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The L_{pq} -Cohomology of SOL^(*)

VLADIMIR GOL'DSHTEIN⁽¹⁾ and MARC TROYANOV⁽²⁾

RÉSUMÉ. — On prouve un résultat de non-annulation de la cohomologie non réduite du groupe de Lie SOL.

ABSTRACT. — We prove a non vanishing result for the unreduced L_{pq} -cohomology of the Lie group SOL.

1. Introduction

SOL is the three dimensional Lie group of 3×3 real matrices of the form

$$\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a solvable and unimodular group; it is diffeomorphic to \mathbb{R}^3 (with coordinates x, y, z). A left invariant Riemannian metric is $ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2$; its volume measure is given by $dx dy dz$ and is bi-invariant. For more information about the geometry of this group, see [9].

Let us recall the definition of the unreduced L_{pq} -cohomology groups, let (M, ds^2) be a complete connected Riemannian manifold of dimension n . We

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note $L^p(M, \Lambda^k)$ the space of differential forms of degree k with measurable coefficients on M such that

$$\|\theta\|_p := \left(\int_M |\theta|^p \right)^{1/p} < \infty .$$

We denote by $Z_p^k(M)$ the set of differential forms in $L^p(M, \Lambda^k)$ which are closed in the sense of current and by $B_{pq}^k(M)$ the set of forms $\theta \in L^p(M, \Lambda^k)$ such that there exists a form $\phi \in L^q(M, \Lambda^{k-1})$ with $d\phi = \theta$. The unreduced L_{pq} -cohomology of (M, ds^2) is by definition the quotient

$$H_{pq}^k(M) := Z_p^k(M) / B_{pq}^k(M) .$$

Other papers dealing with L_{pq} cohomology are [2], [3], [8] and [10]. The goal of this paper is to prove the following result about the unreduced L_{pq} -cohomology of SOL.

THEOREM 1. — *We have $\dim(H_{pq}^2(\text{SOL})) = \infty$ for every $1 < p, q < \infty$.*

2. Auxiliary results

The main ingredient in the proof of Theorem 1 is the next proposition (which is a kind of duality argument in L_{pq} -cohomology).

PROPOSITION 2.1. — *Let $\alpha \in Z_p^k(M)$, and suppose that for every $\epsilon > 0$, there exists a form*

$$\gamma = \gamma_\epsilon \in L^{p'}(M, \Lambda^{n-k}) \cap L^{q'}(M, \Lambda^{n-k})$$

such that

$$\|d\gamma\|_{q'} \leq \epsilon \quad \text{and} \quad \int_M \gamma \wedge \alpha \geq a$$

where $a > 0$ is independent of ϵ (here $1/q + 1/q' = 1/p + 1/p' = 1$). Then $\alpha \notin B_{pq}^k(M)$ (in particular, $H_{pq}^k(M) \neq 0$).

For the proof, we will need the following integration-by-part lemma (for differential forms of class C^1 , this lemma is due to Gaffney [1]).

LEMMA 2.1. — *Let M be a complete Riemannian manifold. Let $\beta \in L^q(M, \Lambda^{k-1})$ be such that $d\beta \in L^p(M, \Lambda^k)$, and $\gamma \in L^{p'}(M, \Lambda^{n-k}) \cap$*

$L^{q'}(M, \Lambda^{n-k})$ be such that $d\gamma \in L^{q'}(M, \Lambda^{n-k+1})$ where $1/p + 1/p' = 1/q + 1/q' = 1$.

Then $d\gamma \wedge \beta$ and $\gamma \wedge d\beta$ are integrable and

$$\int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta.$$

Proof. — By Hölder's inequality, the forms $d\gamma \wedge \beta$, $\gamma \wedge d\beta$ and $\gamma \wedge \beta$ all belong to L^1 . For smooth forms β with compact support, the lemma is true by definition of the weak exterior differential (of γ).

