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Realization of Hölder Complexes(*)

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Résumé. — Un complexe de Hölder est un graphe fini tel qu'à chaque arête est associé un nombre rationnel positif et on sait que c'est un invariant bi-lipschitzien des ensembles semi-algébriques singuliers de dimension 2. On montre dans cet article que tout complexe de Hölder peut être réalisé comme un ensemble semi-algébrique de dimension 2. Pour ce faire on plonge le graphe dans un tore de dimension n qu'on fait contracter sur un point singulier de telle sorte que les générateurs s'évanouissent avec les vitesses rationnelles et différentes.

Abstract. — Hölder Complex, a graph and a rationaly-valued function on the set of the edges of the graph, is a bi-Lipschitz invariant of 2-dimensional semialgebraic singular sets. Here we prove that each Hölder Complex can be realized as a 2-dimensional semialgebraic set. For this purpose we embed the graph to an n-dimensional torus. The torus is vanishing in a singular point such that the generators are vanishing with different rational rates.

1. Introduction

The paper is devoted to the local geometry of 2-dimensional semialgebraic sets. The local bi-Lipschitz classification theorem is proved in [1]. The main notion of the classification is a so-called Geometric Hölder Complex. It is a local version of a simplicial complex with some additional geometric information (see the definition below). A Hölder Complex can be considered as a combinatorial object – a finite graph with a rational-valued function defined on the set of edges.

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The following question is natural. Let us define a Hölder Complex in a combinatorial way. Does it correspond to some semialgebraic set?

The answer is positive. To prove the Realization theorem we define a semialgebraic set $T(\beta_1, \ldots, \beta_k)$. It is a generalization of the real algebraic set which gives an example of the noncoincidence of $L_p$-cohomology and Intersection Homology \cite{2}. The set $T(\beta_1, \ldots, \beta_k)$ has a toric link at the singular point and all generators of the torus have different vanishing rates in this point. It gives us a possibility to separate vanishing rates of all edges of a Hölder Complex.

2. Definitions and notations

Let us recall some definitions from \cite{1}. Let $\Gamma$ be a connected graph without loops, $V_\Gamma = \{a_1, a_2, \ldots, a_k\}$ be the set of vertices and $E_\Gamma = \{g_1, g_2, \ldots, g_r\}$ be the set of edges of the graph.

**DEFINITION 2.1.** A Hölder Complex $(\Gamma, \beta)$ is a graph $\Gamma$ with an associated function $\beta: E_\Gamma \to [1, \infty[ \cap \mathbb{Q}$ (here $\mathbb{Q}$ is the ring of rational numbers).

**DEFINITION 2.2.** A Curvilinear triangle $T$ is a subset of homeomorphic to a 2-dimensional simplex satisfying the following properties.

1) Each internal (in the induced topology) point $t \in T$ has an open neighbourhood $U_t \subset T$ such that $U_t$ is a smooth 2-dimensional submanifold of $\mathbb{R}^n$ at each point $t' \in U_t$.

2) The boundary of $T$ is a union of three analytic curves $\gamma_1$, $\gamma_2$, $\gamma_3$ such that $\gamma_i$ (for $i = 1, 2, 3$) has a neighbourhood at each internal (in the induced from $\mathbb{R}$ topology on $\gamma_i$) point which is a smooth 1-dimensional submanifold of $\mathbb{R}^n$.

3) Locally $T$ is a smooth manifold with a boundary at each smooth point of the boundary.

Boundary points of $\gamma_i$ we call vertices of $T$.

**DEFINITION 2.3.** A standard $\beta$-Hölder triangle $ST_{\beta}$ is a subset of the plane $\mathbb{R}^2$ bounded by the following curves:

$$\{y = 0\}, \quad \{y = x^\beta\}, \quad \{x = 1\}.$$
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Let us consider a cone $CT$ over $\Gamma$. Let $A_0$ be the vertex of $CT$. We can consider $CT$ as a topological space with the standard topology of a simplicial complex.

**Definition 2.4.** — A subset $H(\Gamma, \beta) \subset \mathbb{R}^n$ is called a Geometric Hölder Complex corresponding to $(\Gamma, \beta)$ if it satisfies the following conditions.

