VLADIMIR GOL’DSHITEIN
MARC TROYANOV

The $L_{pq}$-Cohomology of SOL


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The $L_{pq}$-Cohomology of $\text{SOL}^{(*)}$

VLADIMIR GOLO'DSHTEIN(1) and MARC TROYANOV(2)

RÉSUMÉ. — On prouve un résultat de non-annulation de la cohomologie non réduite du groupe de Lie $\text{SOL}$.

ABSTRACT. — We prove a non vanishing result for the unreduced $L_{pq}$-cohomology of the Lie group $\text{SOL}$.

1. Introduction

$\text{SOL}$ is the three dimensional Lie group of $3 \times 3$ real matrices of the form

$$
\begin{pmatrix}
  e^z & 0 & x \\
  0 & e^{-z} & y \\
  0 & 0 & 1
\end{pmatrix}.
$$

This is a solvable and unimodular group; it is diffeomorphic to $\mathbb{R}^3$ (with coordinates $x, y, z$). A left invariant Riemannian metric is $ds^2 = e^{-2z} \, dx^2 + e^{2z} \, dy^2 + dz^2$; its volume measure is given by $dx \, dy \, dz$ and is bi-invariant. For more information about the geometry of this group, see [9].

Let us recall the definition of the unreduced $L_{pq}$-cohomology groups, let $(M, ds^2)$ be a complete connected Riemannian manifold of dimension $n$. We

\begin{itemize}
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  \item[(1)] Département of Mathematics, Ben Gurion University of The Negev, P.O. Box 653, IL-84105 Beer Sheva (Israel)
  \item[(2)] Département de Mathématiques, E.P.F.L., CH-1015 Lausanne (Switzerland)
\end{itemize}
note $L^p(M, \Lambda^k)$ the space of differential forms of degree $k$ with measurable coefficients on $M$ such that

$$\|\theta\|_p := \left( \int_M |\theta|^p \right)^{1/p} < \infty.$$  

We denote by $Z_p^k(M)$ the set of differential forms in $L^p(M, \Lambda^k)$ which are closed in the sense of current and by $B^k_{pq}(M)$ the set of forms $\theta \in L^p(M, \Lambda^k)$ such that there exists a form $\phi \in L^q(M, \Lambda^{k-1})$ with $d\phi = \theta$. The unreduced $L_{pq}$-cohomology of $(M, ds^2)$ is by definition the quotient

$$H^k_{pq}(M) := Z^k_p(M)/B^k_{pq}(M).$$  

Other papers dealing with $L_{pq}$ cohomology are [2], [3], [8] and [10]. The goal of this paper is to prove the following result about the unreduced $L_{pq}$-cohomology of SOL.

**Theorem 1.** We have $\dim(H^2_{pq}(\text{SOL})) = \infty$ for every $1 < p, q < \infty$.

2. **Auxiliary results**

The main ingredient in the proof of Theorem 1 is the next proposition (which is a kind of duality argument in $L_{pq}$-cohomology).

**Proposition 2.1.** Let $\alpha \in Z^k_p(M)$, and suppose that for every $\epsilon > 0$, there exists a form

$$\gamma = \gamma_\epsilon \in L^p(M, \Lambda^{n-k}) \cap L^{q'}(M, \Lambda^{n-k})$$

such that

$$\|d\gamma\|_{q'} \leq \epsilon \quad \text{and} \quad \int_M \gamma \wedge \alpha \geq a$$

where $a > 0$ is independent of $\epsilon$ (here $1/q + 1/q' = 1/p + 1/p' = 1$). Then $\alpha \not\in B^k_{pq}(M)$ (in particular, $H^k_{pq}(M) \neq 0$).

For the proof, we will need the following integration-by-part lemma (for differential forms of class $C^1$, this lemma is due to Gaffney [1]).

**Lemma 2.1.** Let $M$ be a complete Riemannian manifold. Let $\beta \in L^q(M, \Lambda^{k-1})$ be such that $d\beta \in L^p(M, \Lambda^k)$, and $\gamma \in L^{p'}(M, \Lambda^{n-k}) \cap
be such that $dy \in L^{q'}(M, \Lambda^{n-k+1})$ where $1/p + 1/p' = 1/q + 1/q' = 1$.

