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Lower semicontinuity of a class of multiple integrals below the growth exponent (*)

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RÉSUMÉ. — Dans cet article, nous démontrons la semicontinuité inférieure pour la convergence faible dans $W^{1,p}(\Omega)$ de fonctionnelles quasiconvexes de la forme $\int_\Omega f\big(x,u(x),\mathcal{L}u(x)\big)dx$, où $0 \leqslant f(x,s,\xi) \leqslant c(1+|\xi|^q)$, $q \geqslant p > 1$ et où $\mathcal{L}$ est un opérateur différentiel linéaire du premier ordre. Nous démontrons aussi un résultat de semicontinuité inférieure pour des fonctionnelles du type $\int_\Omega g\big(\langle \mathcal{P}v, \mathcal{Q}^*w\rangle\big)dx$, où $g : \mathbb{R} \to [0,+\infty)$ est convexe, où $\mathcal{P}$ et $\mathcal{Q}$ sont des opérateurs différentiels du premier ordre tels que $Im\mathcal{P} = Ker\mathcal{Q}$, et où $\mathcal{Q}^*$ est l'adjoint formel de $\mathcal{Q}$.

ABSTRACT. — In this paper we prove the lower semicontinuity of quasiconvex functionals of the form $\int_\Omega f\big(x,u(x),\mathcal{L}u(x)\big)dx$, with respect to the weak convergence in $W^{1,p}(\Omega)$, where $0 \leqslant f(x,s,\xi) \leqslant c(1+|\xi|^q)$, $q \geqslant p > 1$ and $\mathcal{L}$ is a linear differential operator of first order. We also show a lower semicontinuity result for functionals of the type $\int_\Omega g\big(\langle \mathcal{P}v, \mathcal{Q}^*w\rangle\big)dx$, where $g : \mathbb{R} \to [0,+\infty)$ is convex, $\mathcal{P}$, $\mathcal{Q}$ are linear differential operators of first order such that $Im\mathcal{P} = Ker\mathcal{Q}$ and $\mathcal{Q}^*$ is the formal adjoint operator of $\mathcal{Q}$.

1. Introduction

In this paper we study the lower semicontinuity of an integral functional of the type

$$F(u) = \int_\Omega f\big(x,u(x),\mathcal{L}u(x)\big)dx \tag{1.1}$$

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where $\Omega < \mathbb{R}^N$ is a bounded open set, $u \in W^{1,p}(\Omega, \mathbb{R}^d)$, f is a nonnegative integrand satisfying the growth condition

$$0 \leqslant f(x, s, \xi) \leqslant c(1 + |\xi|^q) \tag{1.2}$$

$q \geqslant p > 1$ and $\mathcal{L}$ is a linear differential operator of first order , $\mathcal{L}$: $C^\infty(\Omega, \mathbb{R}^d) \to C^\infty(\Omega, \mathbb{R}^m)$.

In the special case $\mathcal{L}u = \nabla u$ and $q = p$, there is a vast literature on the subject of lower semicontinuity properties of F (see for instance [21],[22],[2],[19],[17]).

More recently, in connection with the applications to materials exhibiting non standard elastic and magnetic behaviours, people have been interested to study lower semicontinuity also when $p < q$ and $\mathcal{L}$ is a general linear operator of first order (see [11],[12],[10]).

To fix the ideas let us assume that

$$\mathcal{L}u = \sum_{k=1}^{N} A_k \frac{\partial u}{\partial x_k} \tag{1.3}$$

where A_k, $k = 1, \cdots, N$, are given linear operators from $\mathbb{R}^d$ into $\mathbb{R}^m$. Then our main result, when f depends only on ξ, is the following.

THEOREM 1.1.— *Assume $q \geqslant p > max\left\{1, q\frac{N-1}{N}\right\}$. Let $f = f(\xi)$: $\mathbb{R}^m \to [0, +\infty)$ be a function satisfying (1.2) and $\mathcal{L}$ a linear differential operator of the type (1.3). Let us assume that for any $A \in \mathbb{R}^{N\times d}$ and any $u \in C_0^\infty(Q, \mathbb{R}^d)$ we have*

$$\int_Q \Big[f\big(\mathcal{L}(Ax + u(x)\big) - f\big(\mathcal{L}(Ax)\big)\Big] \geqslant 0$$

where $Q = (0,1)^N$ is the unit cube.