1) $H(\Gamma, \beta)$ is a subanalytic subset of $\mathbb{R}^n$.
2) There exists a homeomorphism $F: CT \rightarrow H(\Gamma, \beta)$.
3) The set $H(\Gamma, \beta) \cap S_{F(A_0), r}$ is empty or homeomorphic to $\Gamma$, for every $r$. (We use the notation $S_{F(A_0), r}$ for the sphere centered at the point $F(A_0)$ with the radius $r$.)
4) The image of the triangle $(A_0, a_i, a_j, g)$ (where $a_i$ and $a_j$ are vertices of $\Gamma$, $g$ is the edge connecting $a_i$ and $a_j$, $(A_0, a_i, a_j, g)$ is the subcone of $CT$ over $g$) has the following properties:
   a) $F(A_0, a_i, a_j, g)$ is a subanalytic subset of $\mathbb{R}^n$;
   b) $F(A_0, a_i, a_j, g)$ is subanalytically bi-Lipschitz equivalent to the standard $\beta(g)$-Hölder triangle $ST_{\beta(g)}$;
   c) let $L: ST_{\beta(g)} \rightarrow F(A_0, a_i, a_j, g)$ be this subanalytic bi-Lipschitz map; then
      $$L(0, 0) = F(A_0), \quad L(1, 0) = F(a_i), \quad L(1, 1) = F(a_j).$$

**Definition 2.5.** — A $\beta$-Hölder triangle $HT_\beta$ is a subset of $\mathbb{R}^n$ satisfying the following conditions.

1) $HT_\beta$ is a curvilinear triangle.
2) $HT_\beta$ is bi-Lipschitz equivalent to some standard $\beta$-Hölder triangle $ST_\beta$.
3) The bi-Lipschitz map $L: ST_\beta \rightarrow HT_\beta$ is subanalytic. (The image of the point $(0, 0)$ is called a Hölder vertex of $HT_\beta$.)

**Definition 2.6.** — A standard $\beta$-horn $SH_\beta$ (here $\beta \in \mathbb{Q} \cap [1, +\infty[$) is a semialgebraic set in $\mathbb{R}^3$ defined by the following conditions:

$$ (x_1^2 + x_2^2)^{\beta} = y^p, \quad 0 \leq y \leq 1, $$

$(x_1, x_2, y)$ are coordinates of a point in $\mathbb{R}^3$ and $\beta = p/q$ with $\gcd(p, q) = 1$. 

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We proved in [1] that every 2-dimensional semialgebraic (as well as semianalytic and subanalytic) set $X$ is a Geometric Hölder Complex in a neighbourhood of a given point $a_0 \in X$ corresponding to some Hölder Complex. Here we are going to prove the following result.

**Realization Theorem.** — Let $(\Gamma, \beta)$ be a Hölder Complex. Then there exist a semialgebraic 2-dimensional set $X \subset \mathbb{R}^n$, a point $a_0 \in X$ and $\varepsilon > 0$ such that $X \cap B_{a_0,\varepsilon}$ is a Geometric Hölder Complex corresponding to the Hölder Complex $(\Gamma, \beta)$ (here $B_{a_0,\varepsilon}$ is a closed ball in $\mathbb{R}^n$ centered at the point $a_0$ with the radius $\varepsilon$).

3. The set $T(\beta_1, \ldots, \beta_k)$. Polar maps

We consider the space $\mathbb{R}^{2k+1}$ with coordinates $(x_1, y_1, x_2, y_2, \ldots, x_k, y_k, z)$. Let $D(\beta_1, \ldots, \beta_k)$ (here $\beta_i = \frac{p_i}{q_i}$ with $p_i, q_i \in \mathbb{Z}$ and $\gcd(p_i, q_i) = 1$) be a subvariety of $\mathbb{R}^{2k+1}$ given by the following equations:

$$
\begin{align*}
z^{2p_1} &= (x_1^2 + y_1^2)^{q_1} \\
& \quad \vdots \\
z^{2p_i} &= (x_i^2 + y_i^2)^{q_i} \\
& \quad \vdots \\
z^{2p_k} &= (x_k^2 + y_k^2)^{q_k}.
\end{align*}
$$

(The set described in the paper [2] is a special 3-dimensional example of $D(\beta_1, \beta_2)$.)

