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Persistence of Homoclinic Tangencies for
Area-Preserving Maps(*)

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RÉSUMÉ. — Nous prouvons que dans une variété symplectique bidimensionnelle $M$, l’existence de courbes lisses invariantes dans le monde des applications symplectiques de $M$ est un mécanisme pour créer des ouverts contenant un ensemble dense d’applications exhibant des tangences homocliniques.

ABSTRACT. — In a 2-dimensional symplectic manifold $M$ we show that the presence of smooth invariant curves in the world of symplectic maps of $M$ is a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies.

1. Introduction

In 1970, S. Newhouse [N2] proved the existence of an open set $U \subset \text{Diff}^s(M)$, $s \geq 2$, where $M$ is a 2-dimensional compact manifold, with the following property: there exists a dense subset of $U$ such that each $g : M \rightarrow$ in this subset exhibits homoclinic tangencies (tangential intersections between the stable set and unstable set, $W^s(p)$ and $W^u(p)$ respectively, of a hyperbolic periodic point $p$). We call such a set $U \subset \text{Diff}^s(M)$, with the last property, an open set of “persistence of homoclinic tangencies”, from now on OSPHT.

Later, in 1979 [N3], he proved that a mechanism to create this kind of sets is the unfolding of a dissipative homoclinic tangency. More precisely, for every $f \in \text{Diff}^s(M)$, with a homoclinic tangency associated to a dissipative hyperbolic periodic point $p$ ($|\det Df^n(p)| < 1$, where $n$ is the minimal period of $p$), there exists $U$ an OSPHT such that $f \in \overline{U}$.

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Here we present a mechanism to generate OSPHT's in the world of symplectic diffeomorphisms; we show that the presence of a smooth invariant curve generates, for nearby maps, this kind of open sets. To be more precise, let $M$ be a 2-dimensional compact manifold with $w$ a symplectic 2-form on $M$ and denote by $\text{Diff}_w^s$ the space of $C^s$ diffeomorphisms that preserve $w$, then we have the following result.

**Theorem 1.** Let $f \in \text{Diff}_w^\infty(M)$ admit a $C^\infty$ closed invariant curve $\gamma$ such that the rotation number $\omega = p(f|_\gamma)$ is irrational. Then for every $s \geq 4$ there exists $\mathcal{U} \subset \text{Diff}_w^s(M)$ an OSPHT such that $f \in \overline{\mathcal{U}}$. Moreover, there is a residual subset $\mathcal{V}$ of $\mathcal{U}$ such that every $f \in \mathcal{V}$ has an invariant smooth curve which is accumulated by elliptic points.

The method to prove Theorem 1 is different from the dissipative case. The wild hyperbolic sets mechanism used to produce persistence of homoclinic tangencies is replaced by the rich structure around a smooth invariant curve, obtained from KAM theory [Bo], combined with the following two propositions.

**Proposition 2.** For $f \in \text{Diff}_w^\infty(M)$ and $\gamma$ a $C^\infty$ invariant curve assume that:

(i) $\omega$ satisfies a diophantine condition: there exist $\beta \geq 0$ and $C > 0$ such that for every $p/q \in \mathbb{Q}$ then

$$\left| \omega - \frac{p}{q} \right| > \frac{C}{q^{2+\beta}};$$

(ii) $f$ satisfies a twist condition along $\gamma$ (see Sect. 2),

(iii) there exist $\tilde{\mathcal{U}} \subset \text{Diff}_w^s(M)$, such that for each $g \in \tilde{\mathcal{U}}$ there is a continuation curve $\gamma_g$ of $\gamma$ which is invariant by $g$ and with the same rotation number $\omega$.

Then there exists $\mathcal{U} \subset \tilde{\mathcal{U}}$ an OSPHT and for a residual set in $\mathcal{U}$, the continuation curve $\gamma_g$ is the limit of elliptic periodic orbits.

**Remark.** The same conclusion can be obtained in Proposition 2 if we replace the invariant curve $\gamma$ by a collection of disjoint curves $\{\gamma_i\}_{i=0}^{n-1}$ such that $f(\gamma_i) = \gamma_{i+1}$ and $f(\gamma_{n-1}) = \gamma_0$. Just take $f^n$, apply Proposition 2 and pull back $\mathcal{U}$ by the map $f \rightarrow f^n$. 

