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Persistence of Homoclinic Tangencies for Area-Preserving Maps(*)

LEONARDO MORA(1) and NEPTALÍ ROMERO(2)

Résumé. — Nous prouvons que dans une variété symplectique bidimensionnelle M, l'existence de courbes lisses invariantes dans le monde des applications symplectiques de M est un mécanisme pour créer des ouverts contenant un ensemble dense d'applications exhibant des tangences homocliniques.

Abstract. — In a 2-dimensional symplectic manifold M we show that the presence of smooth invariant curves in the world of symplectic maps of M is a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies.

1. Introduction

In 1970, S. Newhouse [N2] proved the existence of an open set \( U \subset \text{Diff}^s(M) \), \( s \geq 2 \), where \( M \) is a 2-dimensional compact manifold, with the following property: there exists a dense subset of \( U \) such that each \( g : M \to M \) in this subset exhibits homoclinic tangencies (tangential intersections between the stable set and unstable set, \( W^s(p) \) and \( W^u(p) \) respectively, of a hyperbolic periodic point \( p \)). We call such a set \( U \subset \text{Diff}^s(M) \), with the last property, an open set of “persistence of homoclinic tangencies”, from now on OSPHT.

Later, in 1979 [N3], he proved that a mechanism to create this kind of sets is the unfolding of a dissipative homoclinic tangency. More precisely, for every \( f \in \text{Diff}^s(M) \), with a homoclinic tangency associated to a dissipative hyperbolic periodic point \( p (|\det Df^n(p)| < 1, \) where \( n \) is the minimal period of \( p \)), there exists \( U \) an OSPHT such that \( f \in \overline{U} \).

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Here we present a mechanism to generate OSPHT’s in the world of symplectic diffeomorphisms; we show that the presence of a smooth invariant curve generates, for nearby maps, this kind of open sets. To be more precise, let $M$ be a 2-dimensional compact manifold with $\omega$ a symplectic 2-form on $M$ and denote by $\text{Diff}_\omega^s$ the space of $C^s$ diffeomorphisms that preserve $\omega$, then we have the following result.

**THEOREM 1.** — Let $f \in \text{Diff}_\omega^\infty(M)$ admit a $C^\infty$ closed invariant curve $\gamma$ such that the rotation number $\omega = p(f|_{\gamma})$ is irrational. Then for every $s \geq 4$ there exists $U \subset \text{Diff}_\omega^s(M)$ an OSPHT such that $f \in \overline{U}$. Moreover, there is a residual subset $V$ of $U$ such that every $f \in V$ has an invariant smooth curve which is accumulated by elliptic points.

The method to prove Theorem 1 is different from the dissipative case. The wild hyperbolic sets mechanism used to produce persistence of homoclinic tangencies is replaced by the rich structure around a smooth invariant curve, obtained from KAM theory [Bo], combined with the following two propositions.

**PROPOSITION 2.** — For $f \in \text{Diff}_\omega^\infty(M)$ and $\gamma$ a $C^\infty$ invariant curve assume that:

1. $\omega$ satisfies a diophantine condition: there exist $\beta \geq 0$ and $C > 0$ such that for every $p/q \in \mathbb{Q}$ then
   
   \[ |\omega - \frac{p}{q}| > \frac{C}{q^{2+\beta}}, \]

2. $f$ satisfies a twist condition along $\gamma$ (see Sect. 2),

3. there exist $\mathcal{U} \subset \text{Diff}_\omega^s(M)$, such that for each $g \in \mathcal{U}$ there is a continuation curve $\gamma_g$ of $\gamma$ which is invariant by $g$ and with the same rotation number $\omega$.

Then there exists $U \subset \mathcal{U}$ an OSPHT and for a residual set in $U$, the continuation curve $\gamma_g$ is the limit of elliptic periodic orbits.

**Remark.** — The same conclusion can be obtained in Proposition 2 if we replace the invariant curve $\gamma$ by a collection of disjoint curves $\{\gamma_i\}_{i=0}^{n-1}$ such that $f(\gamma_i) = \gamma_{i+1}$ and $f(\gamma_{n-1}) = \gamma_0$. Just take $f^n$, apply Proposition 2 and pull back $U$ by the map $f \rightarrow f^n$. 

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Proposition 3. — Let $f \in \text{Diff}_{w}^{\infty}(M)$. Then for each $s \geq 1$, we have:

(i) if $f$ exhibits a $C^\infty$ invariant curve with an irrational rotation number, then for each $\varepsilon > 0$ there exists $\bar{f} \in C^s_{\varepsilon}$-near to $f$ such that $\bar{f}$ exhibits homoclinic tangencies;

(ii) if $f$ exhibits a homoclinic tangency associated to a hyperbolic periodic orbit, then for each $\varepsilon > 0$ there exists $\bar{f} \in C^s_{\varepsilon}$-near to $f$ such that $\bar{f}$ has a generic (in the KAM sense) elliptic periodic point; in particular $\bar{f}$ exhibits $C^\infty$ invariant curves.