Then for any $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ and any sequence $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^d)$ we have

$$\int_\Omega f\big(\mathcal{L}u(x)\big)dx \leqslant \liminf_{n\to\infty} \int_\Omega f\big(\mathcal{L}u_n(x)\big)dx$$

This result, very much in the spirit of the lower semicontinuity results of Fonseca-Malý and Fonseca-Marcellini, is proved by a blow-up argument.

This type of argument is also used to extend the result to the case when f depends on x and s too.

It is interesting to notice that in this framework it is natural to consider the particular case $u = (v, w)$ and $\mathcal{L}u = (\mathcal{P}v, \mathcal{Q}^* w)$. Here $\mathcal{P}, \mathcal{Q}$ are linear differential operators of first order with constant coefficients forming an elliptic complex, i.e.

$$C^\infty(\mathbb{R}^N, \mathbb{R}^d) \xrightarrow{\mathcal{P}} C^\infty(\mathbb{R}^N, \mathbb{R}^m) \xrightarrow{\mathcal{Q}} C^\infty(\mathbb{R}^N, \mathbb{R}^k)$$

and

$$Im\mathcal{P}(\lambda) = Ker\mathcal{Q}(\lambda) \quad \text{for} \quad \text{all} \quad \lambda \neq 0$$

where the symbols $\mathcal{P}(\lambda) : \mathbb{R}^d \to \mathbb{R}^m$, $\mathcal{Q}(\lambda) : \mathbb{R}^m \to \mathbb{R}^k$ are linear operators (see Section 5). Here we denote by $\mathcal{L}^*$ the formal adjoint operator of $\mathcal{L}$

$$\mathcal{L}^* : C^\infty(\mathbb{R}^N, \mathbb{R}^m) \to C^\infty(\mathbb{R}^N, \mathbb{R}^d),$$

defined by the rule

$$\int_{\mathbb{R}^N} \langle \mathcal{L}^* v, u\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^N} \langle v, \mathcal{L}u\rangle_{\mathbb{R}^m}, \tag{1.4}$$

for any $v \in C^\infty(\mathbb{R}^N, \mathbb{R}^m)$ and any $u \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^d)$. Then one can easily check that any functional of the type

$$G(u) = \int_\Omega g\big(\langle \mathcal{P}v, \mathcal{Q}^* w\rangle\big)dx, \tag{1.5}$$

where $g : \mathbb{R} \to [0, \infty)$ is convex, is quasiconvex in u. Hence Theorem 1.1 implies the lower semicontinuity of G with respect to the weak convergence in $W^{1,p}$ for all $p > \dfrac{2(N-1)}{N}$. Functionals of the type (1.5) can be viewed as a generalization of the usual polyconvex functionals. In fact if $N = 2$, taking $\mathcal{P}u = \nabla u$, $u \in C^\infty(\mathbb{R}^2, \mathbb{R})$, $\mathcal{Q}v = \mathrm{curl}v = \dfrac{\partial v^2}{\partial x} - \dfrac{\partial v^1}{\partial y}$, $v \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ then one has an elliptic complex and $\langle \mathcal{P}u, \mathcal{Q}^* w\rangle$ is equal to the determinant of the matrix whose rows are given by ∇u and ∇w.

2. Notation and preliminaries

The space of infinitely differentiable functions in $\mathbb{R}^N$ which take values in $\mathbb{R}^d$ will be denoted by $C^\infty(\mathbb{R}^N, \mathbb{R}^d)$.

Let $\mathcal{L} : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^m)$ be a linear differential operator of first order of the type

$$\mathcal{L}u = \sum_{k=1}^N A_k \frac{\partial u}{\partial x_k} \tag{2.1}$$

where A_k, $k = 1, \cdots, N$, are given linear operators from $\mathbb{R}^d$ into $\mathbb{R}^m$.

