ANNALES DE LA FACULTÉ DES SCIENCES DE TOULOUSE

ERIC NABANA

Uniqueness for positive solutions of p-Laplacian problem in an annulus

Annales de la faculté des sciences de Toulouse 6e série, tome 8, nº 1 (1999), p. 143-154

<http://www.numdam.org/item?id=AFST_1999_6_8_1_143_0>

Annales de la Faculté des Sciences de Toulouse Vol. VIII, n° 1, 1999

Uniqueness for positive solutions of p-Laplacian problem in an annulus[(*)]

ERIC NABANA[(1)]

RÉSUMÉ. — À l'aide d'une méthode de tir et de quelques identités de comparaison, nous donnons un résultat d'unicité pour les solutions radiales positives d'un problème de Dirichlet avec le p-laplacien dans un anneau.

ABSTRACT. — By means of a shooting method and some comparison identities, we give a uniqueness result of positive radial solutions for p-Laplacian Dirichlet problem in an annulus.

1. Introduction and Results

We give here a uniqueness result for positive radial solutions of the problem

$$\begin{cases} \Delta_p u + f\big(u, |x|\big) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

where $\Omega = \big\{x \in \mathbb{R}^n \mid 0 < a < |x| < b\big\}$, and $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $1 < p < +\infty$.

We start by transforming the radial equation of (P) into the form :

$$\Big(\big|w'(s)\big|^{p-2} w'(s)\Big)' + F\big(w(s), s\big) = 0\,.$$

Our uniqueness result is then derived by using some comparison identities established here for a initial value problem associated to this second order differential equation.

(*) Reçu le 23 mai 1995, accepté le 10 novembre 1998

(1) L.A.M.F.A., Faculté de Mathématiques et Informatique, Université de Picardie Jules-Verne, 33 rue Saint-Leu, F-80039 Amiens Cedex 1 (France)
E-mail : nabana@u-picardie.fr

In Nabana–de Thelin ([4], [5]), we give uniqueness results of positive radial solutions of (P) for Ω a ball in $\mathbb{R}^n$. In this paper we essentially follow the approach used in [5]. The difficulty is reduced to the obtention of the comparison identities (I1), (I2) and (I3) in Section 2.

This kind of technique is associated to several names in the particular case $p = 2$ (see for example : Kolodner [3], Coffman ([1], [2]), Ni [6], Ni–Nussbaum [7]). It is important to remark that this method is made by the first time for $p \neq 2$ in [5].

Let $u(x) = u(|x|)$ a radial solution of (P). Then u verifies the following problem

$$\begin{cases} \left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' + r^{n-1}f(u(r), r) = 0\,, & a < r < b \\ u(a) = 0\,, \quad u(b) = 0\,. \end{cases} \tag{1.1}$$

By making a change of variables of the kind

$$u(r) = w(s) \quad \text{and} \quad s = \rho(r)\,,$$

then problem (1.1) is transformed into

$$\begin{cases} \left(|w'(s)|^{p-2}w'(s)\right)' + F(w(s), s) = 0\,, & \alpha < s < \beta \\ w(\alpha) = 0\,, \quad w(\beta) = 0\,, \end{cases} \tag{1.2}$$

where α and β depend on a and b.

For the remainder of this section, we first give a uniqueness result for the problem (1.2).

THEOREM 1.1.— *Suppose $p \geq 2$. Assume that $F : \mathbb{R} \times [\alpha, \infty[\to \mathbb{R}$ is C^1 with $F(0, s) = 0$ for $s \geq \alpha$. Suppose that there exists $M > 0$ such that, for $0 < w \leq M$ and $\alpha \leq s \leq \beta$, F satisfies the following conditions :*

(C1) $\quad w\dfrac{\partial F}{\partial w}(w, s) - (p-1)F(w, s) > 0,$

(C2) $\quad (s-\alpha)\dfrac{\partial F}{\partial s}(w, s) + \dfrac{p}{p-1}F(w, s) \geq 0,$

(C3) $\quad \dfrac{\partial F}{\partial s}(w, s) \leq 0.$

Then problem (1.2) has at most one positive solution.

Theorem 1.1 is the fundamental tool in the proofs of the following theorems which are generalisations of some results in [7].