The same conclusion of Theorem 1 holds if we replace the assumption of the presence of an invariant curve by the presence of some homoclinic tangency associated to a hyperbolic periodic point $p$.

Corollary 4. — Assume that $f \in \text{Diff}_{w}^{s}(M)$, $s \geq 4$, has a hyperbolic periodic point $p$ and that $f$ exhibits a homoclinic tangency associated to $p$, then there exists $\mathcal{U} \subset \text{Diff}_{w}^{s}(M)$ an OSPHT such that $f \in \overline{\mathcal{U}}$. Moreover, there is a residual subset $\mathcal{V}$ of $\mathcal{U}$ such that every $f \in \mathcal{V}$ has an invariant smooth curve which is accumulated by elliptic points.

A consequence of Corollary 4 is the creation of infinitely many elliptic islands accumulating KAM curves. However, these elliptic points do not accumulate at the hyperbolic point which unfolds the homoclinic tangency. A related question in the unfolding of a homoclinic tangency is whether the OSPHT's can be constructed generating elliptic islands which accumulate at the hyperbolic periodic point. Some partial results concerning the previous question were obtained in [D]. Moreover, it seems possible to answer the question above by using [MR] and the methods of proof in the dissipative case.

This paper is organized as follows: In Section 2, Birkhoff's normal form and KAM theorem are recalled. The proof of Proposition 3, using some tools of [Z], is presented in Section 3. Finally, in Section 4 we prove Proposition 2 and Theorem 1.

2. Birkhoff's normal form theorem and KAM theorem

Let $f$ be an area-preserving $C^r$ diffeomorphism of the annulus $A = S^1 \times \mathbb{R}$, with $r \geq 4k + 4$ and $k \geq 0$; here and in what follows we identify $S^1$ with
\[ S^1 \times \{0\}. \] Assume that \( f(S^1) = S^1 \) and that \( f|_{S^1} = R_\omega \) the rotation with angle \( \omega \). So we can write

\[
 f(\theta, r) = (\theta + \omega + r a(\theta, r), rb(\theta, r)) .
\] (1)

We say that \( \omega \in \mathbb{R} \) satisfies a diophantine condition if there exist \( \beta > 0 \) and \( C > 0 \) such that for every \( p/q \in \mathbb{Q} \) then \(|\omega - p/q| > C/q^{\beta+2}\). Let \( D(C, \beta) \) be the set of these numbers with \( C \) and \( \beta \) fixed. We recall that the set \( D(\beta) = \bigcup_{C \geq 0} D(C, \beta) \) has total Lebesgue measure, i.e., \( m(D(\beta) \cap [0, 1]) = 1 \) when \( \beta > 0 \).

The following version of Birkhoff's normal form theorem says that if \( \omega \) satisfies a diophantine condition then after an area-preserving change of coordinates the term \( r a(\theta, r) \) in (1) can be written as a polynomial function in \( r \) plus higher order terms in \( r \). More precisely, letting

\[ A_\delta = \{ (\theta, r) \mid \theta \in S^1, |r| < \delta \}, \]

we have the following result.

**Theorem 5.** — For each \( n \leq k \) there exists \( h_n : A_\delta \to A_\delta \) a \( C^{-4n} \) area-preserving map letting \( S^1 \) invariant and such that \( \tilde{f}_n = h_n^{-1} \circ f \circ h_n \) has the following form

\[
 \tilde{f}_n(\theta, r) = (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1})) .
\]

**Proof.** — For a proof in the \( C^\infty \) case see appendices 1 and 2 of \([Do]\). The finite-differentiability case follows the same lines as the \( C^\infty \) case but it is necessary to use lemma 8.1 of \([H]\). □

**Remark.** — In the case that \( f \) is \( C^\infty \) all the changes of coordinates are also \( C^\infty \), and we can choose \( n \) as large as we want.

Now consider a \( C^\infty \) symplectic diffeomorphism \( \tilde{f} \in \text{Diff}^\infty_\omega \) with an invariant \( C^\infty \) curve \( \gamma \). We define the twist condition along \( \gamma \) as follows: we say that \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if there exists a transversal unit vector field \( X \) on \( \gamma \) such that \( \omega (\operatorname{D} \tilde{f} X(p), X(\tilde{f}(p))) > 0 \) for all \( p \in \gamma \). When \( \rho(\tilde{f}|_\gamma) \) satisfies a diophantine condition it is well known that after a symplectic change of coordinates, \( \tilde{f} \) restricted to a neighborhood \( V \) of \( \gamma \) has the form (1) with \( X(\theta, 0) = (0, 1) \). In this case a symplectic diffeomorphism of the annulus \( \tilde{f} \) satisfies a twist condition along \( \gamma \) if and only if

\[
 a_1 = \int a(\theta, 0) \, d\theta \neq 0 .
\]
This number does not depend on the symplectic change of coordinates used to put $f$ in the form (1) and it is called the first Birkhoff coefficient.

