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Singular Foliations of Toric Type(*)

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RÉSUMÉ. — Une singularité de type torique est réduite par une suite d'éclatements combinatoires. Nous présentons ici un algorithme qui permet de trouver des coordonnées privilégiées. Si la singularité est de type torique, le polynôme de Newton pour ces coordonnées donnera directement le morphisme de réduction des singularités.

ABSTRACT. — We present here an algorithm to get privilegied coordinates in order both to identify toric type singularities and to describe explicitly the corresponding toric morphism in terms of the Newton Polygon.

1. Introduction

The desingularization of a germ of singular holomorphic foliation $\mathcal{F}$ given by a 1-form $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y$ splits in a natural way into a composition of local toric morphims. In the case of an analytic branch, each step destroys a Puiseux Pair. Call Toric Type Singularities the ones corresponding to a single toric morphism. We present here an algorithm to get privilegied coordinates in order both to identify Toric Type Singularities and to describe explicitly the corresponding toric morphism in terms of the Newton Polygon.

2. Combinatorial blow-ups of $\mathbb{C}^2$

Let us recall here the basic language we need about toric morphisms. For more details, see [1].

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Let $\mathbb{R}_0$ denote the set of nonnegative real numbers. A *strongly convex rational polyhedral cone* $\sigma \subset \mathbb{R}^2$ is a subset of the type $\sigma = n_1\mathbb{R}_0 + n_2\mathbb{R}_0$, where $n_j \in \mathbb{Z}^2$ and $\sigma \cap (-\sigma) = \{0\}$. The faces $\tau < \sigma$ are given by $\tau = r_1\mathbb{R}_0 + \cdots + r_t\mathbb{R}_0$, where $\{r_1, \ldots, r_t\} \subset \{n_1, n_2\}$. A *fan* Δ in $\mathbb{R}^2$ is a collection of strongly convex rational polyhedral cones containing the faces of all its elements and such that the intersections of two elements in Δ are faces of each one.

Denote $(\mathbb{Z}^2)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z})$. Then

$$\mathcal{S}_\sigma = \left\{m \in (\mathbb{Z}^2)^\vee \mid m(x) \geq 0\,,\ \forall x \in \sigma\right\}$$

is a finitely generated additive semigroup in $(\mathbb{Z}^2)^\vee$. From a set of generators $\{m_1, \ldots, m_p\}$ of $\mathcal{S}_\sigma$ we construct the intrinsic algebraic normal variety U_σ given as subset of $\mathbb{C}^p$ by the monomial equations

$$x_1^{a_1} \cdots x_p^{a_p} = x_1^{b_1} \cdots x_p^{b_p}$$

where $(a_1 - b_1)m_1 + \cdots + (a_p - b_p)m_p = 0$. If $\{m_1, \ldots, m_p\}$ are independent (and hence $p \leq 2$) we get $U_\sigma = \mathbb{C}^p$. If $n_1 \neq n_2$, then U_σ is non-singular iff $\{n_1, n_2\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$, in this case U_σ is isomorphic to $\mathbb{C}^2$.

For $\sigma, \tau \in \Delta$, the variety $U_{\sigma\cap\tau}$ is, in a natural way an open set of both U_σ and U_τ. We construct by glueing in this way the toric variety $T(\Delta)$.

If $\sigma' \subset \sigma$ we have that $\mathcal{S}_\sigma \subset \mathcal{S}_{\sigma'}$. Then there is an intrinsic morphism $U_{\sigma'} \to U_\sigma$ that can be viewed as a projection. We say that a fan Δ' refines Δ if for each $\sigma' \in \Delta'$ there is a $\sigma \in \Delta$ such that $\sigma' \subset \sigma$. In this case we get an intrinsic morphism $T(\Delta') \to T(\Delta)$. The refinement Δ' is a subdivision of Δ if each $\sigma \in \Delta$ is the union of several $\sigma' \in \Delta'$, we note this by $\Delta' > \Delta$. Let Ψ be a direct automorphism of $\mathbb{Z}^2$, the fan $\Psi(\Delta)$ has an evident definition and we have a natural isomorphism between $T(\Delta)$ and $T(\Psi(\Delta))$, compatible with the morphism induced by a refinement.