THEOREM 1.2.— *Suppose that $n > p \geq 2$. Suppose that $f : (u,r) \in [0, \infty[\times [a, b] \to f(u,r) \in [0, \infty[$ is C^1 and that $f(0,r) = 0$ for all $r \in [a, b]$.*

Then (P) has at most one positive radial solution provided, for $u > 0$ and $a \leq r \leq b$, the function f satisfies the following conditions

$$u\frac{\partial f}{\partial u}(u,r) - (p-1)f(u,r) > 0\,, \tag{1.3}$$

$$p\left((n-1)b^{-\mu} - \frac{n(p-2)+1}{p-1}\,a^{-\mu}\right) f(u,r) \geq (p-1)(b^{-\mu} - a^{-\mu})r\frac{\partial f}{\partial r}(u,r)\,, \tag{1.4}$$

where $\mu = (n-p)/(p-1) > 0$,

$$p(n-1)f(u,r) + (p-1)r\frac{\partial f}{\partial r}(u,r) \geq 0\,. \tag{1.5}$$

THEOREM 1.3.— *Let $\Omega = \{x \in \mathbb{R}^p \mid 0 < a < |x| < b\}$ with $p \geq 2$. Suppose that $f : (u,r) \in [0, \infty[\times [a, b] \to f(u,r) \in [0, \infty[$ is C^1 and that $f(0,r) = 0$ for all $r \in [a, b]$.*

Then (P) has at most one positive radial solution provided, for $u > 0$ and $a \leq r \leq b$, the function f satisfies (1.3) and the following conditions

$$p\left(\frac{1}{p-1} - \ln\left(\frac{b}{a}\right)\right) f(u,r) \geq \ln\left(\frac{b}{a}\right) r\frac{\partial f}{\partial r}(u,r)\,, \tag{1.6}$$

$$pf(u,r) + r\frac{\partial f}{\partial r}(u,r) \geq 0\,. \tag{1.7}$$

If $\partial f(u,r)/\partial r \equiv 0$ in Theorem 1.2 and in Theorem 1.3, we obtain the following corollary.

COROLLARY 1.4.— *Suppose that $n \geq p \geq 2$ and that*

$$\frac{b}{a} \leq \begin{cases} e^{1/(p-1)} & \text{for } n = p\,, \\ \left(\dfrac{(p-1)(n-1)}{n(p-2)+1}\right)^{(p-1)/(n-p)} & \text{for } n > p\,. \end{cases}$$

Then, the problem

$$\begin{cases} \Delta_p u + f(u) = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

has at most one positive radial solution provided

$$0 < (p-1)f(t) < tf'(t) \quad for\ t > 0\,.$$

2. Proof of Results

We begin this section by giving a theorem on existence and uniqueness of a positive solution of the following initial value problem :

$$\begin{cases} \left(r^{n-1}|u'(r)|^{p-2}u'(r)\right)' + r^{n-1}f\big(u(r), r\big) = 0\,, & a < r \\ u(a) = 0, \quad u'(a) = d > 0\,. \end{cases} \tag{2.1}$$

THEOREM 2.1.— *Suppose that $n \geq p$ and that the function $f : (u, r) \in \mathbb{R} \times \mathbb{R}^+ \to f(u, r) \in \mathbb{R}$ is C^1. Then for each $d > 0$, the problem (2.1) has a unique positive solution $u(r) = u(r, d)$ defined in a maximal interval $J_d \subset [\,a\,, +\infty\,[$. The function $d \to u(r, d)$ is C^1 and $r \to u(r, d)$ is C^2 if $1 < p \leq 2$, and at all points $r \geq a$ such that $u'(r) \neq 0$ if $p > 2$.*

We prove this theorem in the Appendix.

2.1 Proof of Theorem 1.1

Let $w(s, d)$ be a solution of the following initial value problem

$$\begin{cases} \left(|w'(s)|^{p-2}w'(s)\right)' + F\big(w(s), s\big) = 0\,, & \alpha \leq s \\ w(\alpha) = 0\,, \quad w'(\alpha) = d > 0\,. \end{cases} \tag{2.2}$$

Consider the set

$$D = \big\{d > 0 \mid \text{there exists an } s > 0 \text{ such that } w(s, d) = 0\big\}$$

and the map

$$T : d \in D \longrightarrow T(d) = \min\{s > 0 \mid w(s, d) = 0\}\,.$$

The domain D is open in (α, ∞) and since $w'\big(T(d), d\big) \neq 0$, it arises from the implicit function theorem that T is C^1 at the neighbourhood of D's points and that

$$T'(d) = -\frac{\dfrac{\partial w}{\partial d}\,\big(T(d), d\big)}{w'\big(T(d), d\big)}\,.$$