Assume first that β is smooth with non compact support and satisfies the conditions of the lemma. On a complete Riemannian manifold M , we can construct a sequence $\{\lambda_i\}$ of smooth functions with compact support such that $\lambda_i(x) \rightarrow 1$ uniformly on every compact subset, $0 \leq \lambda_i(x) \leq 1$ and $|d\lambda_i|_x \leq 1$ for all $x \in M$. The forms $\lambda_i \beta$ have compact support, thus the lemma holds for each $\lambda_i \beta$. Since

$$\left| \gamma \wedge d(\lambda_i \beta) + (-1)^{n-k} d\gamma \wedge (\lambda_i \beta) \right| \leq |d\gamma \wedge \beta| + |\gamma \wedge d\beta| + |\gamma \wedge \beta| \in L^1,$$

we can apply Lebesgue's dominated convergence theorem. Thus we have

$$\begin{aligned} \int_M \left(\gamma \wedge d\beta + (-1)^{n-k} d\gamma \wedge \beta \right) &= \\ &= \lim_{i \rightarrow \infty} \int_M \left(\gamma \wedge d(\lambda_i \beta) + (-1)^{n-k} d\gamma \wedge (\lambda_i \beta) \right) = 0. \end{aligned}$$

Finally, for any $\beta \in L^q(M, \Lambda^{k-1})$ with $d\beta \in L^p(M, \Lambda^k)$, we can construct a sequence β_j of smooth forms such that $\beta_j \rightarrow \beta$ in L^p -topology and $d\beta_j \rightarrow d\beta$ in L^p -topology (see Corollary 1 of [4]). Thus the same limiting process proves the lemma in all its generality. \square

Proof of Proposition 2.1. — Suppose that $\alpha = d\beta$ for some $\beta \in L^q(M, \Lambda^{k-1})$. We have by Lemma 1,

$$\int_M \gamma \wedge \alpha = \int_M \gamma \wedge d\beta = (-1)^{n-k+1} \int_M d\gamma \wedge \beta.$$

Using Hölder's inequality, we get

$$a \leq \int_M \gamma \wedge \alpha \leq \left| \int_M d\gamma \wedge \beta \right| \leq \|d\gamma\|_q \|\beta\|_q \leq \epsilon \|\beta\|_q.$$

This is impossible since $\epsilon > 0$ is arbitrary. \square

Proposition 2.1 can be completed in the following way.

LEMMA 2.2. — *Let $\alpha_1, \alpha_2, \dots, \alpha_r \in Z_p^k(M)$ and suppose that we can find pairwise disjoint sets $S_i \subset M$ such that for every $\epsilon > 0$ there exists $\gamma_i = \gamma_{i,\epsilon} \in L^{p'}(M, \Lambda^{n-k}) \cap L^{q'}(M, \Lambda^{n-k})$ with $\text{supp}(\alpha_i) \cup \text{supp}(\gamma_i) \subset S_i$ and such that $\|d\gamma_i\|_{q'} \leq \epsilon$ and $\int_M \gamma_i \wedge \alpha_i \geq a$ where $a > 0$ is independent of ϵ and i . Then $[\alpha_1], [\alpha_2], \dots, [\alpha_r]$ are linearly independent elements of $H_{p,q}^k(M)$.*

Proof. — Choose $\lambda_i \in \mathbb{R}$, $i = 1, \dots, r$, and set $\alpha = \sum \lambda_i \alpha_i$ and $\gamma = \gamma_\epsilon = \sum \lambda_i \gamma_i$. The assumption on the supports of these forms implies that

$$\int_M \alpha \wedge \gamma = \sum_i \lambda_i^2 \int_M \alpha_i \wedge \gamma_i \geq a \sum_i \lambda_i^2.$$

This sum vanishes if and only if all $\lambda_i = 0$. Since $\gamma \in L^{p'} \cap L^{q'}$ and

$$\|d\gamma\|_{q'} \leq \sum_i |\lambda_i| \|d\gamma_i\|_{q'} \leq \epsilon \sum_i |\lambda_i|$$

we can deduce from Proposition 1 that $\sum \lambda_i [\alpha_i] = [\alpha] \neq 0 \in H_{p,q}^k(M)$ unless all $\lambda_i = 0$. \square

For all $x_0 \in \mathbb{R}$ the surface

$$\mathcal{H}_{x_0} := \{(x, y, z) \in \text{SOL} \mid x = x_0\} \subset \text{SOL}$$

is a totally geodesic surface isometric to the hyperbolic plane \mathbb{H}^2 . In particular a function $f : \text{SOL} \rightarrow \mathbb{R}$ which is invariant under all x -translations (i.e., $f = f(y, z)$) can be seen as a function on the hyperbolic plane.