Then $d\gamma \wedge \beta$ and $\gamma \wedge d\beta$ are integrable and

$$
\int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta.
$$

**Proof.** — By Hölder’s inequality, the forms $d\gamma \wedge \beta$, $\gamma \wedge d\beta$ and $\gamma \wedge \beta$ all belong to $L^1$. For smooth forms $\beta$ with compact support, the lemma is true by definition of the weak exterior differential (of $\gamma$).

Assume first that $\beta$ is smooth with non compact support and satisfies the conditions of the lemma. On a complete Riemannian manifold $M$, we can construct a sequence $\{\lambda_i\}$ of smooth functions with compact support such that $\lambda_i(x) \to 1$ uniformly on every compact subset, $0 \leq \lambda_i(x) \leq 1$ and $|d\lambda_i| \leq 1$ for all $x \in M$. The forms $\lambda_i \beta$ have compact support, thus the lemma holds for each $\lambda_i \beta$. Since

$$
|\gamma \wedge d(\lambda_i \beta) + (-1)^{n-k} d\gamma \wedge (\lambda_i \beta)| \leq |d\gamma \wedge \beta| + |\gamma \wedge d\beta| + |\gamma \wedge \beta| \in L^1,
$$

we can apply Lebesgue’s dominated convergence theorem. Thus we have

$$
\int_M (\gamma \wedge d\beta + (-1)^{n-k} d\gamma \wedge (\lambda_i \beta)) =
$$

$$
= \lim_{i \to \infty} \int_M (\gamma \wedge d(\lambda_i \beta) + (-1)^{n-k} d\gamma \wedge (\lambda_i \beta)) = 0.
$$

Finally, for any $\beta \in L^q(M, \Lambda^{k-1})$ with $d\beta \in L^p(M, \Lambda^k)$, we can construct a sequence $\beta_j$ of smooth forms such that $\beta_j \to \beta$ in $L^p$-topology and $d\beta_j \to d\beta$ in $L^p$-topology (see Corollary 1 of [4]). Thus the same limiting process proves the lemma in all its generality. $\Box$

**Proof of Proposition 2.1.** — Suppose that $\alpha = d\beta$ for some $\beta \in L^q(M, \Lambda^{k-1})$. We have by Lemma 1,

$$
\int_M \gamma \wedge \alpha = \int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta.
$$

Using Hölder’s inequality, we get

$$
a \leq \int_M \gamma \wedge \alpha \leq \left| \int_M d\gamma \wedge \beta \right| \leq \|d\gamma\|_q \|\beta\|_q \leq \epsilon \|\beta\|_q.
$$
This is impossible since $\epsilon > 0$ is arbitrary. □

Proposition 2.1 can be completed in the following way.

**Lemma 2.2.** — Let $\alpha_1, \alpha_2, \ldots, \alpha_r \in Z_p^k(M)$ and suppose that we can find pairwise disjoint sets $S_i \subset M$ such that for every $\epsilon > 0$ there exists $\gamma_i = \gamma_{i, \epsilon} \in L^{p'}(M, \Lambda^{n-k}) \cap L^{q'}(M, \Lambda^{n-k})$ with $\text{supp}(\alpha_i) \cup \text{supp}(\gamma_i) \subset S_i$ and such that $\|d\gamma_i\|_{q'} \leq \epsilon$ and $\int_M \gamma_i \wedge \alpha_i \geq a$ where $a > 0$ is independent of $\epsilon$ and $i$. Then $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ are linearly independent elements of $H_{p, q}^k(M)$.

*Proof.* — Choose $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, r$, and set $\alpha = \sum \lambda_i \alpha_i$ and $\gamma = \gamma_{\epsilon} = \sum \lambda_i \gamma_i$. The assumption on the supports of these forms implies that

$$
\int_M \alpha \wedge \gamma = \sum \lambda_i^2 \int_M \alpha_i \wedge \gamma_i \geq a \sum \lambda_i^2.
$$

This sum vanishes if and only if all $\lambda_i = 0$. Since $\gamma \in L^{p'} \cap L^{q'}$ and

$$
\|d\gamma\|_{q'} \leq \sum |\lambda_i| \|d\gamma_i\|_{q'} \leq \epsilon \sum |\lambda_i|
$$

we can deduce from Proposition 1 that $\sum \lambda_i [\alpha_i] = [\alpha] \neq 0 \in H_{p, q}^k(M)$ unless all $\lambda_i = 0$. □

For all $x_0 \in \mathbb{R}$ the surface

$$
\mathcal{H}_{x_0} := \{(x, y, z) \in \text{SOL} \mid x = x_0\} \subset \text{SOL}
$$

is a totally geodesic surface isometric to the hyperbolic plane $\mathbb{H}^2$. In particular a function $f : \text{SOL} \to \mathbb{R}$ which is invariant under all $x$-translations (i.e., $f = f(y, z)$) can be seen as a function on the hyperbolic plane.