Put $\varphi(s, d) = \partial w(s, d)/\partial d$. Then φ satisfies the following problem

$$\begin{cases} (p-1)\Big(\big|w'(s)\big|^{p-2}\varphi'(s)\Big)' + \dfrac{\partial F}{\partial w}\,\big(w(s), s\big)\varphi(s) = 0\,, \quad \alpha \le s \\ \varphi(\alpha) = 0\,, \quad \varphi'(\alpha) = 1\,. \end{cases} \tag{2.3}$$

As in the proof of Theorem 2.2 in [6], we are going to show that T is strictly decreasing on D. Since

$$T'(d) = -\frac{\varphi\big(T(d), d\big)}{w'\big(T(d), d\big)} \quad \text{and} \quad w'\big(T(d), d\big) < 0 \text{ for all } d \in D\,,$$

it is sufficient to prove that $\varphi\big(T(d), d\big) < 0$.

Put

$$v = \big|w'\big|^{p-2} w' \quad \text{and} \quad \psi = \frac{\partial v}{\partial d} = (p-1)\big|w'\big|^{p-2}\varphi'\,.$$

Then, ψ satisfies the equation

$$\psi' + \frac{\partial F}{\partial w}\,(w, s)\varphi = 0\,, \quad s \ge \alpha\,.$$

Note that, from the hypothesis $F(0, s) = 0$ for all $s \ge \alpha$ and the uniqueness of $w(s, d)$, we deduce that $v(s, d) \neq 0$ at $s \ge \alpha$ for which $w(s, d) = 0$. In the same way, if $\varphi(s_0) = 0$, then necessarily $\psi(s_0) \neq 0$.

Now, for a given $d > 0$, put $\beta = T(d)$.

LEMMA 2.1.— *Let $X = (p-1)v\varphi - w\psi$. Then*

$$\text{(I1)} \quad X' = \left(w\frac{\partial F}{\partial w}(w, s) - (p-1)F(w, s)\right)\varphi$$

and the function $\varphi(s)$ vanishes at least once in the interval (α, β).

Proof.— The identity (I1) is straightforward.

Suppose that φ does not vanish in (α,β). Then $\varphi(s) > 0$ in (α,β). Integrating (I1) from α to β, we obtain

$$X(\beta) - X(\alpha) = \int_\alpha^\beta \left(w(t) \frac{\partial F}{\partial w} \left(w(t), t\right) - (p-1) F\left(w(t), t\right) \right) \varphi(t)\,\mathrm{d}t\,. \quad (2.4)$$

Since X is continuous on $[\,\alpha\,,\,\beta\,]$ with

$$X(\alpha) = 0 \quad \text{and} \quad X(\beta) = (p-1) v(\beta) \varphi(\beta) < 0\,,$$

then, the left-hand of (2.4) would be negative while the right-hand remains positive because of the condition (C1) of Theorem 1.1; this is a contradiction. □

Since $F(w,s) > 0$ for $0 < w \leq M$, and $\alpha < s < \beta$, there exists a unique point $\gamma_0 \in (\alpha,\beta)$ such that $v(\gamma_0) = 0$, $v > 0$ on $[\,\alpha\,,\,\gamma_0\,[$ and $v < 0$ on $]\,\gamma_0\,,\,\beta\,]$.

LEMMA 2.2.— *Let* $Y = (s-\alpha) w' \psi - \big((s-\alpha)v\big)' \varphi$. *Then, for all* $s \in [\,\alpha\,,\,\beta\,] \setminus \{\gamma_0\}$, *we have*

$$\textit{(I2)} \qquad Y' + \frac{p'-2}{s-\alpha} Y = \frac{2-p'}{s-\alpha} v\varphi + \left\{ (s-\alpha) \frac{\partial F}{\partial s}(w,s) + p' F(w,s) \right\} \varphi$$

with $p' = p/(p-1)$. *Let* $\xi_0 = \min\{ s > \alpha \mid \varphi(s) = 0 \}$. *Then* $\gamma_0 \leq \xi_0$.

Proof.— A straightforward computation gives the comparison identity (I2).