LEMMA 2.3. — *There exists two non negative smooth functions f and g on $\mathbb{H}^2 \simeq \mathcal{H}_{x_0}$ such that:*

- (1) $f(y, z) = g(y, z) = 0$ if $z \leq 0$ or $|y| \geq 1$;
- (2) df and $dg \in L^r(\mathbb{H}^2, \Lambda^1)$ for any $1 < r \leq \infty$;

- (3) the support of $df \wedge dg$ is contained in $\{(y, z) \mid |y| \leq 1, 0 \leq z \leq 1\}$ and $df \wedge dg \geq 0$;
- (4) $\iint_{\mathbb{H}^2} df \wedge dg = 1$;
- (5) $\partial f / \partial y$ and $\partial g / \partial y \in L^\infty(\mathbb{H}^2)$, and $\partial f / \partial z, \partial g / \partial z$ have compact support.

Remark. — The forms df and dg cannot have compact support, otherwise, by Stokes theorem, we would have

$$\int_{\mathbb{H}^2} df \wedge dg = 0.$$

Proof. — Choose non negative smooth functions h_1, h_2 and $k : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- (i) $h_i(y) = 0$ if $|y| \geq 1, h'_1(y)h_2(y) \geq 0$ and $h_1(y)h'_2(y) \leq 0$ for all y ;
- (ii) the function $(h'_1(y)h_2(y) - h_1(y)h'_2(y))$ has non empty support;
- (iii) $k'(z) \geq 0$ for all z , furthermore

$$k(z) = \begin{cases} 1 & \text{if } z \geq 1 \\ 0 & \text{if } z \leq 0. \end{cases}$$

We set $f(y, z) := h_1(y)k(z)$ and $g(y, z) := h_2(y)k(z)$. Property (1) of the lemma is clear. We prove (3) (i.e., that $df \in L^r$ for any $1 < r \leq \infty$), we have

$$df = h_1(y)k'(z) dz + k(z)h'_1(y) dy.$$

The first term $h_1(y)k'(z) dz$ has compact support, and the second term $k(z)h'_1(y) dy$ has its support in the infinite rectangle $Q = \{|y| \leq 1, z \geq 0\}$.

Choose $D < \infty$ such that $|k(z)h'_1(y)| \leq D$ on Ω . We have

$$|k(z)h'_1(y) dy| \leq D|dy| = D e^{-z},$$

thus, since the element of area of \mathbb{H}^2 is $dA = e^z dy dz$, we have

$$\begin{aligned} \int_{\mathbb{H}^2} |k(z)h'_1(y) dy|^r dA &\leq D^r \int_Q e^{-rz} e^z dy dz \\ &\leq 2D^r \int_0^\infty e^{(1-r)z} dz < \infty, \end{aligned}$$

from which one gets $df \in L^r$.

Now observe that

$$df \wedge dg = (k(z)k'(z))(h_1'(y)h_2(y) - h_1(y)h_2'(y)) \, dy \wedge dz,$$

hence the property (3) follows from the construction of h_1 , h_2 and k .

Property (4) is only a normalisation, and property (5) is easy to check. \square

The following is a vanishing result for some kind of “anisotropic weighted capacity”.

LEMMA 2.4. — *Given any numbers δ and q' such that $1 < q' < \infty$ and $0 < \delta < (1/2)(q' - 1)$, we can construct a family of Lipschitz functions $\psi_t = \psi_t(x, z)$, $t \geq 1$, on \mathbb{R}^2 such that:*

- (i) $0 \leq \psi_t \leq 1$, $\text{supp } \psi_t \subset \{(x, z) \mid x^2 + |z|^{2s} \leq 2t\}$, $\psi_t(x, z) = 1$ if $x^2 + |z|^{2s} \leq t$;
- (ii) $\iint_{z>0} \left(\left| \frac{\partial \psi_t}{\partial x} \right|^{q'} + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right|^{q'} \right) dx dz \leq Ct^{-\delta}$,

where the constant $C = C(\delta)$ is independent of t .