Our basic example is that of the gradient operator

$$(2.2) \qquad \nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_N}) : C^\infty(\mathbb{R}^N, \mathbb{R}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^N).$$

Other two interesting examples are the divergence operator

$$(2.3) \qquad \operatorname{div} : C^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \mathbb{R})$$

defined by

$$\operatorname{div} u = \frac{\partial u^1}{\partial x_1} + \cdots + \frac{\partial u^N}{\partial x_N} \quad \text{for} \quad u = (u^1, \cdots, u^N)$$

and the so-called rotation operator

$$\operatorname{curl} : C^\infty(\mathbb{R}^N, \mathbb{R}^N) \to C^\infty(\mathbb{R}^N, \mathbb{R}^{\binom{N}{2}})$$

defined by

$$(2.4) \qquad \operatorname{curl} v = \sum_{1 \leqslant i < j \leqslant N} (\frac{\partial v^i}{\partial x_j} - \frac{\partial v^j}{\partial x_i}) dx_j \wedge dx_i$$

for $v = (v^1, \cdots, v^N)$ (here we have identified in the obvious way $\mathbb{R}^{\binom{N}{2}}$with the space of 2−covectors on $\mathbb{R}^N$).

Many more examples of operators in applied PDEs also fit well into the framework of this paper, but we shall not discuss them here.

Let us denote by $W^{1,p}(\Omega, \mathbb{R}^d)$ the Sobolev space consisting of those functions $u : \Omega \to \mathbb{R}^d$ such that $|u| \in L^p(\Omega)$ and $|\nabla u| \in L^p(\Omega)$, where ∇u denotes the distributional gradient of u. Notice that if $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ then (2.1) makes still sense and $\mathcal{L}u \in L^p(\Omega, \mathbb{R}^m)$.

Let us give the following definition.

DEFINITION 2.1.— *Let $\mathcal{L}$ be a linear differential operator of the type (2.1). Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function. We say that f is quasiconvex with respect to the operator $\mathcal{L}$ if for almost every $x_0 \in \Omega$, for any $s_0 \in \mathbb{R}^d$ and any matrix $A \in \mathbb{R}^{N \times d}$ we have*

$$(2.9) \qquad \int_Q \Big[f\big(x_0, s_0, \mathcal{L}(Ax + u(x))\big) - f\big(x_0, s_0, \mathcal{L}(Ax)\big) \Big] dx \geqslant 0$$

for all $u \in C_0^\infty(Q, \mathbb{R}^d)$, where $Q = (0,1)^N$ is the unit cube.

Notice that by a density argument it follows that if $|f(x, s, \xi)| \leqslant c(1 + |\xi|^q)$, then (2.9) holds with $u \in W_0^{1,q}(Q, \mathbb{R}^d)$.

3. Main result

This section is devoted to the proof of Theorem 1.1. We consider fixed exponents $r, q \geqslant 1$ and $p > max\left\{1, r\frac{N-1}{N}, q\frac{N-1}{N}\right\}$. The following lemma, proved by Fonseca-Malý [11], will be useful in the sequel.

LEMMA 3.1. — *Let $V \subset\subset \Omega$ and $W \subset \Omega$ be open sets, $\Omega = V \cup W, v \in W^{1,q}(V)$ and $w \in W^{1,q}(W)$. Let $m \in \mathbf{N}$. There exist a function $z \in W^{1,q}_{loc}(\Omega)$ and open sets $V' \subset V$ and $W' \subset W$, such that $V' \cup W' = \Omega, z = v$ on $\Omega - W', z = w$ on $\Omega - V'$,*

$$L^N(V' \cap W') \leqslant Cm^{-1} \tag{3.1}$$

and

$$\|z\|_{L^r(V'\cap W')} + \|z\|_{W^{1,q}(V'\cap W')} \leqslant Cm^{-\tau}(\|v\|_{W^{1,p}(V\cap W)} + \|w\|_{W^{1,p}(V\cap W)} + m\|w - v\|_{L^p(V\cap W)}),$$

where $C = C(p, q, r, V, W)$ and $\tau = \tau(N, p, q, r) > 0$.

In the sequel we denote by $B_\varrho(x)$ the ball $\{y \in \mathbb{R}^N : |y - x| < \varrho\}$; if the center of the ball is the origin we will simply write B_ρ instead of $B_\varrho(0)$.

Proof of Theorem 1.1. — The proof falls naturally into two parts.