**Lemma 2.3.** — There exists two non-negative smooth functions $f$ and $g$ on $\mathbb{H}^2 \cong \mathcal{H}_{x_0}$ such that:

1. $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;
2. $df$ and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;

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(3) the support of $df \wedge dg$ is contained in \{(y, z) \mid |y| \leq 1, 0 \leq z \leq 1\} and $df \wedge dg \geq 0$;

(4) $\int_{\mathbb{H}^2} df \wedge dg = 1$;

(5) $\partial f/\partial y$ and $\partial g/\partial y \in L^\infty(\mathbb{H}^2)$, and $\partial f/\partial z$, $\partial g/\partial z$ have compact support.

Remark. — The forms $df$ and $dg$ cannot have compact support, otherwise, by Stokes theorem, we would have

$$\int_{\mathbb{H}^2} df \wedge dg = 0.$$

Proof. — Choose non negative smooth functions $h_1$, $h_2$ and $k : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $h_1(y) = 0$ if $|y| \geq 1$, $h_1'(y)h_2(y) \geq 0$ and $h_1(y)h_2'(y) \leq 0$ for all $y$;

(ii) the function $(h_1'(y)h_2(y) - h_1(y)h_2'(y))$ has non empty support;

(iii) $k'(z) \geq 0$ for all $z$, furthermore

$$k(z) = \begin{cases} 1 & \text{if } z \geq 1 \\ 0 & \text{if } z \leq 0. \end{cases}$$

We set $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$. Property (1) of the lemma is clear. We prove (3) (i.e., that $df \in L^r$ for any $1 < r \leq \infty$), we have

$$df = h_1(y)k'(z)dz + k(z)h_1'(y)dy.$$ The first term $h_1(y)k'(z)dz$ has compact support, and the second term $k(z)h_1'(y)dy$ has its support in the infinite rectangle $Q = \{|y| \leq 1, z \geq 0\}$.

Choose $D < \infty$ such that $|k(z)h_1'(y)| \leq D$ on $\Omega$. We have

$$|k(z)h_1'(y)dy| \leq D|dy| = De^{-z},$$

thus, since the element of area of $\mathbb{H}^2$ is $dA = e^z dy dz$, we have

$$\int_{\mathbb{H}^2} |k(z)h_1'(y)dy|^r dA \leq D^r \int_Q e^{-rz}e^z dy dz \leq 2D^r \int_0^\infty e^{(1-r)z}dz < \infty,$$

from which one gets $df \in L^r$. 

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Now observe that
\[ df \wedge dg = (k(z)k'(z))(h'_1(y)h_2(y) - h_1(y)h'_2(y)) \, dy \wedge dz, \]
hence the property (3) follows from the construction of \( h, h_2 \) and \( k \).

Property (4) is only a normalisation, and property (5) is easy to check. \( \square \)

The following is a vanishing result for some kind of “anisotropic weighted capacity”.

**Lemma 2.4.** — Given any numbers \( \delta \) and \( q' \) such that \( 1 < q' < \infty \) and \( 0 < \delta < (1/2)(q' - 1) \), we can construct a family of Lipschitz functions \( \psi_t = \psi_t(x, z), t \geq 1, \) on \( \mathbb{R}^2 \) such that:

1. \( 0 \leq \psi_t \leq 1, \) \( \text{supp} \, \psi_t \subset \{(x, z) \mid x^2 + |z|^{2s} \leq 2t\} \), \( \psi_t(x, z) = 1 \) if \( x^2 + |z|^{2s} \leq t \);
2. \( \int \int_{z > 0} \left( \left| \frac{\partial \psi_t}{\partial x} \right|^{q'} + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right|^{q'} \right) \, dx \, dz \leq Ct^{-\delta}, \)

where the constant \( C = C(\delta) \) is independent of \( t \).