Now we recall the KAM theorem and remark some facts that we will use in the sequel. Let $f : \mathbb{A}_\delta \to \mathbb{A}$ be a $C^\infty$ map of the annulus. We say that $f$ has the intersection property if for each curve $\gamma$ in $\mathbb{A}_\delta$ non homotopically trivial we have that $f(\gamma) \cap \gamma \neq \emptyset$. If $f$ admits an invariant curve which is non homotopically trivial and preserves a symplectic form $w$ then it is easy to see that $f$ has the intersection property. Let $s \geq 4$ and $t \in C^\infty((-\delta, \delta), \mathbb{R})$.

For each $(\nu, \mu) \in C^s(\mathbb{A}_\delta, \mathbb{R})^2$ let $T_{\nu, \mu} : \mathbb{A}_\delta \to \mathbb{A}$ be the map

$$(\theta, r) \mapsto (\theta + t(r) + \nu(\theta, r), r + \mu(\theta, r)).$$

**Theorem 6.** Let $r_0 \in (-\delta, \delta)$ and assume:

(a) $t > 0$, $T_{\nu, \mu}$ satisfies a twist condition;

(b) $\alpha = t(r_0) \in D(c, \beta)$, $\alpha = t(r_0)$ satisfies a diophantine condition;

(c) $T_{\nu, \mu}$ satisfies the intersection property for every $(\nu, \mu)$ in a neighborhood of $(0, 0)$.

Let $s > 2\beta + 3$, then there exists a neighborhood $W$ in $C^s(\mathbb{A}_\delta, \mathbb{R})^2$ of $(0, 0)$ such that, for all $(\nu, \mu) \in W$, one can find $\gamma \in C^{s-2(1+\beta)}(\mathbb{S}^1, \mathbb{R})$ and $h \in \text{Diff}^{s-2(1+\beta)}(\mathbb{S}^1)$ with

(i) $\Gamma = \{(\theta, \gamma(\theta)) \mid \theta \in \mathbb{S}^1\}$ is invariant under $T_{\nu, \mu}$;

(ii) $T_{\nu, \mu}|_\Gamma$ is $C^{s-2(1+\beta)}$ conjugated to the rotation $R_\alpha(\theta) = \theta + \alpha(\text{mod}1)$ by the following conjugation $\theta \to (h(\theta), \gamma \circ h(\theta))$.

See [Bo] and [SZ] for a proof.

**Remarks**

- The neighborhood $W$ depends *a priori* on $\alpha = t(r_0)$ (in fact on $(dt(r_0)/dr)^{-1}$) but it can be proved that if $r_0$ varies in a compact set $K$, such that $t(K) \subset D(\beta)$ then we can choose $W$ depending just on $K$. Because of $D(\beta)$ has total Lebesgue measure, this is what gives the rich structure (lots of other invariant curves) around an invariant curve.

- We have the following regularity statement: if $\nu, \mu$ are $C^\infty$ then $\gamma$ is $C^\infty$, see [SZ].
3. Invariant curves and homoclinic tangencies

In this section our goal is to give the proof of Proposition 3, which in turn is a consequence of the following proposition.

**Proposition 7.** Let \( f : \mathbb{A}_\delta \to \mathbb{A} \) be a \( C^\infty \) area-preserving map of the annulus which leaves invariant some \( C^\infty \) curve

\[
\Lambda = \left\{ (\theta, \Psi(\theta)) \mid \theta \in \mathbb{S}^1 \right\}
\]

where \( \Psi : \mathbb{S}^1 \to \mathbb{R} \), and such that \( f|_\Lambda \) has an irrational rotation number. Then for \( s > 1 \) and each \( \varepsilon > 0 \), \( f \) can be \( \varepsilon \)-approximated in the \( C^s \)-topology by one \( F \) which exhibits homoclinic tangencies and such that for some \( \delta' < \delta \) we have \( F|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})} = f|_{(\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'})} \).