Proof. — We first choose some number $s > 0$ so large that $(s + 1 - q')/2s < -\delta$ and set $\rho(x, z) := x^2 + |z|^{2s}$. We now define $\psi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\psi_t(x, z) = \begin{cases} 1 & \text{if } \rho(x, z) \leq t \\ \frac{\log(2t) - \log(\rho(x, z))}{\log(2)} & \text{if } t \leq \rho(x, z) \leq 2t \\ 0 & \text{if } \rho(x, z) \geq 2t. \end{cases}$$

We will prove that

$$\iint_{z>0} \left(\left| \frac{\partial \psi_t}{\partial x} \right|^{q'} + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right|^{q'} \right) dx dz \leq Ct^{(s+1-q')s/2s}, \quad (2.1)$$

where the constant C is independent of t .

It will be convenient to introduce new variables $X = x/\sqrt{t}$ and $Z = z^s/\sqrt{t}$ (we assume $z \geq 0$). We have

$$x = t^{1/2} X \quad \text{and} \quad z = t^{1/2s} Z^{1/s}$$

thus $\rho = t(X^2 + Z^2)$. Let us set $\Psi_t(X, Z) := \psi_t(x, z)$, then

$$\Psi_t(X, Z) = \begin{cases} 1 & \text{if } (X^2 + Z^2) \leq 1 \\ \frac{\log(2) - \log(X^2 + Z^2)}{\log(2)} & \text{if } 1 \leq (X^2 + Z^2) \leq 2 \\ 0 & \text{if } (X^2 + Z^2) \geq 2. \end{cases}$$

In particular, Ψ_t is independent of t (and will henceforth be written as Ψ) and its support is the annulus $A = \{(X, Z) \mid 1 \leq (X^2 + Z^2) \leq 2\}$.

The partial derivatives of ψ_t may be written as

$$\frac{\partial \psi_t}{\partial x} = t^{-1/2} \frac{\partial \Psi}{\partial X} \quad \text{and} \quad \frac{\partial \psi_t}{\partial z} = st^{-1/2} z^{s-1} \frac{\partial \Psi}{\partial Z}. \quad (2.2)$$

The maximum of the function $z \rightarrow sz^{s-1}e^{-z}$ on $0 \leq z < \infty$ is achieved at $z = (s - 1)$, hence

$$|z|^{s-1} e^{-z} \leq c_1 := s(s-1)^{s-1} e^{-s+1} \quad (2.3)$$

for all $z \geq 0$. From the second equation in (2.2) and (2.3), we conclude that

$$e^{-z} \left| \frac{\partial \psi_t}{\partial z} \right| \leq c_1 t^{-1/2} \left| \frac{\partial \Psi}{\partial Z} \right|. \quad (2.4)$$

We see from the first equation in (2.2) and the inequality (2.4) that

$$\left(\left| \frac{\partial \psi_t}{\partial x} \right|^{q'} + \left| \frac{\partial \psi_t}{\partial z} e^{-z} \right|^{q'} \right) \leq t^{-q'/2} \left(\left| \frac{\partial \Psi}{\partial X} \right|^{q'} + \left| \frac{\partial \Psi}{\partial Z} c_1 \right|^{q'} \right).$$

Since

$$dx \, dz = \frac{1}{s} t^{(s+1)/2s} Z^{(s-1)/s} dx \, dz$$

we obtain (2.1) with

$$C = \iint_{A^+} \left(\left| \frac{\partial \Psi}{\partial X} \right|^{q'} + \left| \frac{\partial \Psi}{\partial Z} c_1 \right|^{q'} \right) \frac{Z^{(s-1)/s}}{s} dx \, dz < \infty,$$

where the domain of integration is the half annulus $A^+ = \{(X, Z) \mid Z \geq 0 \text{ and } 1 \leq (X^2 + Z^2) \leq 2\}$. \square

3. Proof of the main theorem

The proof is technical and will be divided in five steps: we first fix some arbitrarily $\epsilon > 0$.