**Proof.** — We first choose some number \( s > 0 \) so large that \( (s + 1 - q's)/2s < -\delta \) and set \( \rho(x, z) := x^2 + |z|^{2s} \). We now define \( \psi_t : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
\psi_t(x, z) = \begin{cases} 
1 & \text{if } \rho(x, z) \leq t \\
\frac{\log(2t) - \log(\rho(x, z))}{\log(2)} & \text{if } t \leq \rho(x, z) \leq 2t \\
0 & \text{if } \rho(x, z) \geq 2t.
\end{cases}
\]

We will prove that
\[
\int \int_{z > 0} \left( \left| \frac{\partial \psi_t}{\partial x} \right|^{q'} + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right|^{q'} \right) \, dx \, dz \leq Ct^{s(1-q')/2s}, \tag{2.1}
\]
where the constant \( C \) is independent of \( t \).

It will be convenient to introduce new variables \( X = x/\sqrt{t} \) and \( Z = z^s/\sqrt{t} \) (we assume \( z \geq 0 \)). We have

\[ x = t^{1/2} X \quad \text{and} \quad z = t^{1/2s} Z^{1/s} \]
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thus $\rho = t (X^2 + Z^2)$. Let us set $\Psi_t(X, Z) := \psi_t(x, z)$, then

$$
\Psi_t(X, Z) = \begin{cases} 
1 & \text{if } (X^2 + Z^2) \leq 1 \\
\log(2) - \log(X^2 + Z^2) \over \log(2) & \text{if } 1 \leq (X^2 + Z^2) \leq 2 \\
0 & \text{if } (X^2 + Z^2) \geq 2.
\end{cases}
$$

In particular, $\Psi_t$ is independent of $t$ (and will henceforth be written as $\Psi$) and its support is the annulus $A = \{(X, Z) | 1 \leq (X^2 + Z^2) \leq 2\}$.

The partial derivatives of $\psi_t$ may be written as

$$
\frac{\partial \psi_t}{\partial x} = t^{-1/2} \frac{\partial \Psi}{\partial X} \quad \text{and} \quad \frac{\partial \psi_t}{\partial z} = st^{-1/2} z^{s-1} \frac{\partial \Psi}{\partial Z}.
$$

The maximum of the function $z \rightarrow s z^{s-1} e^{-z}$ on $0 \leq z < \infty$ is achieved at $z = (s-1)$, hence

$$
|z|^{s-1} e^{-z} \leq c_1 := s(s-1)^{s-1} e^{-s+1}
$$

for all $z \geq 0$. From the second equation in (2.2) and (2.3), we conclude that

$$
e^{-z} \left| \frac{\partial \psi_t}{\partial x} \right| \leq c_1 t^{-1/2} \left| \frac{\partial \Psi}{\partial Z} \right|.
$$

We see from the first equation in (2.2) and the inequality (2.4) that

$$
\left( \left| \frac{\partial \psi_t}{\partial x} \right| + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right| \right) \leq t^{-q'/2} \left( \left| \frac{\partial \Psi}{\partial X} \right|^{q'} + \left| \frac{\partial \Psi}{\partial Z} c_1 \right|^{q'} \right).
$$

Since

$$
dx \, dz = \frac{1}{s} t^{(s+1)/2s} Z^{(s-1)/s} \, dx \, dz
$$

we obtain (2.1) with

$$
C = \int_{A^+} \left( \left| \frac{\partial \Psi}{\partial X} \right|^{q'} + \left| \frac{\partial \Psi}{\partial Z} c_1 \right|^{q'} \right) \frac{Z^{(s-1)/s}}{s} \, dx \, dz < \infty,
$$

where the domain of integration is the half annulus $A^+ = \{(X, Z) | Z \geq 0 \text{ and } 1 \leq (X^2 + Z^2) \leq 2\}$. □
3. Proof of the main theorem

The proof is technical and will be divided in five steps: we first fix some arbitrarily \( \epsilon > 0 \).