**Proof of Proposition 3**

Item (ii) follows from [N3], see also [MR], so we will only prove item (i). Because \( f \) and \( \gamma \) are \( C^\infty \), we can find a tubular neighborhood \( U \) of \( \mathbb{S}^1 \) such that there is \( h : U \to \mathbb{A}_\delta \) for which \( h(\gamma) : \mathbb{S}^1 \times \{0\} \subset \mathbb{A}_\delta \) and \( h^*(d\theta \wedge dr) = \omega \). So making use of Proposition 7 the result follows. \( \square \)

To prove Proposition 7 we need first some preliminary results presented in the following subsection.

3.1 Preliminaries

Let \( f : \mathbb{A}_\delta \to \mathbb{A} \) be a \( C^\infty \) area-preserving map of the annulus which leaves \( \mathbb{S}^1 \) invariant, i.e., \( f(\mathbb{S}^1) = \mathbb{S}^1 \). We assume that \( f|_{\mathbb{S}^1} = R_\omega \) with \( \omega = p/n \) where \( p, n \) are relatively prime and

\[
f(\theta, r) = (\theta + \omega + a_1 r + a_2 r^2 + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1}))
\]

with \( a_1 > 0 \). Since \( f \) leaves \( \mathbb{S}^1 \) invariant (see [Do]) we have that locally around \( \mathbb{S}^1 \), \( f(\theta, r) = (\Theta, R) \) is described by a generating function \( h(\theta, R) \) in the following way

\[
f(\theta, r) = (\Theta, R) \quad \text{iff} \quad \begin{cases} r = \frac{\partial h}{\partial \theta} \\ \Theta = \frac{\partial h}{\partial R} \end{cases}
\]
It is easy to check that
\[ h_n(\theta, R) = (\theta, \omega)R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} \]
is the generating function of \( f_n \). From this we get that the generating function of \( f \) has the form
\[ h(\theta, R) = h_n(\theta, R) + O(R^{n+2}). \]

We follow Moser and Zehnder to make a perturbation of \( f \). Consider the following two parameter family of generating functions
\[
h_{\varepsilon, \gamma}(\theta, R) = h(\theta, R) - \varepsilon R + \gamma \cos(2\pi n\theta)R^{n+1} = (\theta + \omega - \varepsilon)R + \frac{a_1}{2} R^2 + \frac{a_2}{3} R^2 + \cdots + \frac{a_n}{n+1} R^{n+1} + \gamma \cos(2\pi n\theta)R^{n+1} + O(R^{n+2}). \tag{2} \]

This family generates, for \( \varepsilon \) and \( \gamma \) small enough, the following two parameter family of diffeomorphisms \( f_{\varepsilon, \gamma} : \mathbb{R}/ \delta/2 \to \mathbb{R} \) with
\[
f_{\varepsilon, \gamma}(\theta, r) = (\theta + \omega - \varepsilon + a_1 r + a_2 r^2 + \cdots + a_n r^n + \gamma \cos(2\pi n\theta) + O(r^{n+1}) + \gamma \cos(2\pi n\theta)(r^{n+1}) + O(r^{n+2})). \tag{3} \]

Observe that by the way we made the perturbation, \( S^1 \) continues to be invariant for the family \( f_{\varepsilon, \gamma} \).

**Proposition 8.** Assume that \( a_1 \neq 0 \), then for \( \varepsilon \) and \( \gamma \) small enough, \( f_{\varepsilon, \gamma} \) has two \( n \)-periodic orbits \( \{ h_i(\varepsilon, \gamma) \}_{i=0}^{n-1}, \{ e_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \), which satisfy:

(a) \( \{ h_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \) is a hyperbolic \( n \)-periodic orbit with
\[ h_i(\varepsilon, \gamma) \to \left( \frac{i}{n}, 0 \right) \]
for \( \gamma \) fixed and \( \varepsilon \to 0 \);

(b) \( \{ e_i(\varepsilon, \gamma) \}_{i=0}^{n-1} \) is an elliptic \( n \)-periodic orbit with
\[ e_i(\varepsilon, \gamma) \to \left( \frac{2i + 1}{2n}, 0 \right) \]
for \( \gamma \) fixed and \( \varepsilon \to 0 \);
(c) there exist $\delta > 0$ and
\[
\psi_{h_i}(\varepsilon, \gamma) \in W^s_{\text{loc}}(h_i(\varepsilon, \gamma)) \cap W^u_{\text{loc}}(h_{i+1}(\varepsilon, \gamma))
\]
for which we have
\[
\psi_{h_i}(\varepsilon, \gamma) \rightarrow \psi_{h_i}(0, \gamma) \in \left(\frac{2i + 1}{2n} - \delta, \frac{2i + 1}{2n} + \delta\right) \times \{0\},
\]
where $\delta$ does not depend on $\varepsilon$ when this is small enough;
(d) the angle
\[
\angle \left( T_{\psi_{h_i}} W^s_{\text{loc}}(h_i(\varepsilon, \gamma)), T_{\psi_{h_i}} W^u_{\text{loc}}(h_{i+1}(\varepsilon, \gamma)) \right) \rightarrow 0
\]
for $\gamma$ fixed and $\varepsilon \rightarrow 0$.