Step 1. We prove the result in the special case that $\Omega = B_1$ and u is linear, $u(x) = Ax$ for $A \in \mathbb{R}^{N\times d}$. According to Rellich's compact imbedding theorem, we may assume that

$$\|u_n - u\|_{L^p} \leqslant n^{-1}$$

Let $R < 1$ and $\rho = \frac{R+1}{2}$. We apply the lemma above to $v = u_n$, $w = u$, $V = B_\varrho$ and $W = B_1 \setminus B_R$ in order to obtain $z_n \in W^{1,q}(B_1, \mathbb{R}^d)$ and open sets $V_n \subset\subset V$, $W_n \subset W$ such that $V_n \cup W_n = B_1$,

$$z_n = u_n \quad on \quad B_1 \setminus W_n, \quad z_n = u \quad on \quad B_1 \setminus V_n$$

and

$$L^N(V_n \cap W_n) \leqslant \frac{c(R)}{n}, \quad \int_{V_n\cap W_n} |\mathcal{L}z_n|^q \leqslant \frac{c(R,M)}{n^{\tau q}}$$

where $M = \sup ||u_n||_{W^{1,p}}$ and $\tau > 0$ is the exponent provided by Lemma 3.1. Since $z_n - u \in W^{1,q}_0(B_1, \mathbb{R}^d)$, from the growth condition and the quasiconvexity of f we have

$$\int_{B_1} f(\mathcal{L}u) \leqslant \int_{B_1} f(\mathcal{L}z_n).$$

Therefore

$$\begin{aligned}\int_{B_1} f(\mathcal{L}u) - \int_{B_1} f(\mathcal{L}u_n) &\leqslant \int_{B_1} f(\mathcal{L}z_n) - \int_{B_1} f(\mathcal{L}u_n)\\ &\leqslant \int_{B_1\setminus V_n} f(\mathcal{L}u) + \int_{V_n\cap W_n} f(\mathcal{L}z_n)\\ &\leqslant cL^N(B_1\setminus V_n) + \int_{V_n\cap W_n}(1+|\mathcal{L}z_n|^q)\\ &\leqslant c\left(L^N(B_1\setminus B_R) + n^{-1} + n^{-\tau q}\right)\\ &\leqslant c(1-R+n^{-1}+n^{-\tau q})\end{aligned}$$

The conclusion follows letting first $n\to\infty$ and then $R\to 1$.

Step 2. Let $u\in W^{1,p}(\Omega,\mathbb{R}^d)$, $u_n\in W^{1,q}(\Omega,\mathbb{R}^d)$, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^d)$. With no loss of generality we may assume that

$$\liminf_{n\to\infty}\int_\Omega f(\mathcal{L}u_n)dx = \lim_{n\to\infty}\int_\Omega f(\mathcal{L}u_n)dx < \infty.$$

Passing, if necessary, to a subsequence, we obtain the existence of finite Radon nonnegative measures μ and ν such that

$$f\Big(\mathcal{L}(u_n(x))\Big)\to\mu \quad w^*-\mathcal{M}(\Omega), \qquad\qquad |\mathcal{L}u_n|^p\to\nu \quad w^*-\mathcal{M}(\Omega),$$

where $\mathcal{M}(\Omega)$ is the space of all Radon measures. Now our purpose is to prove that for L^N-a.e. $x_0\in\Omega$

$$\frac{d\mu}{dL^N}(x_0) = \lim_{\rho\to0^+}\frac{\mu(B_\rho(x_0))}{\omega_N\rho^N}\geqslant f(\mathcal{L}u(x_0)). \tag{3.2}$$

In fact if (3.2) is true, then we have

$$\lim_{n\to\infty}\int_\Omega f(\mathcal{L}u_n)\geqslant\lim_{n\to\infty}\int_\Omega f(\mathcal{L}u_n)=\int_\Omega d\mu\geqslant\int_\Omega\frac{d\mu}{dL^N}dx\geqslant\int_\Omega f(\mathcal{L}u).$$

Let $x_0\in\Omega$ such that the limits

$$\frac{d\mu}{dL^N}(x_0) = \lim_{\rho\to0^+}\frac{\mu(B_\rho(x_0))}{\omega_N\rho^N}, \qquad\qquad \frac{d\nu}{dL^N}(x_0) = \lim_{\rho\to0^+}\frac{\nu(B_\rho(x_0))}{\omega_N\rho^N}$$

exist and are finite and

$$\lim_{\rho\to0^+}\frac{1}{\rho}\fint_{B_\rho(x_0)}|u(y)-u(x_0)-\nabla u(x_0)(y-x_0)|dy=0.$$

Note that the last three conditions are satisfied by all points $x_0 \in \Omega$, except maybe on a set of L^N-measure zero. Then we select $\rho_k \to 0^+$ such that $\mu(\partial B_{\rho_k}(x_0)) = 0$, $\nu(\partial B_{\rho_k}(x_0)) = 0$.