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Proposition 3.— Let \( f \in \text{Diff}_w^{\infty}(M) \). Then for each \( s \geq 1 \), we have:

(i) if \( f \) exhibits a \( C^\infty \) invariant curve with an irrational rotation number, then for each \( \varepsilon > 0 \) there exists \( \tilde{f} \) \( C^s\varepsilon \)-near to \( f \) such that \( \tilde{f} \) exhibits homoclinic tangencies;

(ii) if \( f \) exhibits a homoclinic tangency associated to a hyperbolic periodic orbit, then for each \( \varepsilon > 0 \) there exists \( \tilde{f} \) \( C^s\varepsilon \)-near to \( f \) such that \( \tilde{f} \) has a generic (in the KAM sense) elliptic periodic point; in particular \( \tilde{f} \) exhibits \( C^\infty \) invariant curves.

The same conclusion of Theorem 1 holds if we replace the assumption of the presence of an invariant curve by the presence of some homoclinic tangency associated to a hyperbolic periodic point \( p \).

Corollary 4.— Assume that \( f \in \text{Diff}_w^s(M), s \geq 4 \), has a hyperbolic periodic point \( p \) and that \( f \) exhibits a homoclinic tangency associated to \( p \), then there exists \( U \subset \text{Diff}_w^s(M) \) an OSPHT such that \( f \in \bar{U} \). Moreover, there is a residual subset \( V \) of \( U \) such that every \( f \in V \) has an invariant smooth curve which is accumulated by elliptic points.

A consequence of Corollary 4 is the creation of infinitely many elliptic islands accumulating KAM curves. However, these elliptic points do not accumulate at the hyperbolic point which unfolds the homoclinic tangency. A related question in the unfolding of a homoclinic tangency is whether the OSPHT's can be constructed generating elliptic islands which accumulate at the hyperbolic periodic point. Some partial results concerning the previous question were obtained in [D]. Moreover, it seems possible to answer the question above by using [MR] and the methods of proof in the dissipative case.

This paper is organized as follows: In Section 2, Birkhoff's normal form and KAM theorem are recalled. The proof of Proposition 3, using some tools of [Z], is presented in Section 3. Finally, in Section 4 we prove Proposition 2 and Theorem 1.

2. Birkhoff's normal form theorem and KAM theorem

Let \( f \) be an area-preserving \( C^r \) diffeomorphism of the annulus \( \mathbb{A} = \mathbb{S}^1 \times \mathbb{R} \), with \( r \geq 4k + 4 \) and \( k \geq 0 \); here and in what follows we identify \( \mathbb{S}^1 \) with
\( \mathbb{S}^1 \times \{0\} \). Assume that \( f(\mathbb{S}^1) = \mathbb{S}^1 \) and that \( f|_{\mathbb{S}^1} = R_\omega \) the rotation with angle \( \omega \). So we can write
\[
f(\theta, r) = (\theta + \omega + ra(\theta, r), rb(\theta, r)).
\]
We say that \( \omega \in \mathbb{R} \) satisfies a diophantine condition if there exist \( \beta \geq 0 \) and \( C > 0 \) such that for every \( p/q \in \mathbb{Q} \) then \( |\omega - p/q| > C/q^{2+\beta} \).

Let \( D(C, \beta) \) be the set of these numbers with \( C \) and \( \beta \) fixed. We recall that the set \( D(\beta) = \bigcup_{C \geq 0} D(C, \beta) \) has total Lebesgue measure, i.e.,
\[
m(D(\beta) \cap [0, 1]) = 1 \text{ when } \beta > 0.
\]

The following version of Birkhoff’s normal form theorem says that if \( \omega \) satisfies a diophantine condition then after an area-preserving change of coordinates the term \( ra(\theta, r) \) in (1) can be written as a polynomial function in \( r \) plus higher order terms in \( r \). More precisely, letting
\[
\mathbb{A}_\delta = \{ (\theta, r) \mid \theta \in \mathbb{S}^1, |r| < \delta \},
\]
we have the following result.

**Theorem 5.** For each \( n \leq k \) there exists \( h_n : \mathbb{A}_\delta \to \mathbb{A} \) a \( C^{\infty} \) area-preserving map letting \( \mathbb{S}^1 \) invariant and such that \( \hat{f}_n = h_n^{-1} \circ f \circ h_n \) has the following form
\[
\hat{f}_n(\theta, r) = (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})).
\]

**Proof.** For a proof in the \( C^{\infty} \) case see appendices 1 and 2 of [Do]. The finite-differentiability case follows the same lines as the \( C^{\infty} \) case but it is necessary to use lemma 8.1 of [H].

**Remark.** In the case that \( f \) is \( C^{\infty} \) all the changes of coordinates are also \( C^{\infty} \), and we can choose \( n \) as large as we want.