The proof of this proposition is contained in [Z], so we only present the
construction of the periodic points and shows how the homoclinic points are
found and refer to [Z] for the rest of the details.

Proof. — We begin by making the following change of coordinates
\[
\ell(\theta, \rho) = (\theta, \varepsilon \rho) = (\theta, r)
\]
which allows us to see what happens in a microscopic neighborhood of $S^1$. In terms of $\theta$ and $\rho$, $\tilde{f} = \ell^{-1} \circ f \circ \ell$ is written as
\[
\tilde{f}_{\varepsilon, \gamma}(\theta, \rho) = \left( \theta + \frac{p}{n} - \varepsilon + a_1 \varepsilon \rho + \cdots + a_n(\varepsilon \rho)^n + \right.
\]
\[
+ (n + 1) \gamma \cos(2\pi \theta n) (\varepsilon \rho)^n + O((\varepsilon \rho)^{n+1}) ,
\]
\[
\rho + 2\pi \gamma n \sin(2\pi \theta n) \varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1} \rho^{n+2})
\]
\[
= \left( \theta + \frac{p}{n} - \varepsilon + a_1 \varepsilon \rho + \cdots + a_n(\varepsilon \rho)^n + \right.
\]
\[
+ (n + 1) \gamma \cos(2\pi \theta n) (\varepsilon \rho)^n + O(\varepsilon^{n+1}) ,
\]
\[
\rho + 2\pi \gamma n \sin(2\pi \theta n) \varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1}) \right).
\]
We get for the $n$-th iterate of $\tilde{f}_{\varepsilon, \gamma}$ the following expression
\[
\tilde{f}_{\varepsilon, \gamma}^n(\theta, \rho) = \left( \theta + p - n\varepsilon + na_1 \varepsilon \rho + O(\varepsilon^2), \right.
\]
\[
\rho + 2\pi \gamma n^2 \sin(2\pi \theta n) \varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1}) \right),
\]
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where
\[
O(\varepsilon^2) = \frac{n a_2 \varepsilon^2 \rho^2}{\frac{(\varepsilon \rho)^n}{+ n(n+1)\gamma \cos(2\pi \theta n)}(\varepsilon \rho)^n + O(\varepsilon^{n+1})}.
\]

The fixed points of \( \hat{f}_{\epsilon, \gamma} \) are the solutions of the equations
\[
\begin{align*}
\theta &= \theta + p - n \varepsilon + a_1 \varepsilon \rho + O(\varepsilon^2) \\
\rho &= \rho + 2\pi \gamma n^2 \sin(2\pi \theta n) \varepsilon^n \rho^{n+1} + O(\varepsilon^{n+1}).
\end{align*}
\]

The fact that \( a_1 \neq 0 \) and the implicit function theorem imply that there exists \( \rho(\varepsilon) \) a solution of (6) which equals \( 1/a_1 \) when \( \varepsilon = 0 \). Using this solution in (7) we get \( 2n \) solutions \( \{ h_i(\varepsilon), e_i(\varepsilon) \} \) with \( i = 1, \ldots, n \) which equal
\[
\left( \frac{i}{n}, \frac{1}{a_1} \right), \left( \frac{2i + 1}{2n}, \frac{1}{a_1} \right)
\]
respectively when \( \varepsilon = 0 \). Since \( p, n \) are relative primes, the uniqueness part of the implicit function theorem gives that
\[
\{ e_i(\varepsilon) = (e_i(\varepsilon), \rho(\varepsilon)) \} \quad \text{and} \quad \{ h_i(\varepsilon) = (h_i(\varepsilon), \rho(\varepsilon)) \}
\]
are actually part of a \( n \)-periodic orbit. To determine the nature of these orbits we make another change of coordinates. Let \( \phi \) be any of the points \( \{ e_i, h_i \}_{i=0}^{n-1} \) and let \( \hat{\ell}(\psi, x) = (\psi + \phi, \rho(\varepsilon) + \varepsilon^{(n-1)/2} x) \) then
\( \hat{f}_{\epsilon, \gamma} = \hat{\ell}^{-1} \circ \hat{f}_{\epsilon, \gamma} \circ \hat{\ell} \) takes the following form
\[
\hat{f}(\psi, x) = (\psi + \hat{f}_1(\varepsilon, \psi, \varepsilon^{(n-1)/2} x), x + \hat{f}_2(\varepsilon, \psi, \varepsilon^{(n-1)/2} x)),
\]
where
\[
\hat{f}_1(\varepsilon, \psi, y) = -n \varepsilon + n a_1 \varepsilon (\rho(\varepsilon) + y) + \cdots + n a_n \varepsilon^n (\rho(\varepsilon) + y)^n + n(n+1)\gamma \cos(2\pi \theta n) \varepsilon^n (\rho(\varepsilon) + y)^n + O(\varepsilon^{n+1})
\]
and
\[
\hat{f}_2(\varepsilon, \psi, y) = (-1)^\sigma 2\pi \gamma n^2 \sin(2\pi \theta n) \varepsilon^{n-(n-1)/2} (\rho(\varepsilon) + y)^{n+1} + O(\varepsilon^{n-(n-1)/2+1}),
\]
with \( \sigma = \pm 1 \) depending on the value of \( \phi \). From (9) and the fact \( \hat{f}_1(\varepsilon, 0, 0) = (0, 0) \) we get
\[
\hat{f}_1(\varepsilon, \psi, y) = \varphi_0(\varepsilon, \psi) + \varphi_1(\varepsilon, \psi) y + \varphi_2(\varepsilon, \psi) y^2
\]