Thus

$$\begin{aligned}\lim_{k\to\infty} \frac{\mu(B_{\rho_k}(x_0))}{\omega_N \rho_k^N} &\geqslant \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_{\rho_k}(x_0)} f\big(\mathcal{L}\alpha_n(x)\big)dx \\ &= \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_1} f\big(\mathcal{L}v_{n,k}(y)\big)dy\end{aligned}$$

where

$$v_{n,k} = \frac{u_n(x_0+\rho_k y) - u(x_0)}{\rho_k}$$

It follows that $v_{n,k} \in W^{1,q}(B_1, \mathbb{R}^d)$,

$$\lim_{k\to\infty} \lim_{n\to\infty} \|v_{n,k} - \nabla u(x_0)x\|_{L^1(B_1)} = 0$$

and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|\mathcal{L}v_{n,k}\|_{L^p(B_1)} \leqslant \frac{d\nu}{dL^N}(x_0) < \infty.$$

Hence, we may extract a subsequence such that

$$v_{n_k,k} = v_k \rightharpoonup \nabla u(x_0)x \quad \text{weakly in} \quad W^{1,p}(B_1, \mathbb{R}^d)$$

and

$$\frac{d\mu}{dL^N}(x_0) = \lim_{k\to\infty} \fint_{B_1} f\big(\mathcal{L}v_k(y)\big)dy.$$

Therefore from Step 1 we get

$$\frac{d\mu}{dL^N}(x_0) = \lim_{k\to\infty} \fint_{B_1} f\big(\mathcal{L}v_k(y)\big)dy \geqslant f\big(\mathcal{L}u(x_0)\big)$$

and this concludes the proof.

4. Extensions

In the sequel $f(x,s,\xi)$ will denote a function such that

i) $f(x,s,\xi)$ is quasiconvex;

ii) $0 \leqslant f(x,s,\xi) \leqslant c(1+|\xi|^q)$;

iii) for any $(x_0,s_0) \in \Omega \times \mathbb{R}^d$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x-x_0| < \delta$, $|s-s_0| < \delta$ and $\xi \in \mathbb{R}^{N\times d}$ then $f(x,s,\xi) \geqslant (1-\epsilon)f(x_0,s_0,\xi)$.

THEOREM 4.1. — *Let us suppose that $f(x,s,\xi)$ satisfies conditions i), ii) and iii) . Let $u_n \in W^{1,q}(\Omega, \mathbb{R}^d)$ and $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^d)$. Then*

$$\int_\Omega f(x,u,\mathcal{L}u)dx \leqslant \liminf_{n\to\infty} \int_\Omega f(x,u_n,\mathcal{L}u_n)dx.$$

Proof. — Passing to a subsequence we may assume that

$$\liminf_{n\to\infty} \int_\Omega f(x,u_n,\mathcal{L}u_n)dx = \lim_{n\to\infty} \int_\Omega f(x,u_n,\mathcal{L}u_n)dx < \infty$$

and

$$f\big(x,u_n(x),\mathcal{L}u_n(x)\big) \to \mu \quad w^* - \mathcal{M}(\Omega), \qquad |\mathcal{L}u_n|^p \to \nu \quad w^* - \mathcal{M}(\Omega).$$

Let us observe that for L^N-a.e. $x_0 \in \Omega$ we have

$$\frac{d\mu}{dL^N}(x_0) = \lim_{\rho\to 0} \frac{\mu(B_\rho(x_0))}{\omega_N \rho^N}, \qquad \frac{d\nu}{dL^N}(x_0) = \lim_{\rho\to 0} \frac{\nu(B_\rho(x_0))}{\omega_N \rho^N}. \tag{4.1}$$

If we prove that for a.e. $x_0 \in \Omega$

$$\frac{d\mu}{dL^N}(x_0) \geqslant f\big(x_0,u(x_0),\mathcal{L}u(x_0)\big) \tag{4.2}$$

we have

$$\lim_{n\to\infty} \int_\Omega f(x,u_n,\mathcal{L}u_n)dx \geqslant \mu(\Omega) \geqslant \int_\Omega \frac{d\mu}{dL^N}(x)dx \geqslant \int_\Omega f\big(x,u(x),\mathcal{L}u(x)\big)dx$$

and therefore the conclusion.