Let $\varepsilon > 0$ small enough and consider the function H defined by

$$H(s) = (p'-2) \int_{\alpha+\varepsilon}^{s} \frac{1}{t-\alpha}\,\mathrm{d}t\,.$$

Then, (I2) can be put into the following form

$$\left(e^{H(s)} Y(s)\right)' = e^{H(s)} \left\{ \frac{2-p'}{s-\alpha} v\varphi + \left((s-\alpha) \frac{\partial F}{\partial s}(w,s) + p' F(w,s) \right) \right\} \varphi\,. \quad (2.5)$$

The existence of ξ_0 is given by Lemma 2.1, and we have $\varphi(s) > 0$ for $\alpha < s < \xi_0$ and $\psi(\xi_0) < 0$. Suppose that $\xi_0 < \gamma$; Then

$$v(s) > 0 \quad \text{for all } s \text{ such that } \alpha \leq s \leq \xi_0\,.$$

Integrating (2.5) from $\alpha+\varepsilon$ to ξ_0 and by using condition (C2) in Theorem 1.1, we obtain

$$e^{H(\xi_0)}Y(\xi_0) \geq e^{H(\alpha+\varepsilon)}Y(\alpha+\varepsilon) = Y(\alpha+\varepsilon) > 0\,.$$

Therefore

$$Y(\xi_0) = (\xi_0-\alpha)w'(\xi_0)\psi(\xi_0) > 0\,.$$

thus $w'(\xi_0) < 0$; this is a contradiction. □

LEMMA 2.3.— *Let $Z = w'\psi - v'\varphi$. Then :*

(I3) $\quad Z' = \dfrac{\partial F}{\partial s}(w,s)\varphi$

and ξ_0 is the unique zero of φ in $]\,\alpha\,,\,\beta\,[$.

Proof.— The identity (I3) is straightforward.

Suppose that φ has others zeros in $(\alpha\,,\,\beta\,]$ and let

$$\xi_1 = \min\{s > \xi_0 \mid \varphi(s) = 0\}\,.$$

Then, $\varphi < 0$ in the interval (ξ_0, ξ_1) and $\psi(\xi_1) > 0$. By integrating (I3) frow ξ_0 to ξ_1, and by considering the condition (C3), we have

$$Z(\xi_1) = w'(\xi_1)\psi'(\xi_1) \geq Z(\xi_0) = w'(\xi_0)$$

thus $w'(\xi_1) > 0$; a contradiction with the Lemma 2.2. □

It follows from Lemma 2.3 that $\varphi(s) < 0$ on $\xi_0 < s \leq \beta$. In particular $\varphi\big(T(d), d\big) < 0$, so Theorem 1.1 is proved. □

2.2 Proof of Theorem 1.2

Let $r = |x|$ and $u(x) = u(r)$, a positive radial solution of (P). Then, u satisfies the problem (1.1). Consider the following change of variables

$$u(r) = w(s)\,,$$

$$s = \frac{p-1}{n-p}\left(\frac{1}{r}\right)^{(n-p)/(p-1)} \quad \text{for } n > p\,.$$

Then problem (1.1) is transformed into

$$\begin{cases} \left(|w'(s)|^{p-2}w'(s)\right)' + F(w(s),s) = 0\,, & \alpha < s < \beta \\ w(\alpha) = 0\,, \quad w(\beta) = 0 \end{cases} \tag{2.6}$$

where

$$\alpha = \frac{p-1}{n-p}\left(\frac{1}{b}\right)^{(n-p)/(p-1)} \quad \text{and} \quad \beta = \frac{p-1}{n-p}\left(\frac{1}{a}\right)^{(n-p)/(p-1)}\,;$$

the function F is given by

$$F(w,s) = \left(\frac{p-1}{(n-p)s}\right)^{p(n-1)/(n-p)} f\left(w, \left(\frac{p-1}{n-p}\right)^{(p-1)/(n-p)} s^{(1-p)/(n-p)}\right). \tag{2.7}$$

We are going to apply Theorem 1.1 to the problem (2.6) in order to obtain the uniqueness of u, positive radial solution of (P). It suffices to show that the conditions (C1), (C2) and (C3) of Theorem 1.1 are satisfied by the function F defined at (2.7).

By a straightforward computation, we obtain an equivalence between (C1) and the condition (1.3) of Theorem 1.2 on the one hand and between (C3) and (1.5) on the other hand. Otherwise, (C2) is equivalent to

$$\left((n-1)\alpha - \frac{n(p-2)+1}{p-1}\,s\right) f(u,r) \geq$$

$$\geq \frac{p-1}{p}(s-\alpha) r \frac{\partial f}{\partial r}(u,r) \quad \text{for all } s \in [\alpha\,,\,\beta]\,.$$

This inequality is satisfied if f verifies the relation (1.4). Theorem 1.2 is thus proved. □

2.3 Proof of Theorem 1.3

Let $r = |x|$ and $u(x) = u(r)$, a positive radial solution of (P). The change of variables

$$u(r) = w(s)\,,$$

$$s = -\ln(r) \quad \text{for } n = p\,,$$

transforms the problem (1.1) into

$$\begin{cases} \left(|w'(s)|^{p-2}w'(s)\right)' + e^{-ps} f(w, e^s) = 0\,, & \alpha < s < \beta \\ w(\alpha) = 0\,, \quad w(\beta) = 0 \end{cases} \tag{2.8}$$

where $\alpha = -\ln(b)$, and $\beta = -\ln(a)$.