Take $\sigma = n\mathbb{R}_0 + n'\mathbb{R}_0$ and denote by Θ the convex hull in $\mathbb{R}^2$ of $\{\sigma \cap \mathbb{Z}^2\} \setminus \{(0,0)\}$. Let $\{n = n_0, n_1, \ldots, n_s, n_{s+1} = n'\}$ be, in its order, the points of $\mathbb{Z}^2$ contained in the compact edges of the boundary of Θ. Each $\sigma_j = n_{j-1}\mathbb{R}_0 + n_j\mathbb{R}_0$ is non-singular. Applying this procedure to each $\sigma \in \Delta$ we get a fan Δ' which is the coarsest non-singular subdivision of Δ: denote it by $\Delta' \gg \Delta$. The toric morphism $T(\Delta') \to T(\Delta)$ is the minimal desingularization of $T(\Delta)$.

Given a non-singular $\sigma = n\mathbb{R}_0 + n'\mathbb{R}_0$ with $n \neq n'$, the blowing-up at the origin of U_σ corresponds to the subdivision induced by

$$\sigma_1 = n\mathbb{R}_0 + (n + n')\mathbb{R}_0\,, \quad \sigma_2 = (n + n')\mathbb{R}_0 + n'\mathbb{R}_0\,.$$

Consider the standard fan Δ_{st}, associated to $(0,1)\mathbb{R}_0 + (1,0)\mathbb{R}_0$ and a non-singular subdivision $\Delta' > \Delta_{\mathrm{st}}$. We get a sequence

$$\Delta' = \Delta_k > \ldots > \Delta_1 > \Delta_0 = \Delta_{\mathrm{st}}$$

such that each morphism $T(\Delta_j) \to T(\Delta_{j-1})$ is a quadratic blowing-up as follows. Denote by $\{n_0 = (0,1), n_1, \ldots, n_{k+1} = (1,0)\}$ the points in $\mathbb{Z}^2$ such that $\sigma_j = n_{j-1}\mathbb{R}_0 + n_j\mathbb{R}_0$ are the two dimensional strongly convex rational polyhedral cones of Δ'. Each $\{n_{i-1}, n_i\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Pick n_j, $0 < j < k+1$, with biggest module. Then $n_j = n_{j-1} + n_{j+1}$ and $\{n_{j-1}, n_{j+1}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Define Δ_{k-1} to be the fan obtained from Δ_k substituting σ_j and σ_{j+1} by $n_{j-1}\mathbb{R}_0 + n_{j+1}\mathbb{R}_0$. Repeat.

3. Minimal coordinates

Consider a germ of singular holomorphic foliation $\mathcal{F}$ over $(\mathbb{C}^2, 0)$ given by a 1-form $\omega = f\,\mathrm{d}x + g\,\mathrm{d}y$ and let D be a normal crossing divisor invariant by $\mathcal{F}$. Choose *adapted* coordinates (x,y) such that $D \subset (xy = 0)$. Hence $D = (x^\epsilon y^\eta = 0)$ for $\epsilon, \eta \in \{0,1\}$. Write

$$\begin{aligned} \omega &= x^\epsilon y^\eta \left[a(x,y)\frac{\mathrm{d}x}{x} + b(x,y)\frac{\mathrm{d}y}{y} \right] \\ &= x^\epsilon y^\eta \sum_{i,j} \left[a_{ij}\frac{\mathrm{d}x}{x} + b_{ij}\frac{\mathrm{d}y}{y} \right] x^i y^j . \end{aligned}$$

The Newton Polygon $\nabla(\mathcal{F}, D; x, y)$ is defined to be the convex hull of

$$\{(i,j); (a_{ij}, b_{ij}) \neq (0,0)\} + \mathbb{R}_0 .$$

Associated to the Newton Polygon we get a fan $\Delta(\mathcal{F}, D; x, y)$ that is a subdivision of the standard fan Δ_{st} just taking the ortogonals of the sides of $\nabla(\mathcal{F}, D; x, y)$. The subdivision $\Delta'(\mathcal{F}, D; x, y) \gg \Delta(\mathcal{F}, D; x, y)$ provides a toric morphism

$$\tau_{\mathcal{F},D,x,y} : T\big(\Delta'(\mathcal{F}, D; x, y)\big) \longrightarrow T(\Delta_{\mathrm{st}}) = \mathbb{C}^2$$

that is a sequence of quadratic blow-ups. It is a natural question to ask under what conditions this morphism desingularizes the foliation.