To prove (4.2), let us consider $x_0 \in \Omega$ such that the limits in (4.1) exist and are finite and

$$\lim_{\rho\to 0} \frac{1}{\rho} \fint_{B_\rho(x_0)} |u(y) - u(x_0) - \nabla u(x_0)(y-x_0)|dy = 0. \tag{4.3}$$

Let us choose $\rho_k \to 0$ such that

$$\mu(\partial B_{\rho_k}(x_0)) = 0 \quad \text{and} \quad \nu(\partial B_{\rho_k}(x_0)) = 0.$$

It is well known that conditions (4.1) and (4.3) are satisfied in each point $x_0 \in \Omega$ except at most on a set whose L^N-measure is zero.

Hence

$$\begin{aligned}
\lim_{k\to\infty} \frac{\mu(B_{\rho_k}(x_0))}{\omega_N \rho_k^N} &\geqslant \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_{\rho_k}(x_0)} f\big(x, u_n(x), \mathcal{L}u_n(x)\big)dx \\
&= \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_1} f\big(x_0+\rho_k y, u_n(x_0+\rho_k y), \mathcal{L}u_n(x_0+\rho_k y)\big)dy\,, \\
&= \lim_{k\to\infty} \limsup_{n\to\infty} \fint_{B_1} f\big(x_0+\rho_k y, u(x_0)+\rho_k v_{n,k}(y), \mathcal{L}v_{n,k}(y)\big)dy
\end{aligned}$$

where

$$v_{n,k}(y) = \frac{u_n(x_0+\rho_k y) - u(x_0)}{\rho_k}.$$

We get $v_{n,k} \in W^{1,q}(B_1, \mathbb{R}^d)$

$$\lim_{k\to\infty} \lim_{n\to\infty} \|v_{n,k} - \nabla u(x_0)x\|_{L^1(B_1)} = 0$$

and

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|\mathcal{L}v_{n,k}\|_{L^p(B_1)} \leqslant \frac{d\nu}{dL^N}(x_0) < \infty.$$

Take now a subsequence such that

$$v_{n_k,k} = v_k \rightharpoonup \nabla u(x_0)x \quad \text{in} \quad W^{1,p}(B_1, \mathbb{R}^d)$$

and

$$\begin{aligned}
\frac{d\mu}{dL^N}(x_0) &\geqslant \liminf_{k\to\infty} \fint_{B_1} f\big(x_0+\rho_k y, u(x_0)+\rho_k v_k, \mathcal{L}v_k(y)\big)dy \\
&\geqslant (1-\epsilon) \liminf_{k\to\infty} \fint_{B_1} f\big(x_0, u(x_0), \mathcal{L}v_k(y)\big)dy \\
&\geqslant (1-\epsilon) f\big(x_0, u(x_0), \mathcal{L}u(x_0)\big)
\end{aligned}$$

The last inequality follows from Theorem 1.1. Letting $\epsilon \to 0$, we get the conclusion.

5. Polyconvex case

In this section the operator $\mathcal{L}$ will be defined by means of a pair of differential operators of first order in N independent variables with constant coefficients

$$C^\infty(\mathbb{R}^N, \mathbb{R}^d) \xrightarrow{\mathcal{P}} C^\infty(\mathbb{R}^N, \mathbb{R}^m) \xrightarrow{\mathcal{Q}} C^\infty(\mathbb{R}^N, \mathbb{R}^k) \tag{5.1}$$

The symbols $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ are linear operators in $\lambda = (\lambda_1, \cdots, \lambda_N) \in \mathbb{R}^N$ respectively valued in $L(\mathbb{R}^d, \mathbb{R}^m)$ and in $L(\mathbb{R}^m, \mathbb{R}^k)$ and given explicitely by

$$\mathcal{P}(\lambda) = \sum_{k=1}^N \lambda_k A_k, \quad \mathcal{Q}(\lambda) = \sum_{k=1}^N \lambda_k B_k. \tag{5.2}$$