Step 1. We construct a closed 2-form $\alpha \in Z_p^2(\text{SOL})$

We start by choosing a pair of functions $f = f(y, z)$ and $g = g(y, z)$ with the properties of Lemma 2.3. We then choose a smooth function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lambda(u) = \begin{cases} 0 & \text{if } u \leq -1 \\ 1 & \text{if } u \geq 1 \\ 0 \leq \lambda'(u) \leq 1 & \text{for all } u \in \mathbb{R}. \end{cases}$$

Then we set $\varphi(x, z) = \lambda(e^{-z}x)$, and note that

$$d\varphi = (\lambda'(e^{-z}x)e^{-z})(dx - x dz).$$

We finally define

$$\begin{aligned} \alpha &:= d\varphi \wedge df = d(\varphi df) \\ &= (\lambda'(e^{-z}x)e^{-z}) \left(\frac{\partial f}{\partial y} dx \wedge dy + \frac{\partial f}{\partial z} dx \wedge dz + x \frac{\partial f}{\partial y} dy \wedge dz \right). \end{aligned}$$

Observe that $d\alpha = 0$ and

- $\text{supp}(\alpha) \subset \Omega := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, z > 0, |x| < e^z\}$;
- $\lambda'(e^{-z}x) \frac{\partial f}{\partial z}$ has compact support;
- $\left| \frac{\partial f}{\partial y} \right| |dx \wedge dy|$ is bounded (since $|dx \wedge dy| = 1$ and $\partial f / \partial y$ is bounded);
- $\left| \frac{\partial f}{\partial y} \right| |x| |dy \wedge dz|$ is bounded (since $|dy \wedge dz| = e^z$ and $|x| \leq e^{-z}$ on Ω).

From these estimates and $0 \leq \lambda' \leq 1$, we deduce easily that $|\alpha| \leq \text{const } e^{-z}$ on Ω and

$$\int_{\Omega} |\alpha|^p \leq \text{const} \int_0^{\infty} e^{(1-p)z} dz < \infty$$

for all $1 < p \leq \infty$. It follows that $\alpha \in Z_p^2(\text{SOL})$.

Step 2. We construct a family of almost closed forms $\gamma_t \in L^r(\text{SOL}, \Lambda^1)$

Fix $0 < \delta < (1/2)(q' - 1)$ and choose a function $\psi_t = \psi_t(x, z)$ as in Lemma 2.4. Define $\gamma_t := \psi_t(x, z) dg$. In order to show that $\gamma_t \in L^r(\text{SOL}, \Lambda^1)$, observe that γ_t has its support contained in the box

$$Q_t := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, z \geq 0, |x| \leq \sqrt{2t}\}.$$

Recall that $0 \leq \psi_t(x, z) \leq 1$ and the volume form of SOL is $d(\text{vol}) = dx dy dz$. We thus have

$$\begin{aligned} \|\gamma_t\|_r^r &= \int_{x=-\sqrt{2t}}^{\sqrt{2t}} \int_{y=-1}^1 \int_{z=0}^{\infty} |\psi_t(x, z)|^r |dg|^r dx dy dz \\ &\leq \int_{x=-\sqrt{2t}}^{\sqrt{2t}} dx \int_{y=-1}^1 \int_{z=0}^{\infty} |dg|^r dy dz \\ &\leq (2\sqrt{2t}) \int_{y=-1}^1 \int_{z=0}^{\infty} |dg|^r e^z dy dz. \end{aligned}$$

By Lemma 2.3, we know that

$$\int_{\mathbb{R}^2} |dg|^r e^z dy dz < \infty \quad \text{for any } 1 < r < \infty,$$

from which one gets an estimates $\|\gamma_t\|_r \leq C_1(r) t^{1/2r}$. In particular

$$\gamma_t \in \bigcap_{1 < r < \infty} L^r(\text{SOL}, \Lambda^1). \quad (3.1)$$

Step 3. We estimate $\|d\gamma_t\|_{q'}$

We have

$$d\gamma_t = \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial y} dx \wedge dy + \frac{\partial \psi_t}{\partial x} \frac{\partial g}{\partial z} dx \wedge dz + \frac{\partial \psi_t}{\partial z} \frac{\partial g}{\partial y} dz \wedge dy$$

and

$$|dx \wedge dy| = 1, \quad |dx \wedge dz| = e^z, \quad |dz \wedge dy| = e^{-z}.$$

Recall that $\partial g / \partial y$ is bounded, $\partial g / \partial z$ has compact support and $d\gamma_t$ has its support in the region Q_t . Thus

$$|d\gamma_t| \leq C_2 \left(\left| \frac{\partial \psi_t}{\partial x} \right| + \left| \frac{\partial \psi_t}{\partial z} \right| e^{-z} \right).$$