Now consider a \( C^{\infty} \) symplectic diffeomorphism \( \tilde{f} \in \text{Diff}^{\infty}_\omega \) with an invariant \( C^{\infty} \) curve \( \gamma \). We define the twist condition along \( \gamma \) as follows: we say that \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if there exists a transversal unit vector field \( X \) on \( \gamma \) such that \( \omega(\text{D}\tilde{f}X(p), X(\tilde{f}(p))) > 0 \) for all \( p \in \gamma \). When \( \rho(\tilde{f}|_\gamma) \) satisfies a diophantine condition it is well known that after a symplectic change of coordinates, \( \tilde{f} \) restricted to a neighborhood \( V \) of \( \gamma \) has the form (1) with \( X(\theta, 0) = (0, 1) \). In this case a symplectic diffeomorphism of the annulus \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if and only if
\[
a_1 = \int a(\theta, 0) \, d\theta \neq 0.
\]
This number does not depend on the symplectic change of coordinates used

Now we recall the KAM theorem and remark some facts that we will use

For each \((\nu, \mu) \in C^s(A_\delta, \mathbb{R})^2\) let \(T_{\nu, \mu} : A_\delta \rightarrow A\) be the map

\[
(\theta, r) \mapsto (\theta + t(r) + \nu(\theta, r), r + \mu(\theta, r)).
\]

**Theorem 6.** Let \(r_0 \in (-\delta, \delta)\) and assume:

(a) \(t > 0\), \(T_{\nu, \mu}\) satisfies a twist condition;

(b) \(\alpha = t(r_0) \in D(c, \beta)\), \(\alpha = t(r_0)\) satisfies a diophantine condition;

(c) \(T_{\nu, \mu}\) satisfies the intersection property for every \((\nu, \mu)\) in a neighborhood of \((0, 0)\).

Let \(s > 2\beta + 3\), then there exists a neighborhood \(W\) in \(C^s(A_\delta, \mathbb{R})^2\) of

\[
(0, 0)
\]

such that, for all \((\nu, \mu) \in W\), one can find \(\gamma \in C^{s-2(1+\beta)}(S^1, \mathbb{R})\) and \(h \in \text{Diff}^{s-2(1+\beta)}(S^1)\) with

(i) \(\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in S^1\}\) is invariant under \(T_{\nu, \mu}\);

(ii) \(T_{\nu, \mu}|_{\Gamma}\) is \(C^{s-2(1+\beta)}\) conjugated to the rotation \(R_\alpha(\theta) = \theta + \alpha(\text{mod}1)\) by the following conjugation \(\theta \rightarrow (h(\theta), \gamma \circ h(\theta)).\)

See [Bo] and [SZ] for a proof.

**Remarks**

- The neighborhood \(W\) depends *a priori* on \(\alpha = t(r_0)\) (in fact on \((dt(r_0)/dr)^{-1}\)) but it can be proved that if \(r_0\) varies in a compact set \(K\), such that \(t(K) \subset D(\beta)\) then we can choose \(W\) depending just on \(K\). Because of \(D(\beta)\) has total Lebesgue measure, this is what gives the rich structure (lots ot other invariant curves) around an invariant curve.

- We have the following regularity statement: if \(\nu, \mu\) are \(C^\infty\) then \(\gamma\) is \(C^\infty\), see [SZ].

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3. Invariant curves and homoclinic tangencies

In this section our goal is to give the proof of Proposition 3, which in turn is a consequence of the following proposition.

**Proposition 7.** Let \( f : \mathbb{A}_\delta \to \mathbb{A} \) be a \( C^\infty \) area-preserving map of the annulus which leaves invariant some \( C^\infty \) curve where \( \Psi : \mathbb{S}^1 \to \mathbb{R} \), and such that \( f|_\Lambda \) has an irrational rotation number. Then for \( s > 1 \) and each \( \varepsilon > 0 \), \( f \) can be \( \varepsilon \)-approximated in the \( C^s \)-topology by one \( F \) which exhibits homoclinic tangencies and such that for some \( \delta' < \delta \) we have \( F|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})} = f|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})} \).

**Proof of Proposition 3**

Item (ii) follows from [N3], see also [MR], so we will only prove item (i). Because \( f \) and \( \gamma \) are \( C^\infty \), we can find a tubular neighborhood \( U \) of \( \gamma \) such that there is \( h : U \to \mathbb{A}_\delta \) for which \( h(\gamma) : \mathbb{S}^1 \times \{0\} \subset \mathbb{A}_\delta \) and \( h^*(d\theta \wedge dr) = \omega \). So making use of Proposition 7 the result follows. \( \square \)

To prove Proposition 7 we need first some preliminary results presented in the following subsection.