The complex (5.1) is said to be elliptic if the sequence of symbols

$$\mathbb{R}^d \xrightarrow{\mathcal{P}(\lambda)} \mathbb{R}^m \xrightarrow{\mathcal{Q}(\lambda)} \mathbb{R}^k \tag{5.3}$$

is exact, i.e.

$$Im\mathcal{P}(\lambda) = Ker\mathcal{Q}(\lambda) \quad \text{for} \quad \text{all} \quad \lambda \neq 0. \tag{5.4}$$

The dual sequence consists of the formal adjoint operators

$$C^\infty(\mathbb{R}^N, \mathbb{R}^d) \xleftarrow{\mathcal{P}^*} C^\infty(\mathbb{R}^N, \mathbb{R}^m) \xleftarrow{\mathcal{Q}^*} C^\infty(\mathbb{R}^N, \mathbb{R}^k) \tag{5.5}$$

where the formal adjoint of a linear operator $\mathcal{L}$ is defined by the formula (1.4).

It is immediate to check that the dual complex is elliptic if the original complex is so.

A pair

$$\mathcal{F} = (A, B) = (\mathcal{P}\alpha, \mathcal{Q}^*\beta) \tag{5.7}$$

where $\alpha \in W^{1,p}(\Omega, \mathbb{R}^d)$, $\beta \in W^{1,p}(\Omega, \mathbb{R}^k)$ and Ω is any domain in $\mathbb{R}^N$, $N \geqslant 2$, is said to be an elliptic couple associated to the complex (5.1). It is worth pointing out that , if $N = 2$, the complex

$$C^\infty(\mathbb{R}^2, \mathbb{R}) \xrightarrow{\nabla} C^\infty(\mathbb{R}^2, \mathbb{R}^2) \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^2, \mathbb{R})$$

is elliptic and the associated elliptic couple is $\mathcal{F} = (\nabla u, R(\nabla w))$, where $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Notice that

$$\langle \nabla u, R(\nabla w) \rangle = \frac{\partial(u, w)}{\partial(x, y)}.$$

This example can be easily generalized in higher dimension considering the complex

$$C^\infty(\mathbb{R}^N, \mathbb{R}) \xrightarrow{\nabla} C^\infty(\mathbb{R}^N, \mathbb{R}^m) \xrightarrow{curl} C^\infty(\mathbb{R}^N, \mathbb{R}^{\binom{N}{2}})$$

where (up to the standard identification of $\mathbb{R}^{\binom{N}{2}}$ with the space of 2-covectors) curl$v$ is defined as in (2.4).

Notice that the complex is elliptic since it can be easily checked that for any $\lambda \in \mathbb{R}^N$ the linear operators $\mathcal{P}(\lambda) : \mathbb{R} \to \mathbb{R}^N$, $\mathcal{Q}(\lambda) : \mathbb{R}^N \to \mathbb{R}^{\binom{N}{2}}$ are given by

$$\mathcal{P}(\lambda)(t) = t\lambda, \quad t \in \mathbb{R}, \quad \mathcal{Q}(\lambda)(z) = \lambda \wedge z, \quad z \in \mathbb{R}^N.$$

Thus if $\lambda \neq 0$

$$Ker\mathcal{Q}(\lambda) = \{z \in \mathbb{R}^N : \lambda \wedge z = 0\} = \{t\lambda : t \in \mathbb{R}\} = Im\mathcal{P}(\lambda).$$

In the following we shall consider variational integrals defined on elliptic couples. The integrals in question take the form

$$I(\alpha, \beta) = \int_\Omega f(\mathcal{P}\alpha, \mathcal{Q}^*\beta) \quad \text{for} \quad \alpha \in W^{1,p}(\Omega, \mathbb{R}^d), \beta \in W^{1,p}(\Omega, \mathbb{R}^k),$$

where the integrand $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ is at least continuous and $\mathcal{P}$, $\mathcal{Q}$ are linear operators forming an elliptic complex. In [15] the following definition of polyconvexity is given.

DEFINITION 5.1. — *f is said to be polyconvex if it can be expressed as :*

$$f(X, Y) = g(X, Y, \langle X, Y \rangle) \tag{5.8}$$

where $g : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is convex.

The notion of polyconvex integrands, already given in the book of Morrey [22], was deeply studied by Ball [3] providing a better understanding of several problems, especially those concerning the theory of finite elasticity.