Let

$$
T(\beta_1, \ldots, \beta_k) = D(\beta_1, \ldots, \beta_k) \cap \{z \geq 0\}. \tag{2}
$$

**Lemma 3.1**

1) $\dim T(\beta_1, \ldots, \beta_k) = k + 1$.

2) The link of $T(\beta_1, \ldots, \beta_k)$ at the point $(0, \ldots, 0)$ is homeomorphic to $T^k$ (a $k$-dimensional torus).

(Remind that the link of $T(\beta_1, \ldots, \beta_k)$ is the intersection of $T(\beta_1, \ldots, \beta_k)$ with a small sphere centered at $(0, \ldots, 0)$.)
Proof
1) Consider a section of $T(\beta_1, \ldots, \beta_k)$ by the plane $z = c$. We obtain the equations
\[ x_i^2 + y_i^2 = c_i, \]
where $c_i = c^{2^p_i / q_i}$. Clearly, these equations define a $k$-dimensional torus. The variety $T(\beta_1, \ldots, \beta_k)$ we obtain as a suspension of it. So, (1) is proved.

2) Let $r(z)$ be a function defined in the following way:
\[ r(z) = \sqrt{z^2 + \sum_{i=1}^{k} z^{\beta_i}}. \]
This function $r(z)$ is a one-to-one function, for small $z$. Thus, for sufficiently small $\varepsilon > 0$, the link $T(\beta_1, \ldots, \beta_k) \cap S_{0, \varepsilon}$ is equal to the torus $T(\beta_1, \ldots, \beta_k) \cap \{(x_1, y_1, \ldots, x_k, y_k, z) \in \mathbb{R}^{2k+1} | z = r^{-1}(\varepsilon)\}$. □

Each point of $T(\beta_1, \ldots, \beta_k)$ has uniquely defined polar coordinates $(\psi_1, \psi_2, \ldots, \psi_k, z)$: $\psi_i$ is the angle coordinate of the corresponding point of the circle $x_i^2 + y_i^2 = c_i$ and $z$ is a $z$-coordinate in $\mathbb{R}^{2k+1}$. Let $x^0 = (\psi^0, z^0) = (\psi_1^0, \ldots, \psi_k^0, z^0)$ be a point of $T(\beta_1, \ldots, \beta_k)$. Let $L_{x^0}$ be a curve on $T(\beta_1, \ldots, \beta_k)$ defined as follows:
\[ L_{x^0} = \{(\psi_1, \psi_2, \ldots, \psi_k, z) | \psi_1 = \psi_1^0, \ldots, \psi_k = \psi_k^0\}. \]
We call $L_{x^0}$ a polar line generated by $x^0$. Now we can define a polar map in the following way.

Denote, for $\varepsilon > 0$, the set
\[ T(\beta_1, \ldots, \beta_k) \cap \{(x_1, y_1, \ldots, x_k, y_k, z) \in \mathbb{R}^{2k+1} | z \leq \varepsilon\} \]
by $T^\varepsilon(\beta_1, \ldots, \beta_k)$. Let $P_{\varepsilon_1, \varepsilon_2} : T^{\varepsilon_1}(\beta_1, \ldots, \beta_k) \to T^{\varepsilon_2}(\beta_1, \ldots, \beta_k)$ be a map defined as follows:
\[ P_{\varepsilon_1, \varepsilon_2}(\psi_1, \ldots, \psi_k, z) = \left(\psi_1, \ldots, \psi_k, \frac{\varepsilon_1}{\varepsilon_2} z\right). \]
We call $P_{\varepsilon_1, \varepsilon_2}$ a polar map. Observe that $P_{\varepsilon_1, \varepsilon_2}$ is a bi-Lipschitz map.
Remark 3.1. — $T(\beta_1)$ is an usual $\beta_1$-horn.

Remark 3.2. — $T(\beta_1, \ldots, \beta_k)$ is included to $T(\beta_1, \ldots, \beta_k, \ldots, \beta_n)$ (here $n \geq k+1$) as a semialgebraic subset defined by the following equations $\psi_{k+1} = b_1$, $\psi_{k+2} = b_2$, \ldots, $\psi_n = b_{n-k}$, $b_1$, \ldots, $b_{n-k} \in \mathbb{R}$.