### 3.1 Preliminaries

Let \( f : \mathbb{A}_\delta \to \mathbb{A} \) be a \( C^\infty \) area-preserving map of the annulus which leaves \( \mathbb{S}^1 \) invariant, i.e., \( f(\mathbb{S}^1) = \mathbb{S}^1 \). We assume that \( f|_{\mathbb{S}^1} = R_\omega \) with \( \omega = p/n \) where \( p, n \) are relatively prime and

\[
\begin{align*}
f(\theta, r) &= (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})) \\
&= f_n(\theta, r) + O(r^{n+1}),
\end{align*}
\]

with \( a_1 > 0 \). Since \( f \) leaves \( \mathbb{S}^1 \) invariant (see [Do]) we have that locally around \( \mathbb{S}^1 \), \( f(\theta, r) = (\Theta, R) \) is described by a generating function \( h(\theta, R) \) in the following way

\[
f(\theta, r) = (\Theta, R) \quad \text{iff} \quad \begin{cases} r = \frac{\partial h}{\partial \theta}, \\
\Theta = \frac{\partial h}{\partial R}.
\end{cases}
\]
It is easy to check that

\[ h_n(\theta, R) = (\theta, \omega) R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} \]

is the generating function of \( f_n \). From this we get that the generating function of \( f \) has the form \( h(\theta, R) = h_n(\theta, R) + O(R^{n+2}) \).

We follow Moser and Zehnder to make a perturbation of \( f \). Consider the following two parameter family of generating functions

\[ h_{\varepsilon, \gamma}(\theta, R) = h(\theta, R) - \varepsilon R + \gamma \cos(2\pi n \theta) R^{n+1} \]

\[ = (\theta + \omega - \varepsilon) R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} + \gamma \cos(2\pi n \theta) R^{n+1} + O(R^{n+2}) \].

This family generates, for \( \varepsilon \) and \( \gamma \) small enough, the following two parameter family of diffeomorphisms \( f_{\varepsilon, \gamma} : \mathbb{A} \to \mathbb{A} \) with

\[ f_{\varepsilon, \gamma}(\theta, r) = (\theta + \omega - \varepsilon + a_1 r + a_2 r^2 + \cdots + a_n r^n +
\quad + (n + 1) \gamma \cos(2\pi \theta) r^n + O(r^{n+1}),
\quad r + 2\pi \gamma n \sin(2\pi \theta) (r^{n+1}) + O(r^{n+2})) \].

Observe that by the way we made the perturbation, \( S^1 \) continues to be invariant for the family \( f_{\varepsilon, \gamma} \).

**Proposition 8.** Assume that \( a_1 \neq 0 \), then for \( \varepsilon \) and \( \gamma \) small enough, \( f_{\varepsilon, \gamma} \) has two \( n \)-periodic orbits \( \{h_i(\varepsilon, \gamma)\}_{i=0}^{n-1}, \{e_i(\varepsilon, \gamma)\}_{i=0}^{n-1} \), which satisfy:

\( a \) \( \{h_i(\varepsilon, \gamma)\}_{i=0}^{n-1} \) is a hyperbolic \( n \)-periodic orbit with

\[ h_i(\varepsilon, \gamma) \rightarrow \left( \frac{i}{n}, 0 \right) \]

for \( \gamma \) fixed and \( \varepsilon \to 0 \);

\( b \) \( \{e_i(\varepsilon, \gamma)\}_{i=0}^{n-1} \) is an elliptic \( n \)-periodic orbit with

\[ e_i(\varepsilon, \gamma) \rightarrow \left( \frac{2i + 1}{2n}, 0 \right) \]

for \( \gamma \) fixed and \( \varepsilon \to 0 \);
(c) there exist $\delta > 0$ and
\[
\psi_{h_i}(\epsilon, \gamma) \in W^s_{\text{loc}}(h_i(\epsilon, \gamma)) \cap W^u_{\text{loc}}(h_{i+1}(\epsilon, \gamma))
\]
for which we have
\[
\psi_{h_i}(\epsilon, \gamma) \rightarrow \psi_{h_i}(0, \gamma) \in \left(\frac{2i+1}{2n} - \delta, \frac{2i+1}{2n} + \delta\right) \times \{0\},
\]
where $\delta$ does not depend on $\epsilon$ when this is small enough;
(d) the angle \[ L \left( T_{\psi_{h_i}} W^s_{\text{loc}}(h_i(\epsilon, \gamma)), T_{\psi_{h_i}} W^u_{\text{loc}}(h_{i+1}(\epsilon, \gamma)) \right) \rightarrow 0 \]
for $\gamma$ fixed and $\epsilon \to 0$.