Put

$$F(w,s) = e^{-ps} f(w, e^s)\,. \tag{2.9}$$

It is easy to see that F given at (2.9) satisfies the conditions (C1), and (C3) of Theorem 1.1 if f verifies respectively the relations (1.3) and (1.7). The condition (C2) is verified by F if and only if

$$p\left(\frac{1}{p-1} - (s-\alpha)\right) f(u,r) \geq (s-\alpha) r \frac{\partial f}{\partial r}(u,r) \quad \text{for all } s \in [\,\alpha\,,\,\beta\,]\,.$$

We observe that this last inequality is valid if f satisfies (1.6).

Then, the Theorem 1.3 follows immediately from Theorem 1.1 applied to the problem (2.8). □

3. Appendix – Proof of Theorem 2.1

Let $\mathbf{K}$ be a compact in $\mathbb{R}^2$ defined by

$$\mathbf{K} = \big\{(u,r) \mid a \leq r \leq a + \delta_0 \text{ and } |u| \leq \varepsilon_0\big\}$$

where $\delta_0 > 0$ and $\varepsilon_0 > 0$ are fixed. Put

$$M = \sup\left\{\max_{(u,r)\in K} \big|f(u,r)\big|\,;\ \max_{(u,r)\in K}\left|\frac{\partial f}{\partial u}(u,r)\right|\right\}.$$

Consider E the Banach space defined by

$$E = \left\{(u,v) \in \Big(C^0([\,a\,,\,a+\delta\,])\,,\,\mathbb{R}\Big)^2 \;\middle|\; u(a) = 0\,,\ v(a) = d^{p-1} > 0\right\}$$

with the norm

$$\big\|(u,v)\big\| = \max\big(\|u\|_0\,;\ \|v\|_0\big)$$

where

$$\|u\|_0 = \max_{a\leq r\leq a+\delta} \big|u(r)\big|\,.$$

Select $d_0 > 0$, such that $|d - d_0| \leq \varepsilon \leq \varepsilon_0$.

Existence and uniqueness for (2.1) result from the study of fixed point of the application $\mathcal{F}_d : (u,v) \in E \longrightarrow (\bar{u},\bar{v}) \in E$ where $\bar{u}$ and $\bar{v}$ are defined by

$$\bar{u}(r) = \int_a^r \frac{a^k}{t^k} \left|v(t)\right|^{p'-2} v(t)\,\mathrm{d}t \quad \text{with } k = \frac{n-1}{p-1} \text{ and } p' = \frac{p}{p-1},$$
$$\bar{v}(r) = d^{p-1} - \int_a^r \frac{t^{n-1}}{a^{n-1}} f\big(u(t),t\big)\,\mathrm{d}t .$$

Let $B = \big\{(u,v) \in E \mid \|u\|_0 \le \varepsilon \text{ and } \|v - d^{p-1}\|_0 \le \varepsilon\big\}$ a closed, convex bounded subset in E.

Then for $\delta \le \delta_0$ small enough and $|d - d_0| \le \varepsilon$, $\mathcal{F}_d$ is a Lipschitz map of B into itself with Lipschitz constant less than one. In fact for all $(u,v) \in B$ we have

$$\left\|\bar{u}\right\|_0 \le \max_{a \le r \le a+\delta} \int_a^r \frac{a^k}{t^k} \left|v(t)\right|^{p'-1} \mathrm{d}t \le \delta \left\|v\right\|_0^{p'-1} \le \delta\big(\varepsilon + d_0^{p-1}\big)^{p'-1}$$

and then

$$\left\|\mathcal{F}_d(u,v) - (0, d^{p-1})\right\| \le \sup\Big(\delta\big(\varepsilon + d_0^{p-1}\big)^{p'-1} ;\, \delta M\Big) .$$

Therefore $\mathcal{F}_d(B) \subset B$ for δ small enough.