Let us describe now an algorithm to obtain a set of *minimal coordinates* starting with an adapted system of coordinates (x, y). The coordinates (x, y) will be called *minimal* if the algorithm stops at the initial step.

- If $D = (xy = 0)$, the algorithm stops.
- Assume $D = (x = 0)$. Denote by $\ell_1, \ell_2, \ldots, \ell_k$ the sides of $\nabla(\mathcal{F}, D; x, y)$ having slope of the type $-1/\eta_j$, $1 \leq j \leq k$, where $\eta_j \in \mathbb{Z}$ and $\eta_j < \eta_{j+1}$. If $k = 0$ the algorithm stops. Do a coordinate change of the type $y_1 = y + \zeta x^{\eta_1}$ such that the side of slope $-1/\eta_1$ is strictly shorter or disappears. If it is not possible, try a change of the type $y_1 = y + \zeta x^{\eta_2}$ such that the side of slope $-1/\eta_2$ is strictly shorter or disappears; try this until the side ℓ_k. If it is impossible to do that change in all those cases, the algorithm stops. Once you are allowed to do a coordinate change, restart the algorithm at the beginning. We get as limit a (possibly formal) coordinate system that is a minimal one.
- Assume $D = (y = 0)$. Interchange at the beginning the coordinates (x, y) and do the precedent procedure.
- If $D = \emptyset$. First proceed as in the case $D = (x = 0)$, second change the order of the obtained coordinates and repeat the procedure.

PROPOSITION.— *Take (x, y) minimal for $\mathcal{F}, D$. Let $\pi : X \to (\mathbb{C}^2, 0)$ be the blow-up at the origin. Put $D' = \pi^{-1}(D)$ and let Q_1, Q_2 be the points in X corresponding to the strict transforms of $y = 0$, respectively $x = 0$. Take coordinates $(x_1 = x\,,\, y_1 = y/x)$, $(x_2 = x/y\,,\, y_2 = y)$ respectively in Q_1 and Q_2. Denote by $\mathcal{F}'$ the strict transform of $\mathcal{F}$ and assume that π is non-dicritical. Then (x_i, y_i), $i = 1, 2$, are minimal for $\mathcal{F}', D'$.*

Proof.— In the case $D = (xy = 0)$ the result is obvious. Assume $D = (x = 0)$. The only interesting point is Q_1. Let m be the minimum of the orders of $a(x, y)$ and $b(x, y)$ at the origin. We have that

$$\nabla(\mathcal{F}', D'; x_1, y_1) = \alpha\big(\nabla(\mathcal{F}, D; x, y)\big) + \mathbb{R}_0^2$$

where $\alpha(u, v) = (u + v - m, v)$. Let us put $y_1' = y_1 + \zeta x_1^{\eta}$, where $\eta \geq 1$ and $y' = y + \zeta x^{\eta+1}$. Since $y_1' = y'/x$ we also have that

$$\nabla(\mathcal{F}', D'; x_1, y_1') = \alpha\big(\nabla(\mathcal{F}, D; x, y')\big) + \mathbb{R}_0^2.$$

In particular, if we can make shorter the side of $\nabla(\mathcal{F}', D'; x_1, y_1)$ of slope $-1/\eta$, we can do the same with the side of $\nabla(\mathcal{F}, D; x, y)$ of slope $-1/(\eta+1)$. The case $D = (y = 0)$ is symmetric to the above one. Also the case $D = \emptyset$ is treated in the same way. The proof is ended.

4. Toric Type Singularities

The choice of coordinates (x, y) in $(\mathbb{C}^2, 0)$ provides a *toric structure* over $(\mathbb{C}^2, 0)$ that identifies it with $T(\Delta_{\mathrm{st}})$ (we do not pay attention to the fact that we work in a germified way). We say that a germ of singular foliation $\mathcal{F}$ over $(\mathbb{C}^2, 0)$ is of *Toric Type* iff there is a toric structure over $(\mathbb{C}^2, 0)$, given by a choice of coordinates, and a non-singular subdivision $\Delta > \Delta_{\mathrm{st}}$ such that the toric morphism

$$f : T(\Delta) \longrightarrow T(\Delta_{\mathrm{st}}) = (\mathbb{C}^2, 0)$$

desingularizes $\mathcal{F}$. That is, all the singularities of the strict transform of $\mathcal{F}$ by f are *simple singularities* in the sense that they have linear part with two eigenvalues $\lambda \neq \mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}_+$.