The proof of this proposition is contained in [Z], so we only present the construction of the periodic points and shows how the homoclinic points are found and refer to [Z] for the rest of the details.

Proof. — We begin by making the following change of coordinates $\ell(\theta, \rho) = (\theta, \epsilon \rho) = (\theta, r)$ which allows us to see what happens in a microscopic neighborhood of $S^1$. In terms of $\theta$ and $\rho$, $\tilde{f} = \ell^{-1} \circ f \circ \ell$ is written as

\[
\tilde{f}_{\epsilon, \gamma}(\theta, \rho) = \left(\theta + \frac{\rho}{n} - \epsilon + a_1 \epsilon \rho + \cdots + a_n(\epsilon \rho)^n +
+ (n+1) \gamma \cos(2\pi \theta n)(\epsilon \rho)^n + O((\epsilon \rho)^{n+1}), \right.
\]

\[
\left. \rho + 2\pi \gamma n \sin(2\pi \theta n)\epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right).
\]

\[
= \left(\theta + \frac{\rho}{n} - \epsilon + a_1 \epsilon \rho + \cdots + a_n(\epsilon \rho)^n +
+ (n+1) \gamma \cos(2\pi \theta n)(\epsilon \rho)^n + O(\epsilon^{n+1}), \right.
\]

\[
\left. \rho + 2\pi \gamma n \sin(2\pi \theta n)\epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right).
\]

We get for the $n$-th iterate of $\tilde{f}_{\epsilon, \gamma}$ the following expression

\[
\tilde{f}_{\epsilon, \gamma}^n(\theta, \rho) = \left(\theta + p - n\epsilon + na_1 \epsilon \rho + O(\epsilon^2), \right.
\]

\[
\left. \rho + 2\pi \gamma n^2 \sin(2\pi \theta n)\epsilon^n \rho^{n+1} + O(\epsilon^{n+1}) \right),
\]

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where
\[
O(\varepsilon^2) = na_2 \varepsilon^2 \rho^2 + \cdots + na_n (\varepsilon \rho)^n + \\
+ n(n+1)\gamma \cos(2\pi \theta n)(\varepsilon \rho)^n + O(\varepsilon^{n+1}).
\]

The fixed points of \( \tilde{f}_{\varepsilon, \gamma}^n \) are the solutions of the equations
\[
\begin{align*}
\theta &= \theta + p - n\varepsilon + na_1 \varepsilon \rho + O(\varepsilon^2) \\
\rho &= \rho + 2\pi \gamma n^2 \sin(2\pi \theta n)\varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1}).
\end{align*}
\]