Note that, for δ small enough, there exists two constants $c_1 > 0$ and $c_2 > 0$ such that for all $v \in C^0\big([\,a\,,\,a+\delta\,]\big)$ with $v(a) > 0$, we have

$$c_1 \le v(r) \le c_2\,, \quad a \le r \le a + \delta .$$

Then, for $v_i \in C^0\big([\,a\,,\,a+\delta\,]\big)$ with $v_i(a) = d^{p-1} > 0$, $i = 1, 2$, we obtain

$$\Big\| \left|v_1\right|^{p'-1} v_1 - \left|v_2\right|^{p'-1} v_2 \Big\|_0 \le c(p) \left\|v_1 - v_2\right\|_0$$

where $c(p) = (p'-1)c_2^{p'-2}$ if $1 < p \le 2$, and $c(p) = (p'-1)c_1^{p'-2}$ if $p > 2$.

For all $X_i = (u_i, v_i) \in B$, $i = 1, 2$, we have

$$\begin{aligned} \left\|\mathcal{F}_d(X_1) - \mathcal{F}_d(X_2)\right\| &= \max\Big(\left\|\bar{u}_1 - \bar{u}_2\right\|_0 ;\, \left\|\bar{v}_1 - \bar{v}_2\right\|_0\Big) \\ &\le \delta \max\Big(c(p)\left\|v_1 - v_2\right\|_0 ;\, M\left\|u_1 - u_2\right\|_0\Big) \\ &\le \delta \max(c(p) ;\, M)\left\|X_1 - X_2\right\| . \end{aligned}$$

The contraction mapping principle implies that $\mathcal{F}_d$ has a unique fixed point $X = (u, v)$ in B. The functions u and v are in $C^1\big([\,a\,,\,a+\delta\,],\mathbb{R}\big)$ and we have

$$v(r) = \Big(\frac{r}{a}\Big)^{n-1} \big|u'(r)\big|^{p-2} u'(r)$$

$$v'(r) = -\Big(\frac{r}{a}\Big)^{n-1} f\big(u(r), r\big)$$

$$u(a) = 0\,, \quad u'(a) = d > 0$$

and thus, $u(r) = u(r, d)$ is the unique solution of (2.1).

By using the same arguments of classical theory on existence and uniqueness for initial value problem for ordinary differential equations, we show that u is uniquely defined on a maximal interval J_d in $[\,a\,,\,+\infty\,[$.

It is immediate to see that $r \longrightarrow u(r)$ is C^2 on J_d if $1 < p \leq 2$, and that $r \longrightarrow u(r)$ is C^2 only at the points r such that $u'(r) \neq 0$.

To prove that $d \longrightarrow u(r, d)$ is C^1, we consider the following map

$$\mathcal{G} : (d, X) \in \,]\,0, \infty\,[\, \times E \longrightarrow X - \mathcal{F}_d(X) \in E\,.$$

For all $X \in B$ and δ small enough, we have $\mathrm{Ker}\big(D_X \mathcal{G}(d, X)\big) = \{0_E\}$. Therefore, by the implicit function theorem for Banach spaces, we conclude that the functions $d \longrightarrow u(r, d)$ and $d \longrightarrow u'(r, d)$ are C^1. □

Bibliography

[1] COFFMAN (C. V.) .— *On the positive solutions of boundary value problems for a class of nonlinear differential equations*, J. Diff. Eqns **3** (1967), pp. 92-111.

[2] COFFMAN (C. V.) .— *Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization for other solutions*, Arch. Rational Mech. Anal. **46** (1972), pp. 81-95.

[3] KOLODNER (I. I.) .— *Heavy rotating string — a nonlinear eigenvalue problem*, Comm. Pure Appl. Math. **8** (1955), pp. 395-408.

[4] NABANA (E.) and DE THELIN (F.) .— *Unicité de la solution radiale positive de l'équation quasilinéaire : $\Delta_p u + f(u, |x|) = 0$*, C. R. Acad. Sc. Paris **307** (1988), pp. 763-766.

[5] NABANA (E.) and DE THELIN (F.) .— *On uniqueness of positive solution for quasilinear elliptic equation*, Nonlinear Analysis TMA **31**, No 3/4 (1998), pp. 413-430.

[6] Ni (W. M.) .— *Uniqueness of solutions of nonlinear Dirichlet problems*, J. Diff. Eqns **50** (1983), pp. 289-304.

[7] Ni (W. M.) and Nussbaum (R. D.) .— *Uniqueness and nonuniqueness for positive radial solutions of* $\Delta u + f(u,r) = 0$, Comm. Pure Appl. Math. **38** (1985), pp. 67-108.