**Step 1. We construct a closed 2-form \( \alpha \in Z^2_p(\text{SOL}) \)**

We start by choosing a pair of functions \( f = f(y, z) \) and \( g = g(y, z) \) with the properties of Lemma 2.3. We then choose a smooth function \( \lambda : \mathbb{R} \to \mathbb{R} \) such that

\[
\lambda(u) = \begin{cases} 
0 & \text{if } u \leq -1 \\
1 & \text{if } u \geq 1 \\
0 \leq \lambda'(u) \leq 1 & \text{for all } u \in \mathbb{R}.
\end{cases}
\]

Then we set \( \varphi(x, z) = \lambda(e^{-z} x) \), and note that

\[
d\varphi = (\lambda'(e^{-z} x) e^{-z}) (dx - x \, dz).
\]

We finally define

\[
\alpha := d\varphi \wedge df = d(\varphi \, df) = (\lambda'(e^{-z} x) e^{-z}) \left( \frac{\partial f}{\partial y} \, dx \wedge dy + \frac{\partial f}{\partial z} \, dx \wedge dz + x \frac{\partial f}{\partial y} \, dy \wedge dz \right).
\]

Observe that \( d\alpha = 0 \) and

- \( \text{supp}(\alpha) \subset \Omega := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, z > 0, |x| < \epsilon^z\} \);
- \( \lambda'(e^{-z} x) \frac{\partial f}{\partial z} \) has compact support;
- \( \left| \frac{\partial f}{\partial y} \right| \) is bounded (since \( |dx \wedge dy| = 1 \) and \( \partial f/\partial y \) is bounded);
- \( \left| \frac{\partial f}{\partial z} \right| \) is bounded (since \( |dy \wedge dz| = e^z \) and \( |x| \leq e^{-z} \) on \( \Omega \)).

From these estimates and \( 0 \leq \lambda' \leq 1 \), we deduce easily that \( |\alpha| \leq \text{const} \, e^{-z} \) on \( \Omega \) and

\[
\int_\Omega |\alpha|^p \leq \text{const} \int_0^\infty e^{(1-p)z} \, dz < \infty
\]

for all \( 1 < p \leq \infty \). It follows that \( \alpha \in Z^2_p(\text{SOL}) \).
Step 2. We construct a family of almost closed forms $\gamma_t \in L^r(\text{SOL}, \Lambda^1)$

Fix $0 < \delta < (1/2)(q' - 1)$ and choose a function $\psi_t = \psi_t(x, z)$ as in Lemma 2.4. Define $\gamma_t := \psi_t(x, z) \, dg$. In order to show that $\gamma_t \in L^r(\text{SOL}, \Lambda^1)$, observe that $\gamma_t$ has its support contained in the box

$$Q_t := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, \ z \geq 0, \ |x| \leq \sqrt{2t}\}.$$ 

Recall that $0 \leq \psi_t(x, z) \leq 1$ and the volume form of SOL is $d(\text{vol}) = dx \, dy \, dz$. We thus have

$$\|\gamma_t\|_r^r = \int_{x=-\sqrt{2t}}^{\sqrt{2t}} \int_{y=-1}^{1} \int_{z=0}^{\infty} |\psi_t(x, z)|^r \, |dg|^r \, dx \, dy \, dz$$

$$\leq \int_{x=-\sqrt{2t}}^{\sqrt{2t}} dx \int_{y=-1}^{1} \int_{z=0}^{\infty} |dg|^r \, dy \, dz$$

$$\leq (2\sqrt{2t}) \int_{y=-1}^{1} \int_{z=0}^{\infty} |dg|^r \, e^z \, dy \, dz.$$ 

By Lemma 2.3, we know that

$$\int_{\mathbb{R}^2} |dg|^r \, e^z \, dy \, dz < \infty \quad \text{for any } 1 < r < \infty,$$

from which one gets an estimates $\|\gamma_t\|_r \leq C_1(r) \, t^{1/2r}$. In particular

$$\gamma_t \in \bigcap_{1 < r < \infty} L^r(\text{SOL}, \Lambda^1). \quad (3.1)$$

Step 3. We estimate $\|d\gamma_t\|_{q'}$

We have $d\gamma_t = \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial y} \, dx \wedge dy + \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial z} \, dx \wedge dz + \frac{\partial \psi_t}{\partial z} \frac{\partial g}{\partial y} \, dz \wedge dy$ and

$$|dx \wedge dy| = 1, \quad |dx \wedge dz| = e^z, \quad |dz \wedge dy| = e^{-z}.$$ 