4. Proof of the Realization theorem

We use the induction on the number of edges. Suppose that each H"{o}lder Complex $(\Gamma, \beta)$ whose graph $\Gamma$ has less or equal than $k$ edges is realized as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k)$ such that all vertices of $\Gamma$ belong to the section by the plane $z = 1$ and, for each vertex $a$, we have $\psi_i(a) = 0$ or $\psi_i(a) = \pi$. (We can identify the graph $\Gamma$ and its image by the map $F$; see Definition 2.4.)

For $k = 1$, the assertion is trivial: $\Gamma$ has two vertices $a_1$ and $a_2$. Set $\psi(a_1) = 0$, $\psi(a_2) = \pi$ and the edge connecting $a_1$ and $a_2$ be a half-circle. So, $(\Gamma, \beta)$ is realized as a half of the standard $\beta$-horn.

Now consider a H"{o}lder Complex $(\Gamma, \beta)$ such that $\Gamma$ has $(k + 1)$ edges. Let $g$ be an edge such that $\beta(g) = \min_{g \in E_{\Gamma}} \beta(g)$. Let us consider a graph $\Gamma' = \Gamma - g$. We have two possibilities: $\Gamma'$ is a connected graph or $\Gamma'$ is not connected.

Suppose that $\Gamma'$ is not connected. Then it is a union of two connected components $\tilde{\Gamma} = \tilde{\Gamma}^1 \cup \tilde{\Gamma}^2$ (we include also a case when one of these components is just a vertex). We can suppose that $g_1, \ldots, g_\ell \in E_{\tilde{\Gamma}^1}$, $g_{\ell+1}, \ldots, g_k \in E_{\tilde{\Gamma}^2}$, $g_{k+1} = g$. Now consider a set $T(\beta_1, \ldots, \beta_k, \beta(g))$ and a section of that by the plane $z = 1$. This section is a $(k + 1)$-dimensional torus (see the proof of the Lemma 3.1). By the induction hypotheses, the subcomplex $(\tilde{\Gamma}^1, \tilde{\beta}^1)$, where $\tilde{\beta}^1 = \beta \big|_{\tilde{\Gamma}^1}$, can be realized as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k)$ which can be considered as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k, \beta(g))$ given by the equation $\psi_{k+1} = 0$ (see the Remark 3.2). By the same way, $(\tilde{\Gamma}^2, \tilde{\beta}^2)$, where $\tilde{\beta}^2 = \beta \big|_{\tilde{\Gamma}^2}$, can be realized as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k)$ which can be considered as a semialgebraic subset of $T(\beta_1, \ldots, \beta_k, \beta(g))$ given by the equation $\psi_{k+1} = \pi$. Suppose that $g$ connects vertices $a_1 \in \tilde{\Gamma}^1$ and $a_2 \in \tilde{\Gamma}^2$.
let $a_1$ has polar coordinates $(\psi_1(a_1), \ldots, \psi_k(a_1), 0)$ and let $a_2$ has polar coordinates $(\psi_1(a_2), \ldots, \psi_k(a_2), \pi)$. We connect these two vertices by the following curve $\Psi(\theta) = \{\psi_1(\theta), \psi_2(\theta), \ldots, \psi_{k+1}(\theta), 1\}$ where

$$
\psi_{k+1}(\theta) = \theta, \quad \psi_i(\theta) = \begin{cases}
\psi_i(a_1) & \text{if } \psi_i(a_1) = \psi_i(a_2) \\
\theta & \text{if } \psi_i(a_1) = 0 \text{ and } \psi_i(a_2) = \pi \\
\pi + \theta & \text{if } \psi_i(a_1) = \pi \text{ and } \psi_i(a_2) = 0,
\end{cases}
$$

(3) $1 \leq i \leq k, \theta \in [0, \pi]$. Clearly, $\Psi(0) = a_1$ and $\Psi(\pi) = a_2$. Define

$$
H_{\beta(g)} := \bigcup_{\theta} L_{\Psi(\theta)},
$$

the union of the polar lines generated by $\Psi(\theta)$.