The fact that \( a_1 \neq 0 \) and the implicit function theorem imply that there exists \( \rho(\varepsilon) \) a solution of (6) which equals \( 1/a_1 \) when \( \varepsilon = 0 \). Using this solution in (7) we get \( 2n \) solutions \( \{ h_i(\varepsilon), e_i(\varepsilon) \} \) with \( i = 1, \ldots, n \) which equal
\[
\left( \frac{i}{n}, \frac{1}{a_1} \right), \quad \left( \frac{2i+1}{2n}, \frac{1}{a_1} \right)
\]
respectively when \( \varepsilon = 0 \). Since \( p, n \) are relative primes, the uniqueness part of the implicit function theorem gives that
\[
\{ e_i(\varepsilon) = (e_i(\varepsilon), \rho(\varepsilon)) \} \quad \text{and} \quad \{ h_i(\varepsilon) = (h_i(\varepsilon), \rho(\varepsilon)) \}
\]
are actually part of a \( n \)-periodic orbit. To determine the nature of these orbits we make another change of coordinates. Let \( \phi \) be any of the points \( \{ e_i, h_i \}_{i=0}^{n-1} \) and let \( \ell(\psi, \rho) = (\psi + \phi, \rho(\varepsilon) + \varepsilon^{(n-1)/2}x) \) then
\[
\tilde{f}_{\varepsilon, \gamma} = \tilde{\ell}^{-1} \circ \tilde{f}_{\varepsilon, \gamma}^n \circ \tilde{\ell}
\]
takes the following form
\[
\tilde{f}(\psi, x) = \left( \psi + \tilde{f}_1(\varepsilon, \psi, \varepsilon^{(n-1)/2}x), x + \tilde{f}_2(\varepsilon, \psi, \varepsilon^{(n-1)/2}x) \right),
\]
where
\[
\tilde{f}_1(\varepsilon, \psi, y) = -n\varepsilon + na_1 \varepsilon (\rho(\varepsilon) + y) + \cdots + na_n \varepsilon^n (\rho(\varepsilon) + y)^n + \\
+ n(n+1)\gamma \cos(2\pi \psi n)\varepsilon^n (\rho(\varepsilon) + y)^n + O(\varepsilon^{n+1})
\]
and
\[
\tilde{f}_2(\varepsilon, \psi, y) = (-1)^\sigma 2\pi \gamma n^2 \sin(2\pi \psi n)\varepsilon^{n-(n-1)/2} (\rho(\varepsilon) + y)^{n+1} + \\
+ O(\varepsilon^{n-(n-1)/2+1}),
\]
with \( \sigma = \pm 1 \) depending on the value of \( \phi \). From (9) and the fact \( \tilde{f}_1(\varepsilon, 0, 0) = (0, 0) \) we get
\[
\tilde{f}_1(\varepsilon, \psi, y) = \varphi_0(\varepsilon, \psi) + \varphi_1(\varepsilon, \psi)y + \varphi_2(\varepsilon, \psi)y^2
\]
with
\[ \varphi_0(\varepsilon, \psi) = \tilde{f}_1(\varepsilon, \psi, 0) = O(\varepsilon^{n+1}), \]
\[ \varphi_1(\varepsilon, \psi) = \frac{\partial \tilde{f}_1}{\partial y}(\varepsilon, \psi, 0) = a_1 \varepsilon + O(\varepsilon^2), \]
\[ \varphi_2(\varepsilon, \psi) = \frac{\partial^2 \tilde{f}_1}{\partial^2 y}(\varepsilon, \psi, \tilde{y}) = O(1) \]
and \( 0 \leq \tilde{y} \leq y \). All of these together imply that we can write
\[
\tilde{f}(\psi, x) = \left( \psi + n a_1 \varepsilon^{(n+1)/2} x + O(\varepsilon^{n/2+1}), \right.
\]
\[ x + (-1)^\sigma 2\pi \left( \frac{1}{a_1} \right)^{n+1} \gamma n^2 \sin(2\pi \psi n) \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1}) \) \]
\[
\text{(12)}
\]
Now from (12) the Jacobian matrix at \((0, 0)\) equals to
\[
J(\varepsilon) = \begin{pmatrix} 1 + O(\varepsilon^{n/2+1}) & na_1 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1}) \\ \mathcal{R} + O(\varepsilon^{n/2+1}) & 1 + O(\varepsilon^{n/2+1}) \end{pmatrix}
\]
where
\[
\mathcal{R} = (-1)^\sigma 4 \pi^2 \left( \frac{1}{a_1} \right)^{n+1} \gamma n^3 \varepsilon^{(n+1)/2}, \quad \sigma = \begin{cases} 1 & \text{at } \varepsilon_i \\ 0 & \text{at } \h_i \end{cases}
\]
From here it follows that
\[
\text{tr } J(\varepsilon) = 2 + O(\varepsilon^{n/2+1})
\]
\[
\det J(\varepsilon) = 1 - (-1)^\sigma 4 \pi^2 \left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{n+1} + O(\varepsilon^{n/2+1}) \}
\]
\[
\text{(13)}
\]
So we conclude from (13) that we have an elliptic orbit at \( \{ \varepsilon_i \}_{i=0}^{n-1} \) and a hyperbolic orbit at \( \{ \h_i \}_{i=0}^{n-1} \) with eigenvalues given by
\[
\lambda_s = 1 - \pi \sqrt{\left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1})}
\]
\[
\lambda_u = 1 + \pi \sqrt{\left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1})}
\]
\[
\text{(14)}
\]
The local stable (unstable) manifold $W_{\text{loc}}^{s(u)}(0)$ of $(0,0)$ is described by the following proposition, whose proof follows immediately from Proposition 1 and 2 of [Z].

**PROPOSITION 9.** There exist $C_1$, $C_2$ and $\varepsilon_0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the local stable (unstable) manifolds $W_{\text{loc}}^{s(u)}(0)$ are given in $|\psi| \leq 3/4n$ by

$$W_{\text{loc}}^{s(u)}(0) = \text{graph } g^{s(u)}$$

$$g^s(\varepsilon, \psi) = -\frac{2}{n} \sqrt{\frac{\gamma}{a_1^{n+2}}} \sin(\pi n \psi) + u^s(\varepsilon, \psi), \quad u^s(\varepsilon, 0) = 0$$

$$g^u(\varepsilon, \psi) = \frac{2}{n} \sqrt{\frac{\gamma}{a_1^{n+2}}} \sin(\pi n \psi) + u^u(\varepsilon, \psi), \quad u^u(\varepsilon, 0) = 0$$

where $|u^{s(u)}(\varepsilon, \psi)| < C_1 \varepsilon$, $\text{Lip}(u^{s(u)}) < C_2 \varepsilon$ and $u^{s(u)}(\varepsilon, 0) = 0$.