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with
\[
\varphi_0(\varepsilon, \psi) = \hat{f}_1(\varepsilon, \psi, 0) = O(\varepsilon^{n+1}),
\]
\[
\varphi_1(\varepsilon, \psi) = \frac{\partial \hat{f}_1}{\partial y}(\varepsilon, \psi, 0) = a_1 \varepsilon + O(\varepsilon^2),
\]
\[
\varphi_2(\varepsilon, \psi) = \frac{\partial^2 \hat{f}_1}{\partial^2 y}(\varepsilon, \psi, \tilde{y}) = O(1)
\]
and \(0 \leq \tilde{y} \leq y\). All of these together imply that we can write
\[
\hat{f}(\psi, x) =
\]
\[
= \left( \psi + na_1 \varepsilon^{(n+1)/2} x + O(\varepsilon^{n/2+1}) \right),
\]
\[
x + (-1)^\sigma 2\pi \left( \frac{1}{a_1} \right)^{n+1} \gamma n^2 \sin(2\pi \psi \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1}) \right).
\]
(12)

Now from (12) the jacobian matrix at (0, 0) equals to
\[
J(\varepsilon) = \begin{pmatrix}
1 + O(\varepsilon^{n/2+1}) & na_1 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1}) \\
\mathcal{R} + O(\varepsilon^{n/2+1}) & 1 + O(\varepsilon^{n/2+1})
\end{pmatrix}
\]
where
\[
\mathcal{R} = (-1)^\sigma 4\pi^2 \left( \frac{1}{a_1} \right)^{n+1} \gamma n^3 \varepsilon^{(n+1)/2}, \quad \sigma = \begin{cases}
1 & \text{at } \varepsilon_i, \\
0 & \text{at } h_i.
\end{cases}
\]

From here it follows that
\[
\text{tr } J(\varepsilon) = 2 + O(\varepsilon^{n/2+1})
\]
\[
det J(\varepsilon) = 1 - (-1)^\sigma 4\pi^2 \left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{n+1} + O(\varepsilon^{n/2+1}).
\]
(13)

So we conclude from (13) that we have an elliptic orbit at \(\{\varepsilon_i\}_{i=0}^{n-1}\) and a hyperbolic orbit at \(\{h_i\}_{i=0}^{n-1}\) with eigenvalues given by
\[
\lambda_s = 1 - \pi \sqrt\left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1})
\]
\[
\lambda_u = 1 + \pi \sqrt\left( \frac{1}{a_1} \right)^n \gamma n^4 \varepsilon^{(n+1)/2} + O(\varepsilon^{n/2+1}).
\]
(14)
The local stable (unstable) manifold $W^{s(u)}_{loc}(0)$ of $(0,0)$ of $\hat{f}_{\epsilon,\gamma}$ is described by the following proposition, whose proof follows immediately from Proposition 1 and 2 of [Z].

PROPOSITION 9. — There exist $C_1$, $C_2$ and $\epsilon_0$ such that for $0 < \epsilon \leq \epsilon_0$, the local stable (unstable) manifolds $W^{s(u)}_{loc}(0)$ are given in $|\psi| \leq 3/4n$ by

$$W^{s(u)}_{loc}(0) = \text{graph } g^{s(u)}$$

$$g^{s}(\epsilon, \psi) = \frac{2}{n} \sqrt{\frac{\gamma}{\alpha_1^{n+2}}} \sin(\pi n \psi) + u^{s}(\epsilon, \psi), \quad u^{s}(\epsilon, 0) = 0 \quad (15)$$

$$g^{u}(\epsilon, \psi) = \frac{2}{n} \sqrt{\frac{\gamma}{\alpha_1^{n+2}}} \sin(\pi n \psi) + u^{u}(\epsilon, \psi), \quad u^{u}(\epsilon, 0) = 0$$

where $|u^{s(u)}(\epsilon, \psi)| < C_1 \epsilon$, $\text{Lip}(u^{s(u)}) < C_2 \epsilon$ and $u^{s(u)}(\epsilon, 0) = 0$.