**Lemma 4.1.** — The set $H_{\beta(g)}$ is a $\beta(g)$-Hölder triangle.

**Proof.** — $H_{\beta(g)}$ is a semialgebraic set because it is defined by the system (3) which can be written as a system of algebraic equations and inequalities in terms of variables $x_i, y_i$, for $1 \leq i \leq k+1$, and by the inequalities $0 \leq z \leq 1$. Hence, $H_{\beta(g)} \cap B_{0,\varepsilon}$ (here $B_{0,\varepsilon}$ is a closed ball in $\mathbb{R}^{2k+3}$ centered at 0 with the radius $\varepsilon$) is a Geometric Hölder Complex $H(\overline{\Gamma}, \alpha)$ corresponding to some graph $\overline{\Gamma}$ with some rational-valued function $\alpha$ defined on its edges [1]. Since $H_{\beta(g)}$ is a curvilinear triangle (by the construction), $H_{\beta(g)} \cap B_{0,\varepsilon_0}$, for sufficiently small $\varepsilon_0 \leq \varepsilon$, is bi-Lipschitz equivalent to the standard $\alpha_0$-Hölder triangle where $\alpha_0 = \min_{\overline{\Gamma}} \alpha(\overline{\gamma})$ [1, Second Structural Lemma]. But $H_{\beta(g)} \cap B_{0,\varepsilon_0}$ is bi-Lipschitz equivalent to $H_{\beta(g)}$ (the bi-Lipschitz equivalence is given by the polar map $P_{\varepsilon_0,1}$).

To complete the proof of the lemma we must show that $\alpha_0 = \beta(g)$. Let $\gamma_\varepsilon$ be the equidistant line in $H_{\beta(g)}$, namely $\gamma_\varepsilon = H_{\beta(g)} \cap S_{0,\varepsilon}$. By [1], there exists a subanalytic bi-Lipschitz map $\Upsilon: H_{\beta(g)} \to S\alpha_0$ such that $\Upsilon(\gamma_\varepsilon) = S\alpha_0 \cap \{(x, y) \in \mathbb{R}^2 \mid x = \varepsilon\}$. Denote by $\ell(\gamma_\varepsilon)$ the length of $\gamma_\varepsilon$. Since $\Upsilon$ is a bi-Lipschitz map, we have

$$
c_1 \varepsilon^{\alpha_0} \leq \ell(\gamma_\varepsilon) \leq c_2 \varepsilon^{\alpha_0},
$$

(4) for some positive constants $c_1$ and $c_2$. To prove that $\alpha_0 = \beta(g)$ we will compute the length of $\gamma_\varepsilon$ from another side. Consider the function

$$
r(z) = \sqrt{z^2 + \sum_{i=1}^{k+1} \varepsilon^{p_i/q_i} },
$$
which is a one-to-one function, for small $z$. So, $r^{-1}(\varepsilon)$ is a well-defined function, for small $\varepsilon$. By the Lemma 3.1,

$$\gamma_\varepsilon = H_\beta(g) \cap \{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}, z) \in \mathbb{R}^{2k+3} \mid z = r^{-1}(\varepsilon)\}.$$ 

Consider the following set

$$T^\varepsilon = T(\beta_1, \ldots, \beta_k, \beta(g)) \cap \{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}, z) \in \mathbb{R}^{2k+1} \mid z = r^{-1}(\varepsilon)\}.$$ 

It is a smooth manifold homeomorphic to a $(k + 1)$-dimensional torus. The equidistant line $\gamma_\varepsilon$ belongs to this set. There are $(k + 1)$ differential 1-forms $d\psi_1^\varepsilon, \ldots, d\psi_k^\varepsilon$ and $d\psi_{k+1}^\varepsilon$ on $T^\varepsilon$ corresponding to the coordinate system $\{\psi_1, \ldots, \psi_k, \psi_{k+1}\}$. By (3), we have

$$\ell(\gamma_\varepsilon) = \int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \, d\psi_i^\varepsilon$$

where $m_i = \begin{cases} 1 & \text{if } \psi_i(a_1) \neq \psi_i(a_2) \\ 0 & \text{if } \psi_i(a_1) = \psi_i(a_2), \end{cases}$

$$\int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \, d\psi_i^\varepsilon \leq \int_{\gamma_\varepsilon} \sum_{i=1}^{k+1} m_i \, d\psi_i^\varepsilon.$$

By the definition of the equidistant line $\gamma_\varepsilon$,

$$\int_{\gamma_\varepsilon} m_i \, d\psi_i^\varepsilon = m_i \pi z^{\beta_i}.$$

Using the above formula we obtain

$$\ell(\gamma_\varepsilon) \leq \sum_{i=1}^{k+1} m_i \pi z^{\beta_i}.$$

If $z$ sufficiently small ($z < 1$) there exists $C_2 > 0$ such that

$$\ell(\gamma_\varepsilon) \leq \sum_{i=1}^{k+1} m_i \pi z^{\beta_i} \leq C_2 z^{\beta(g)},$$

because $\beta(g) = \min_{1 \leq i \leq k+1} \beta_i$.