We say that $\mathcal{F}$ has a singularity of *Toric Type Related to D* iff, in addition, the coordinate system (x, y) is adapted to D and D is maximal between the normal crossing divisor invariant by $\mathcal{F}$. Actually, if $\mathcal{F}$ has a singularity of Toric Type, there is always a normal crossings divisor D such that $\mathcal{F}$ has a singularity of Toric Type Related to D. To see this, note that the desingularization of $\mathcal{F}$ is necessarily a linear chain (the dual graph has a single branch). Looking at the ends of this chain, we eventually modify our coordinates in such a way that $(xy = 0)$ contains a maximum of integral curves.

THEOREM.— *Assume that $\mathcal{F}$ has a non-dicritical singularity of Toric Type Related to D. Consider a coordinate system (x, y) adapted to D. Then the following statements are equivalent.*

(1) The toric morphism $\tau_{\mathcal{F},D,x,y}$ desingularizes $\mathcal{F}$.

(2) The coordinate system (x, y) is minimal.

In particular, we detect a Toric Type Singularity by choosing first the finitely many possible divisors and second a minimal system of coordinates.

Proof of the Theorem

Induction on the number k of quadratic blowing-ups corresponding to $\tau_{\mathcal{F},D,x,y}$.

(1)$\Rightarrow$(2) If $k=0$, then $\Delta(\mathcal{F},D;x,y)$ is the standard fan and hence $\nabla(\mathcal{F},D;x,y)$ has a single vertex and thus (x,y) is minimal. Assume $k>0$. If $D=(xy=0)$ there is nothing to prove. Consider the case $D=(x=0)$. Let us reason by contradiction, assuming that (x,y) is not minimal. Then a change $y'=y+\zeta x^{\eta}$ makes shorter the side of the Newton Polygon of slope $-1/\eta$. Consider the following two possible cases: $\eta=1$ and $\eta\geq 2$.

If $\eta=1$, let Q be the point in the exceptional divisor of the first blow-up π corresponding to $y'=0$. Then $Q\neq Q_1$, where Q_1 is the point corresponding to $y=0$. The hypothesis that $\tau_{\mathcal{F},D,x,y}$ desingularizes implies that Q is either a non-singular point for the strict transform $\mathcal{F}'$ of $\mathcal{F}$ or a simple singularity. In that second possibility, we get an invariant curve Γ', non-singular and transversal to the divisor, that projects over a non-singular invariant curve Γ transversal to D. This contradicts the maximality of D. To see that Q is necessarily singular, put $(x_1=x, y_1'=y'/x)$. We get that

$$\nabla\big(\mathcal{F}',\pi^{-1}(D);x_1,y_1'\big)=\alpha\big(\nabla(\mathcal{F},D;x,y')\big)+\mathbb{R}_0^2$$

where $\alpha(u,v)=(u+v-m,v)$. Since $\nabla(\mathcal{F},D;x,y')$ has not its longest possible side of slope -1, we deduce that $(0,0)$ is not a vertex of $\nabla\big(\mathcal{F}',\pi^{-1}(D);x_1,y_1'\big)$ and thus Q is a singular point of $\mathcal{F}'$.

Consider the case $\eta\geq 2$. Now $Q=Q_1$. Put $(x_1=x, y_1=y/x)$. Then (x_1,y_1) is not minimal since the change $y_1'=y_1+\zeta x^{\eta-1}$ makes shorter the side of slope $-1/(\eta-1)$. Recall that the toric morphism $\tau_{\mathcal{F},D,x,y}$ corresponds to a sequence of subdivisions

$$\Delta'(\mathcal{F},D;x,y)=\Delta_k>\ldots>\Delta_1>\Delta_0=\Delta_{\mathrm{st}}$$