From this proposition it follows that the $W^{s(u)}_{loc}(h_i)$ are the graphs of functions $g^{s(u)}_i$ defined on an interval with center at $h_i$ and length equals $3\pi/4n$. To prove part (c) of Proposition 8, let us show that $W^{u}_{loc}(h_i(\epsilon)) \cap W^{s}_{loc}(h_{i+1}(\epsilon)) \neq \emptyset$. We argue by contradiction. Observe that, since $S^1$ is left invariant by $f_{\epsilon,\gamma}$, the annulus is decomposed in two regions and the periodic orbit $\{h_i\}^{n-1}_{0}$ lies in one of these sides (fig. 1).

Now following [Z], we build a curve $C_0$ in the annulus in the following way: the vertical line $(1/2n, x)$ intersects $W^{u}_{loc}(h_0(\epsilon))$ and $W^{s}_{loc}(h_1(\epsilon))$ in the points $P$ and $Q$ respectively (fig. 1); let $C_0$ be the path that goes from $h_0$ until $P$ through $W^{u}_{loc}(h_0(\epsilon))$, then follows by the vertical segment from $P$ until $Q$ and then continues from this point until $h_1$ through $W^{s}_{loc}(h_1(\epsilon))$. Define now the curve $C$ as being $\bigcup_{0}^{n-1} f_{\epsilon,\gamma}(C_0)$. This curve is a non
homotopically trivial Jordan curve. Let \( G \) be the region bounded by \( S^1 \) and this curve. It is easy to see, using the properties of the stable and unstable manifolds described in Proposition 8, that \( m(G) > m(f_{\varepsilon, \gamma}(G)) \) therefore contradicting the area-preserving property.

By (15) the angle between these manifolds at the intersection point goes to zero when \( \varepsilon \) goes to zero. \( \square \)

3.2 Proof of Proposition 7

The proof of the proposition will be made through a sequence of steps that consist in making some reductions and perturbations. We dedicate one item to each one.

- We change coordinates with \( h(\theta, r) = (\theta, r - \Psi(\theta)) = (\theta, \bar{r}) \) so that \( \bar{f}(\theta, \bar{r}) = h \circ f \circ h^{-1} \) has \( h(\Lambda) = S^1 \) as an invariant curve. Observe that \( \bar{f} \) is \( C^\infty \) and

\[
\|h(\theta, r)\|_{C^s} \leq 1 + \|\Psi\|_{C^s},
\]

\[
\|h^{-1}(\theta, r)\|_{C^s} \leq 1 + \|\Psi^{-1}\|_{C^s},
\]

so if we prove the proposition for \( \bar{f} \) then we will also have it proved for \( f \).

- Thus we assume that \( f(S^1) = S^1 \) and \( f \mid S^1 \) is conjugated to \( R_\omega \) with \( \omega \) an irrational number. Consider \( f_\beta(\theta, r) = f(\theta, r) + (\beta, 0) \) then by [H] we can find \( \beta_n \to 0 \) with \( n \to \infty \) such that \( f_{\beta_n}(S^1) = S^1 \) and \( f_{\beta_n} \mid S^1 \) has a rotation number \( \omega_n = \omega + \beta_n \) satisfying a diophantine condition, and once more by [H] we know that there exists \( h_n : S^1 \to \mathbb{R}^2 \) a \( C^\infty \) diffeomorphism, conjugating \( f_{\beta_n} \mid S^1 \) with \( R_{\omega_n} \). Consider \( H_n(\theta, r) = (h_n(\theta), r/h_n'(\theta)) \), then \( H_n^{-1} \circ f_{\beta_n} \circ H_n = \hat{f} \) satisfies \( \hat{f}(S^1) = S^1 \) and \( \hat{f} \mid S^1 = R_{\omega_n} \). Also these changes of coordinates can be made uniformly in the sense that there is some constant \( M_n > 0 \) such that

\[
\max \left\{ \|H_n(\theta, r)\|_{C^s}, \|H_n^{-1}(\theta, r)\|_{C^s} \right\} < M_n.
\]

So once more, it is enough to prove the proposition for this map.

