathcal{H} \) over the circle \( S_{f(0)} \) of radius \( 1/f(0) \) centered at the origin.

**Claim 3.** There is a point \( p \in \partial M \) such that

\[
\langle N, -p \rangle_p \geq f(0) \quad \text{for} \quad N = \frac{\overrightarrow{H}}{|H|}.
\]

This means there is a point \( p \in \partial M \) where the surface \( M \) has bigger (or equal) inclination respect to \( xy \) plane than the small spherical cap of radius \( 1/f(0) \) bounding \( \partial M \).

**Proof of Claim 3**

Let \( p \in \partial M \) be a point of \( \partial M \) where \( F_2(p) \) is tangent to \( \partial M \) at \( p \) (proof of Claim 1). Then, at this point \( p \) we have

\[
\langle -p, N \rangle_p = \langle \nabla_v v, N \rangle_p = \lambda_2(p) \geq f(0).
\]

**Claim 4.** If \( \text{ext} \ C_{f(0)} \) denotes the exterior of the cylinder \( C_{f(0)} \) (i.e. it is the connected region of \( \mathbb{R}^3 - C_{f(0)} \) not containing the origin of \( \mathcal{H} \)), if \( M \cap \text{ext} \ C_{f(0)} = \emptyset \), then \( M \) is a spherical cap.

**Proof of Claim 4**

The proof follows by using Claim 3 and the maximum principle (for special surfaces), comparing \( M_1 \) with a half sphere of radius \( 1/f(0) \) (see for instance [1]).

**Claim 5.** If \( M_1 \cap \text{int} \partial M = \emptyset \), then \( M \) is a spherical cap.

**Proof of Claim 5**

First notice, if \( M_1 \cap \text{int} \partial M = \emptyset \) then, by Claim 2 it follows \( M_1 \cap \mathcal{H} = \partial M \) and \( M \) is globally contained in \( \mathcal{H}^+ \). Now, using Alexandrov Reflection Principle for planes normal to \( \mathcal{H} \), we conclude \( M \) is rotationally symmetric (see, for instance [10]). Therefore, the round boundary is everywhere parallel to one of the principal curvature directions for \( M \). Now because \( M \) is a topological closed disk, we conclude, by the same index reasons as before, that \( M \) is totally umbilical. This shows that \( M \) is a spherical cap (of radius \( 1/f(0) \)).

We finish the proof of Theorem 2 supposing, by contradiction, that

\[
M_1 \cap (\text{ext} \ C_{f(0)}) \neq \emptyset \quad \text{and} \quad M_1 \cap \text{int} \partial M \neq \emptyset.
\]
At this point we may suppose $M$ to be globally transverse to $\mathcal{H}$ without loss of generality. Therefore $M \cap \mathcal{H}$ is a finite collection of closed simple curves of $\mathcal{H}$.

Notice first that under the contradiction hypothesis there should be a curve in $\gamma \in M \cap \mathcal{H} \setminus \partial M$ which is homotopically non trivial in $\mathcal{H} \setminus \partial M$. This follows directly from the extended Graph Lemma for special surfaces (Lemma 3, Remark and final Remarks in [2, pp. 12, 14]).

Let $\gamma_L \in M \cap \mathcal{H}$ be the outermost homotopically non trivial curve in $\mathcal{H} \setminus \partial M$. Observe that $\gamma_L$ bounds a topological disk $D_L \subset M$. Moreover, $D_L$ is locally contained in the upper half-space $\mathcal{H}^+$ along its boundary $\gamma_L$. In fact, if the disk $D_L$ were locally contained in the lower halfspace $\mathcal{H}^-$ we would have a connected component, say $C$, of $M \setminus (M \cap \text{Int}\partial M)$ such that $C \cap \mathcal{H}$ contains at least two distinct closed curves both of them homotopically non trivial in $\mathcal{H} \setminus \partial M$. This is a consequence of the fact that $M_1$ is locally contained in $\mathcal{H}^+$ along its boundary together with the hypothesis that the mean curvature vector $\vec{H}$ never vanishes and the maximum principle. This would lead to a contradiction by applying Alexandrov Reflection Principle by vertical planes as in [6].

Notice that $D_L \cap \mathcal{H}$ is the union of $\gamma_L$ with null homotopic closed curves on $\mathcal{H} \setminus \gamma_L$, and as a consequence of the Graph Lemma proved in [2, Lemma 3, pp. 12-14, Remark, p. 14] each curve on $D_L \cap \mathcal{H} \setminus \gamma_L$ other than $\gamma_L$ bounds a graph over its Jordan interior. We denote the Jordan interior of $\gamma_L$ in $\mathcal{H}$ by $\text{int} \gamma_L$. Now a standard orientation argument yields (since $H \neq 0$ on $M$):

$$D_L \cap (\text{int} \gamma_L) = \emptyset.$$ 

So $D_L \cup \text{int} \gamma_L$ is embedded (non smooth over $\gamma_L$) compact surface without boundary. Moreover $M_1$ is clearly contained in the closed compact solid $S$ determined by $D_L \cup \text{int} \gamma_L = \partial S$ (fig. 2).

Let $M_1(\theta)$, $0 \leq \theta \leq 2\pi$, be the 1-parameter family of surfaces obtained by rotating $M_1 = M_1(0)$ around an axis $z$ normal to $\mathcal{H}$ and passing by the center of the round circle $S_1$ bounding $M$. Clearly $M_1(\theta) \cap D_L = \emptyset$, for every $\theta \in [0, 2\pi]$. Otherwise there would be a first parameter $\theta_0 > 0$ such that $M_1(\theta_0)$ would be tangent to $D_L \setminus \gamma_L$, and contained inside $S$, contradicting the maximum principle for special surfaces.
Now, let $p \in M_1$ be a point of maximum distance of $M_1$ to the $z$-axis, contained in the interior of the solid $S$. The radius of this circle $C_1$ is bigger than $1/f(0)$ because of the hypothesis of contradiction. Also $D_L \cap D_1 = \emptyset$, where $D_1$ is the horizontal disk bounding $C_1$. This is again a consequence of mean curvature orientation and maximum principle.

We now finish the contradiction argument by comparing $D_L$ with a sphere of radius $1/f(0)$ which we can actually introduce through the barrier disk $D_1$. This proves Theorem 2. □

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