From this proposition it follows that the $W_{\text{loc}}^{s(u)}(h_i)$ are the graphs of functions $g_1^{s(u)}$ defined on an interval with center at $h_i$ and length equals $3\pi/4n$. To prove part (c) of Proposition 8, let us show that $W_{\text{loc}}^{u}(h_i(\varepsilon)) \cap W_{\text{loc}}^{s}(h_{i+1}(\varepsilon)) \neq \emptyset$. We argue by contradiction. Observe that, since $S^1$ is left invariant by $f_{\varepsilon, \gamma}$, the annulus is decomposed in two regions and the periodic orbit $\{h_i\}_{0}^{n-1}$ lies in one of these sides (fig. 1).

![Fig. 1](image)

Now following [Z], we build a curve $C_0$ in the annulus in the following way: the vertical line $(1/2n, x)$ intersects $W_{\text{loc}}^{u}(h_0(\varepsilon))$ and $W_{\text{loc}}^{s}(h_1(\varepsilon))$ in the points $P$ and $Q$ respectively (fig. 1); let $C_0$ be the path that goes from $h_0$ until $P$ through $W_{\text{loc}}^{u}(h_0(\varepsilon))$, then follows by the vertical segment from $P$ until $Q$ and then continues from this point until $h_1$ through $W_{\text{loc}}^{s}(h_1(\varepsilon))$. Define now the curve $C$ as being $\bigcup_{0}^{n-1} f_{\varepsilon, \gamma}(C_0)$. This curve is a non
homotopically trivial Jordan curve. Let $G$ be the region bounded by $\mathbb{S}^1$ and this curve. It is easy to see, using the properties of the stable and unstable manifolds described in Proposition 8, that $m(G) > m(f_{\varepsilon, \gamma}(G))$ therefore contradicting the area-preserving property.

By (15) the angle between these manifolds at the intersection point goes to zero when $\varepsilon$ goes to zero. □

### 3.2 Proof of Proposition 7

The proof of the proposition will be made through a sequence of steps that consist in making some reductions and perturbations. We dedicate one item to each one.

- We change coordinates with $h(\theta, r) = (\theta, r - \Psi(\theta)) = (\bar{\theta}, \bar{r})$ so that $\bar{f}(\bar{\theta}, \bar{r}) = h \circ f \circ h^{-1}$ has $h(\Lambda) = \mathbb{S}^1$ as an invariant curve. Observe that $\bar{f}$ is $C^{\infty}$ and

\[
\|h(\theta, r)\|_{C^s} \leq 1 + \|\Psi\|_{C^s},
\]

\[
\|h^{-1}(\theta, r)\|_{C^s} \leq 1 + \|\Psi^{-1}\|_{C^s},
\]

so if we prove the proposition for $\bar{f}$ then we will also have it proved for $f$.

- Thus we assume that $f(S^1) = S^1$ and $f|S^1$ is conjugated to $R_\omega$ with $\omega$ an irrational number. Consider $f_\beta(\theta, r) = f(\theta, r) + (\beta, 0)$ then by [H] we can find $\beta_n \to 0$ with $n \to \infty$ such that $f_{\beta_n}(S^1) = S^1$ and $f_{\beta_n}|S^1$ has a rotation number $\omega_n = \omega + \beta_n$ satisfying a diophantine condition, and once more by [H] we know that there exists $h_n : S^1 \to S^1$ a $C^{\infty}$ diffeomorphism, conjugating $f_{\beta_n}|S^1$ with $R_{\omega_n}$. Consider $H_n(\theta, r) = (h_n(\theta), r/h_n'(\theta))$, then $H_n^{-1} \circ f_{\beta_n} \circ H_n = \hat{f}$ satisfies $\hat{f}(S^1) = S^1$ and $\hat{f}|S^1 = R_{\omega_n}$. Also these changes of coordinates can be made uniformly in the sense that there is some constant $M_n > 0$ such that

\[
\max \left\{ \|H_n(\theta, r)\|_{C^s}, \|H_n^{-1}(\theta, r)\|_{C^s} \right\} < M_n.
\]

So once more, it is enough to prove the proposition for this map.