Note that our definition of polyconvexity agrees with the one given by Ball in dimension two, provided that we take $\mathcal{P}u = \nabla u$, $\mathcal{Q}v = \mathrm{curl} v$.

In the sequel we prove that Theorem 1.1 still holds if the function f is polyconvex.

Let $f(x, y, z, \eta, \xi) : \Omega \times \mathbb{R}^{d+k} \times \mathbb{R}^{2m} \to [0, \infty)$ be a Carathéodory function such that

i) for all $x \in \Omega$, $(y, z) \in \mathbb{R}^d \times \mathbb{R}^k$ the function $(\eta, \xi) \to f(x, y, z, \eta, \xi)$ is polyconvex;

ii) for any $(x_0, y_0, z_0) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^k$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$,

$|(y, z) - (y_0, z_0)| < \delta$ and $\xi, \eta \in \mathbb{R}^{N \times d}$ then $f(x, y, z, \eta, \xi) \geqslant (1 - \epsilon) f(x_0, y_0, z_0, \eta, \xi)$.

THEOREM 5.1. — *Suppose that $f(x, y, z, \eta, \xi)$ satisfies conditions i) and ii) and suppose $p > 2\dfrac{N-1}{N}$.*

Let $\alpha_n \in W^{1,2}(\Omega, \mathbb{R}^d), \beta_n \in W^{1,2}(\Omega, \mathbb{R}^k)$ *and* $\alpha \in W^{1,p}(\Omega, \mathbb{R}^d), \beta \in W^{1,p}(\Omega, \mathbb{R}^k)$ *such that* $\alpha_n \rightharpoonup \alpha$ *in* $W^{1,p}(\Omega, \mathbb{R}^d)$ *and* $\beta_n \rightharpoonup \beta$ *in* $W^{1,p}(\Omega, \mathbb{R}^k)$. *Then*

$$\int_\Omega f\big(x, \alpha, \beta, \mathcal{P}\alpha, \mathcal{Q}^*\beta\big)dx \leqslant \liminf_{n\to\infty} \int_\Omega f\big(x, \alpha_n, \beta_n, \mathcal{P}\alpha_n, \mathcal{Q}^*\beta_n\big)dx.$$

Proof. — If f verifies the assumptions, there exists a sequence of continuous nonnegative functions

$g_j\big(x, y, z, \eta, \xi\big)$ such that each g_j is polyconvex in (η, ξ) and

$$0 \leqslant g_j\big(x, y, z, \eta, \xi\big) \leqslant c_j\big(1 + |\langle\eta, \xi\rangle|\big), \quad g_j(x, (y, z), (\eta, \xi)) \leqslant g_{j+1}\big(x, y, z, \eta, \xi\big)$$

$$f\big(x, y, z, \eta, \xi\big) = \sup_j g_j\big(x, y, z, \eta, \xi\big)$$

(see Lemma 3.2 in [13]). Observe that polyconvexity implies quasiconvexity (see [15]) and that

$$g_j(x, y, z, \eta, \xi) \leqslant c(1 + |\eta|^2 + |\xi|^2).$$

Therefore Theorem 1.1 holds and we have

$$\begin{aligned}\int_\Omega g_j\big(x, \alpha, \beta, \mathcal{P}\alpha, \mathcal{Q}^*\beta\big)dx &\leqslant \liminf_n \int_\Omega g_j\big(x, \alpha_n, \beta_n, \mathcal{P}\alpha_n, \mathcal{Q}^*\beta_n\big)dx \\ &\leqslant \liminf_n \int_\Omega f\big(x, \alpha_n, \beta_n, \mathcal{P}\alpha_n, \mathcal{Q}^*\beta_n\big)dx\end{aligned}.$$

Now notice that since g_j is increasing, we get

$$\begin{aligned}\int_\Omega f\big(x, \alpha, \beta, \mathcal{P}\alpha, \mathcal{Q}^*\beta\big)dx &= \lim_j \int_\Omega g_j\big(x, \alpha, \beta, \mathcal{P}\alpha, \mathcal{Q}^*\beta\big)dx \\ &\leqslant \liminf_n \int_\Omega f\big(x, \alpha_n, \beta_n, \mathcal{P}\alpha_n, \mathcal{Q}^*\beta_n\big)dx.\end{aligned}$$

This concludes the proof.

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