Recall that $\partial g/\partial y$ is bounded, $\partial g/\partial z$ has compact support and $d\gamma_t$ has its support in the region $Q_t$. Thus

$$|d\gamma_t| \leq C_2 \left( \left| \frac{\partial \psi_t}{\partial x} \right| + \left| \frac{\partial \psi_t}{\partial z} \right| \, e^{-z} \right).$$
Since $0 < \delta < (1/2)(q' - 1)$, Lemma 2.4 implies

$$
\|d\gamma_t\|_{q'} \leq C_3 t^{-\delta/q'}.
$$

(3.2)

**Step 4. We estimate the integral of $\alpha \wedge \gamma_t$**

Let

$$
A_t := \int_{\text{SOL}} \alpha \wedge \gamma_t.
$$

We have

$$
\alpha \wedge \gamma_t = \psi_t(x, z) \, d\varphi \wedge df \wedge dg = (\lambda'(e^{-z}x) e^{-z} \psi_t(x, z)) \, dx \wedge df \wedge dg \tag{4}
$$

(since $dz \wedge df \wedge dg = 0$). By Lemma 2.3, $df \wedge dg \geq 0$, and since $\lambda'(e^{-z}x) \geq 0$ we see that $\alpha \wedge \gamma_t$ is a non-negative 3-form. In particular $A_t \geq \int_{\Delta} \alpha \wedge \gamma_t$ for every measurable subset $\Delta \subset \text{SOL}$.

We set

$$
\Delta_t := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, \ 0 \leq z \leq 1, \ |x| \leq \sqrt{t}\}.
$$

Recall that if $t \geq 1$, $0 \leq z \leq 1$ and $|x| \leq \sqrt{t}$, then $\psi_t(x, z) = 1$, we thus get

$$
A_t \geq \int_{\Delta_t} \alpha \wedge \gamma_t = \int_{y=-1}^{+1} \int_{z=0}^{1} \int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'(e^{-z}x) e^{-z} \, dx \wedge df \wedge dg.
$$

Now set $u = e^{-z}x$, $du = e^{-z} \, dx$, $u_0 = -e^{-z}\sqrt{t}$ and $u_1 = e^{-z}\sqrt{t}$. We have

$$
\int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'(e^{-z}x) e^{-z} \, dx = \int_{u_0}^{u_1} \lambda'(u) \, du = 1
$$

if $t$ is large enough (i.e., $e^{-1}\sqrt{t} \geq 1$). Thus

$$
A_t \geq C_4 := \int_{y=-1}^{+1} \int_{z=0}^{1} df \wedge dg > 0. \tag{3.3}
$$

Observe that the constant $C_4$ is positive and independent of $t$ (in fact, using equation (4) of Lemma 2.3 and (i) of Lemma 2.4, we see that $A_t \to 1$ as $t \to \infty$).
Step 5. Recapitulation

Let us summarize the previous estimates (3.1), (3.2) and (3.3):

$$\| \gamma_t \|_{p'} + \| \gamma_t \|_{q'} < \infty, \quad \| d \gamma_t \|_{q'} \leq C_3 t^{-\delta/q'} \quad \text{and} \quad \int_{\text{SOL}} \alpha \wedge \gamma_t \geq C_4 > 0.$$ 

If we let $t \to \infty$ and apply Proposition 2.1, we obtain $\alpha \notin B_{pq}^2(M)$.

By the construction

$$S_t := \text{supp}(\alpha) \cup \text{supp}(\gamma_t) \subset Q := \{(x, y, z) \mid |y| \leq 1, \ 0 \leq z\}.$$ 

Using the group of isometries $T : (x, y, z) \to (x, y + 2k, z), \ k \in \mathbb{Z}$, we can produce an infinite family of forms $\alpha_i \in Z^k_p(\text{SOL})$ satisfying the hypothesis of Lemma 2.2. Therefore

$$\dim H^2_{pq}(\text{SOL}) = \infty$$

for all $1 < p, q < \infty$. The proof is complete.

4. Final remark

The above proof of Theorem 1 is only true for unreduced cohomology. In fact, the work of Jeff Cheeger and Mikhael Gromov gives us the following result in the reduced case (for $p = q = 2$).

THEOREM 2. — The reduced $L_2$-cohomology of SOL is trivial.

Proof. — The Lie group SOL admits uniform lattices (i.e., discrete cocompact subgroups), see [11] for explicit constructions. The result thus follows from [5], [6] and [7] since every lattice in SOL is amenable.

Acknowledgments

Finally, we thank the referee who discovered a mistake in an earlier version of Lemma 2.1.

References