- We assume there that $f(S^1) = S^1$ and $f|S^1 = R_\omega$ with $\omega$ satisfying a diophantine condition. By Theorem 5, we can write after a change of coordinates

\[
f(\theta, r) = (0 + \omega + a_1 r + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})),
\]
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we may assume that $a_1 \neq 0$ unless we perturb $f$ in such a way that the new $f$ has $a_1 \neq 0$, even more we choose $a_1 > 0$ (in the case $a_1 < 0$ we take $f^{-1}$). After this we perturb once again so the rotation number of $f|_{S^1}$ becomes rational. We apply now the Proposition 8 to get a sequence of maps $f_k \to f$ such that $f_k$ has a hyperbolic periodic orbit \( \{ h_i(k) \}_{i=1}^n \) with $\psi_{h_i}(k) \in W_{loc}^s(h_i(k)) \cap W_{loc}^u(h_{i+1}(k))$, and the angle at point goes to zero as $k \to \infty$. Moreover, $h_i(k) = i/n$ and

$$\psi_{h_i}(k) \to \psi_{h_i}' \in \left( \frac{2i+1}{2n} - \delta, \frac{2i+1}{2n} + \delta \right) \times \{0\}.$$

So we can use the following lemma (see [N1]).

**Lemma.** — Let $\varepsilon > 0$ and $s \in \mathbb{N}$. There exists $C(s) > 0$ such that given $\delta$ and a linear subspace $H \subset \{ v = (v_1, v_2) \mid |v_2| \leq C(s)\delta^{s-1}\varepsilon|v_1| \}$, there exists a $C^s$ area-preserving diffeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$, $D\varphi\{v_2 = 0\} = H$ and $\varphi(\theta, r) = (\theta, r)$ for $\text{dist}((\theta, r), (0, 0)) \geq \delta$ and $\|\varphi - \text{id}\|_{C^s} \leq \varepsilon$.

So, we can get perturbations $\tilde{f}_k$ of $f_n$ with the property that $\tilde{f}_k$ exhibits homoclinic tangencies and $\tilde{f}_k \to f$. If the tangency it not quadratic, with a new perturbation, we make it quadratic. □

4. Proof of Theorem 1

**Proof of Proposition 2**

Let $\tilde{U}$ be an open neighborhood of $f$ where the continuation of $\gamma$ exists, i.e., for each $g \in \tilde{U}$ there exists an invariant curve $\gamma_g$ such that the rotation number of $g \mid \gamma_g$ equals that of $f \mid \gamma$; this neighborhood is provided by KAM theory. Since $f$ and $\gamma$ are $C^\infty$ we apply Theorem 6 and the remark which follows to conclude the existence of a subset $\mathcal{U}$ of $\tilde{U}$ for which the following property holds: for each $g \in \mathcal{U}$ such that $g$ is a $C^\infty$ map, the invariant curve $\gamma_g$ prolongation of $\gamma$ is also $C^\infty$. Now Proposition 3 allows us to conclude that this neighborhood is an OSPHT. To see the existence of the residual set we observe first that, by the remark following Theorem 6, for each $g \in \tilde{U}$ there are lots of invariant curves, in particular $\gamma_g$ is the limit of other invariant curves satisfying the twist condition and whose rotation numbers satisfy diophantine conditions. We also notice that each $C^\infty$ map

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f with an $C^\infty$ invariant curve can be approximated by another one having an elliptic periodic orbit with arbitrary large period. This follows from the proof of Proposition 3. Now in $U$ consider the subset $U_m$ of all $g \in U$ having some elliptic periodic orbit in the $1/m$-neighborhood for $\gamma_g$. This set is obvious open and $U_{m+1} \subset U_m$. Also each $U_{m+1}$ is dense in $U_m$, because of the two previous observations. So the set $R = \bigcap U_m$ is a residual set satisfying the conclusion of Proposition 2, so we are done.

Proof of Theorem 1

We approximate $f$ by $\tilde{f}$, a $C^\infty$ map having a generic elliptic periodic orbit; it is a consequence of the proof of Proposition 3. Let $U_1$ be a set containing $\tilde{f}$ and for which this elliptic periodic point survives. Choose an invariant $C^\infty$ curve of $\tilde{f}$ associated to this elliptic periodic point. Observe that this curve is invariant by $f^n$ where $n$ is the period of the elliptic periodic point. By KAM theorem we have a subset $U$ of $U_1$, in which the curve survives. Now the remark after Proposition 2 allows us to conclude the proof.

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