where Δ_1 contains the two strongly convex rational polyhedral cones $\sigma_1=(1,0)\mathbb{R}_0+(1,1)\mathbb{R}_0$ and $\sigma_2=(1,1)\mathbb{R}_0+(0,1)\mathbb{R}_0$. The point Q_1 is the origin of the chart corresponding to σ_1. Denote by $\overline{\Delta_j}$ the fans obtained by taking the strongly convex rational polyhedral cones in Δ_j contained in σ_1. Let Ψ be the direct automorphism of $\mathbb{Z}^2$ given by $\Psi(u,v)=(u-v,v)$. The restriction to the first chart around Q_1 of the toric morphism $\tau_{\mathcal{F},D,x,y}$ may be interpreted as corresponding to the sequence

$$\Psi\big(\overline{\Delta'(\mathcal{F},D;x,y)}\big)=\Psi(\overline{\Delta_k})>\ldots>\Psi(\overline{\Delta_2})>\Psi(\overline{\Delta_1})=\Delta_{\mathrm{st}}\,.$$

The fact that

$$\nabla(\mathcal{F}', \pi^{-1}(D); x_1, y_1) = \alpha\big(\nabla(\mathcal{F}, D; x, y)\big) + \mathbb{R}_0^2$$

implies by a direct computation that

$$\Psi\big(\overline{\Delta'(\mathcal{F}, D; x, y)}\big) = \Delta'\big(\mathcal{F}', \pi^{-1}(D); x_1, y_1\big)\,.$$

Now, we apply induction hypothesis to get a contradiction.

The cases $D = (y = 0)$ and $D = \emptyset$ are similar to the above one.

(2)$\Rightarrow$(1) If $k = 0$, then $\nabla(\mathcal{F}, D; x, y)$ has a single vertex (s, t). Assume that the origin $P \in \mathbb{C}^2$ is a singularity. Then $\mathcal{F}$ is given by

$$\omega = x^{\epsilon+s} y^{\eta+t}\left[a'(x,y)\frac{\mathrm{d}x}{x} + b'(x,y)\frac{\mathrm{d}y}{y}\right]$$

where $(a'(0,0),\ b'(0,0)) = (\lambda, -\mu) \neq (0,0)$. By a blowing-up we get the same situation with $(\lambda - \mu, -\mu)$ and $\big(\lambda, -(\mu - \lambda)\big)$ in the respective origins of the two charts. If $\lambda/\mu \in \mathbb{Q}_+$, in a finite number of steps we have $\lambda = \mu$ and thus a dicritical singularity. Hence P is a simple singularity.

Assume that $k > 0$. Let Q_1, respectively Q_2, be the points corresponding respectively to $(y = 0)$ and $(x = 0)$ after the first blowing-up π. Consider the case $D = (xy = 0)$. Since we have a Toric Type singularity relatively to D, the points Q_1, Q_2 are the only possibly non-simple singularities of $\mathcal{F}'$ in the exceptional divisor E of π. The precedent proposition and the above arguments allow us to apply induction in Q_1 and Q_2. Consider now the case $D = (x = 0)$. If all the singularities of $\mathcal{F}'$ in $E \setminus \{Q_1, Q_2\}$ are simple, we apply induction as above. Assume that there is a non-simple singularity $Q \in E \setminus \{Q_1, Q_2\}$. Then Q is in the strict transform of $y' = y + \zeta x = 0$, for some $\zeta \neq 0$. The fact that we have a Toric Type singularity implies that Q_1 is either non-singular or a simple singularity. The second possibility does not occur, by the maximality of D. Then $\nabla\big(\mathcal{F}', \pi^{-1}(D); x_1, y_1\big)$ has the only vertex $(0,0)$, where $(x_1 = x, y_1 = y/x)$. Since

$$\nabla(\mathcal{F}', \pi^{-1}(D); x_1, y_1) = \alpha\big(\nabla(\mathcal{F}, D; x, y)\big) + \mathbb{R}_0^2$$

with $\alpha(u,v) = (u + v - m, v)$, we deduce that the side of slope -1 in $\nabla(\mathcal{F}, D; x, y)$ has its lower point of ordinate 0. This implies that

$$\nabla(\mathcal{F}, D; x, y') = \nabla(\mathcal{F}, D; x, y)$$

and thus $\nabla(\mathcal{F}', \pi^{-1}(D); x_1, y_1' = y'/x)$ also has the single vertex $(0,0)$. Hence Q is either a simple singularity or a non-singular point, contradiction.

The cases $D = (y = 0)$ and $D = \emptyset$ are similar to the above one. This ends the proof.

Reference

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