Since $0 < \delta < (1/2)(q' - 1)$, Lemma 2.4 implies

$$\|d\gamma_t\|_{q'} \leq C_3 t^{-\delta/q'}. \tag{3.2}$$

Step 4. We estimate the integral of $\alpha \wedge \gamma_t$

Let

$$A_t := \int_{\text{SOL}} \alpha \wedge \gamma_t.$$

We have

$$\alpha \wedge \gamma_t = \psi_t(x, z) d\varphi \wedge df \wedge dg = (\lambda'(e^{-z}x) e^{-z} \psi_t(x, z)) dx \wedge df \wedge dg$$

(since $dz \wedge df \wedge dg = 0$). By Lemma 2.3, $df \wedge dg \geq 0$, and since $\lambda'(e^{-z}x) \geq 0$ we see that $\alpha \wedge \gamma_t$ is a non negative 3-form. In particular $A_t \geq \int_{\Delta} \alpha \wedge \gamma_t$ for every measurable subset $\Delta \subset \text{SOL}$.

We set

$$\Delta_t := \{(x, y, z) \in \text{SOL} \mid |y| \leq 1, 0 \leq z \leq 1, |x| \leq \sqrt{t}\}.$$

Recall that if $t \geq 1$, $0 \leq z \leq 1$ and $|x| \leq \sqrt{t}$, then $\psi_t(x, z) = 1$, we thus get

$$A_t \geq \int_{\Delta_t} \alpha \wedge \gamma_t = \int_{y=-1}^{+1} \int_{z=0}^1 \int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'(e^{-z}x) e^{-z} dx \wedge df \wedge dg.$$

Now set $u = e^{-z}x$, $du = e^{-z} dx$, $u_0 = -e^{-z}\sqrt{t}$ and $u_1 = e^{-z}\sqrt{t}$. We have

$$\int_{x=-\sqrt{t}}^{\sqrt{t}} \lambda'(e^{-z}x) e^{-z} dx = \int_{u_0}^{u_1} \lambda'(u) du = 1$$

if t is large enough (i.e., $e^{-1}\sqrt{t} \geq 1$). Thus

$$A_t \geq C_4 := \int_{y=-1}^1 \int_{z=0}^1 df \wedge dg > 0. \tag{3.3}$$

Observe that the constant C_4 is positive and independent of t (in fact, using equation (4) of Lemma 2.3 and (i) of Lemma 2.4, we see that $A_t \rightarrow 1$ as $t \rightarrow \infty$).

Step 5. Recapitulation

Let us summarize the previous estimates (3.1), (3.2) and (3.3):

$$\|\gamma_t\|_{p'} + \|\gamma_t\|_{q'} < \infty, \quad \|d\gamma_t\|_{q'} \leq C_3 t^{-\delta/q'} \quad \text{and} \quad \int_{\text{SOL}} \alpha \wedge \gamma_t \geq C_4 > 0.$$

If we let $t \rightarrow \infty$ and apply Proposition 2.1, we obtain $\alpha \notin B_{pq}^2(M)$.

By the construction

$$S_t := \text{supp}(\alpha) \cup \text{supp}(\gamma_t) \subset Q := \{(x, y, z) \mid |y| \leq 1, 0 \leq z\}.$$

Using the group of isometries $T : (x, y, z) \rightarrow (x, y + 2k, z)$, $k \in \mathbb{Z}$, we can produce an infinite family of forms $\alpha_i \in Z_p^k(\text{SOL})$ satisfying the hypothesis of Lemma 2.2. Therefore

$$\dim H_{pq}^2(\text{SOL}) = \infty$$

for all $1 < p, q < \infty$. The proof is complete \square

4. Final remark

The above proof of Theorem 1 is only true for unreduced cohomology. In fact, the work of Jeff Cheeger and Mikhael Gromov gives us the following result in the reduced case (for $p = q = 2$).

THEOREM 2. — *The reduced L_2 -cohomology of SOL is trivial.*

Proof. — The Lie group SOL admits uniform lattices (i.e., discrete cocompact subgroups), see [11] for explicit constructions. The result thus follows from [5], [6] and [7] since every lattice in SOL is amenable. \square

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