- We assume there that \( f(S^1) = S^1 \) and \( f \mid S^1 = R_\omega \) with \( \omega \) satisfying a diophantine condition. By Theorem 5, we can write after a change of coordinates

\[
f(\theta, r) = (0 + \omega + a_1 r + \cdots + a_n r^n + O(r^{n+1}), r + O(r^{n+1}))
\]
we may assume that $a_1 \neq 0$ unless we perturb $f$ in such a way that the new $f$ has $a_1 \neq 0$, even more we choose $a_1 > 0$ (in the case $a_1 < 0$ we take $f^{-1}$). After this we perturb once again so the rotation number of $f|_{\Sigma_1}$ becomes rational. We apply now the Proposition 8 to get a sequence of maps $f_k \to f$ such that $f_k$ has a hyperbolic periodic orbit $(h_i(k))_{i=1}^n$ with $\psi_{h_i}(k) \in W^s_{\text{loc}}(h_i(k)) \cap W^u_{\text{loc}}(h_{i+1}(k))$, and the angle at point goes to zero as $k \to \infty$. Moreover, $h_i(k) = i/n$ and

$$\psi_{h_i}(k) \to \psi'_{h_i} \in \left(\frac{2i+1}{2n} - \delta, \frac{2i+1}{2n} + \delta\right) \times \{0\}.$$ 

So we can use the following lemma (see [N1]).

LEMMA. — Let $\varepsilon > 0$ and $s \in \mathbb{N}$. There exists $C(s) > 0$ such that given $\delta$ and a linear subspace $H \subset \{v = (v_1,v_2) \mid |v_2| \leq C(s)\delta^{-1}\varepsilon |v_1|\}$ : there exists a $C^s$ area-preserving diffeomorphism $\varphi : \mathbb{A} \to \mathbb{A}$ such that $\varphi(0) = 0$, $D\varphi\{v_2 = 0\} = H$ and $\varphi(\theta,r) = (\theta,r)$ for $\text{dist}((\theta, r), (0, 0)) \geq \delta$ and $\|\varphi - \text{id}\|_{C^s} \leq \varepsilon$.

So, we can get perturbations $\tilde{f}_k$ of $f_n$ with the property that $\tilde{f}_k$ exhibits homoclinic tangencies and $\tilde{f}_k \to f$. If the tangency is not quadratic, with a new perturbation, we make it quadratic. \(\square\)

4. Proof of Theorem 1

Proof of Proposition 2

Let $\tilde{U}$ be an open neighborhood of $f$ where the continuation of $\gamma$ exists, i.e., for each $g \in \tilde{U}$ there exists an invariant curve $\gamma_g$ such that the rotation number of $g \mid \gamma$ equals that of $f \mid \gamma$; this neighborhood is provided by KAM theory. Since $f$ and $\gamma$ are $C^\infty$ we apply Theorem 6 and the remark which follows to conclude the existence of a subset $\mathcal{U}$ of $\tilde{U}$ for which the following property holds : for each $g \in \mathcal{U}$ such that $g$ is a $C^\infty$ map, the invariant curve $\gamma_g$ prolongation of $\gamma$ is also $C^\infty$. Now Proposition 3 allows us to conclude that this neighborhood is an OSPHT. To see the existence of the residual set we observe first that, by the remark following Theorem 6, for each $g \in \tilde{U}$ there are lots of invariant curves, in particular $\gamma_g$ is the limit of other invariant curves satisfying the twist condition and whose rotation numbers satisfy diophantine conditions. We also notice that each $C^\infty$ map

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f with an $C^\infty$ invariant curve can be approximated by another one having an elliptic periodic orbit with arbitrary large period. This follows from the proof of Proposition 3. Now in $\mathcal{U}$ consider the subset $\mathcal{U}_m$ of all $g \in \mathcal{U}$ having some elliptic periodic orbit in the $1/m$-neighborhood for $\gamma_g$. This set is obvious open and $\mathcal{U}_{m+1} \subset \mathcal{U}_m$. Also each $\mathcal{U}_{m+1}$ is dense in $\mathcal{U}_m$, because of the two previous observations. So the set $R = \bigcap \mathcal{U}_m$ is a residual set satisfying the conclusion of Proposition 2, so we are done. $\square$

**Proof of Theorem 1**

We approximate $f$ by $\tilde{f}$, a $C^\infty$ map having a generic elliptic periodic orbit; it is a consequence of the proof of Proposition 3. Let $\mathcal{U}_1$ be a set containing $f$ and for which this elliptic periodic point survives. Choose an invariant $C^\infty$ curve of $\tilde{f}$ associated to this elliptic periodic point. Observe that this curve is invariant by $f^n$ where $n$ is the period of the elliptic periodic point. By KAM theorem we have a subset $\mathcal{U}$ of $\mathcal{U}_1$, in which the curve survives. Now the remark after Proposition 2 allows us to conclude the proof. $\square$

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**References**


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