By the definition of the function $r(\varepsilon)$, we have $r(\varepsilon) = a\varepsilon + o(\varepsilon)$, with $a > 0$. 

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Hence, \( \ell(\gamma_\epsilon) \leq C'_2 \epsilon^{\beta(g)} \), where \( C'_2 = a C_2 \). To obtain an estimate of \( \ell(\gamma_\epsilon) \) from below let us go back to the formulas (3)

\[
\ell(\gamma_\epsilon) = \int_{\gamma_\epsilon} \sum_{i=1}^{k+1} m_i \, d\psi_i^\epsilon \geq \int_{\gamma_\epsilon} m_{k+1} \, d\psi_{k+1}^\epsilon.
\]

By (3), \( m_{k+1} = 1 \). Thus,

\[
\ell(\gamma_\epsilon) \geq \int_{\gamma_\epsilon} d\psi_{k+1}^\epsilon = \pi z^{\beta(g)} \geq C'_1 \epsilon^{\beta(g)},
\]

for some positive constant \( C'_1 \). So,

\[
C'_1 \epsilon^{\beta(g)} \leq \ell(\gamma_\epsilon) \leq C'_2 \epsilon^{\beta(g)}. \tag{5}
\]

From (4) and (5) we obtain that \( \beta(g) = a_0 \).

Lemma 4.1 is proved. \( \square \)

Thus, the realization of \( (\Gamma, \beta) \) is given by the union of the realizations of \((\tilde{\Gamma}^1, \tilde{\beta}^1), (\tilde{\Gamma}^2, \tilde{\beta}^2)\) and \( H_{\beta(g)} \). It is a semialgebraic set because it is a finite union of semialgebraic sets.

Now consider the second case: \( \tilde{\Gamma} \) is a connected graph. In this case, by the induction hypotheses, \((\tilde{\Gamma}, \tilde{\beta})\) (where \( \tilde{\beta} = \beta|_{\tilde{\Gamma}} \)) can be realized as a semialgebraic subset of \( T(\beta_1, \ldots, \beta_k) \) which can be considered as a semialgebraic subset of \( T(\beta_1, \ldots, \beta_k, \beta(g)) \) defined by the equation \( \psi_{k+1} = 0 \). The edge \( g \) connects two vertices \( a_1 \) and \( a_2 \). Now we can glue the realization of \((\tilde{\Gamma}, \tilde{\beta})\) and the curvilinear triangle \( H_{\beta(g)} \) generated by the curve \( \Psi(\theta) = \{\psi_1(\theta), \psi_2(\theta), \ldots, \psi_{k+1}(\theta)\} \):

\[
\psi_{k+1}(\theta) = \theta \text{ and } \psi_i(\theta) = \begin{cases} 
\psi_i(a_1) & \text{if } \psi_i(a_1) = \psi_i(a_2) \\
\frac{\theta}{2} & \text{if } \psi_i(a_1) = 0 \text{ and } \psi_i(a_2) = \pi \\
\pi + \frac{\theta}{2} & \text{if } \psi_i(a_1) = \pi \text{ and } \psi_i(a_2) = 0,
\end{cases} \tag{6}
\]

for \( 1 \leq i \leq k \), \( \theta \in [0, 2\pi] \), \( a_1 = (\psi_1(a_1), \ldots, \psi_k(a_1), 0) \) and \( a_2 = (\psi_1(a_2), \ldots, \psi_k(a_2), \pi) \).

Set \( H_{\beta(g)} := \bigcup_{\theta} L_{\Psi(\theta)} \). By the same arguments as in the Lemma 4.1, we can prove that \( H_{\beta(g)} \) is a \( \beta(g) \)-Hölder triangle.
The union of the realization of $(\tilde{\Gamma}, \tilde{\beta})$ and $H_{\beta(g)}$ is a semialgebraic realization of $(\Gamma, \beta)$.

The Realization theorem is proved. □

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