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A large deviation result for the subcritical Bernoulli percolation (*)

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Abstract. — We consider subcritical Bernoulli percolation in dimensions two and more. If C is the open cluster containing the origin, we prove that the law of C/N satisfies a large deviation principle with respect to the Hausdorff metric.

Résumé. — Nous considérons la percolation de Bernoulli dans les dimensions supérieures ou égales à deux. Si C est le cluster d'arêtes ouvertes contenant l'origine, nous prouvons que la loi de C/N satisfait un principe de grandes déviations par rapport à la métrique de Hausdorff.

1. Introduction

Consider the cluster C of the origin in the subcritical phase of Bernoulli percolation in $\mathbb{Z}^d$. This is a random object of the space $\mathcal{K}_c$ of connected compact sets in $\mathbb{R}^d$. We let D_H be the Hausdorff distance on $\mathcal{K}_c$. Let

$$\xi = \lim_{N\to\infty} \frac{1}{N} \ln P(0 \text{ is connected to } Nx).$$

be the inverse correlation length. Assume that $\mathcal{H}^1_\xi$ is the one-dimensional Hausdorff measure on $\mathbb{R}^d$ constructed from ξ.

In the supercritical regime, large deviation principles have been proved for the law of C/N [3, 4]. In two dimensions, it relies on estimates of the law of dual clusters, which are subcritical. More precisely, let Γ be a contour in $\mathbb{R}^2$ enclosing an area. The probability that a dual cluster is close for the Hausdorff distance to $N\Gamma$ behaves like $\exp(-N\mathcal{H}^1_\xi(\Gamma))$. But what happens if we consider more general connected sets than contours ?

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In this note we establish a large deviation principle for the law of C/N in the subcritical regime in dimensions two and more. Let $\overline{\mathcal{K}}_c$ denote the set of connected compact sets of $\mathbb{R}^d$ quotiented by the translation equivalence. The usual distance between compact sets is the Hausdorff distance. We denote it by $\overline{D}_H$ when considered as a distance on $\overline{\mathcal{K}}_c$. Let C be still the open cluster containing the origin. Write $\overline{C}$ for the equivalent class of C in $\overline{\mathcal{K}}_c$. Let P be the measure and p_c be the critical point of the Bernoulli percolation process. The formulation of our large deviation principle is the following:

THEOREM 1.1. — *Let $p < p_c$. Under P, the family of the laws of $(\overline{C}/N)_{N\geqslant 1}$ on the space $\overline{\mathcal{K}}_c$ equipped with the Hausdorff metric $\overline{D}_H$ satisfies a large deviation principle with good rate function $\mathcal{H}^1_\xi$ and speed N: for any borel subset $\overline{\mathcal{U}}$ of $\overline{\mathcal{K}}_c$,*

$$\begin{aligned} -\inf\{\mathcal{H}^1_\xi(\overline{U}) : \overline{U} \in \mathit{interior}(\overline{\mathcal{U}})\} &\leqslant \liminf_{N\to\infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}}) \\ &\leqslant \limsup_{N\to\infty} \frac{1}{N} \ln P(\overline{C}/N \in \overline{\mathcal{U}}) \\ &\leqslant -\inf\{\mathcal{H}^1_\xi(\overline{U}) : \overline{U} \in \mathit{closure}(\overline{\mathcal{U}})\}, \end{aligned}$$

where the interior and the closure are taken with respect to the Hausdorff metric on $\overline{\mathcal{K}}_c$.

The proof of the lower bound relies on the FKG inequality; we use it to construct a cluster close to a given large connected set with a sufficient high probability. Concerning the upper bound, the proof is based on the skeleton coarse graining technique and on the BK inequality; it follows the lines of the proof in [3] with slight adaptations.

We underline that in supercritical percolation the large deviation principles lead to estimates of the shape of large finite clusters. In fact, there exists a shape called the Wulff crystal, which minimizes the rate function under a volume constraint. Unfortunately, the large deviation principle does not allow us to describe the typical shape of a large cluster in the subcritical phase. In this regime, computing simulations of large clusters show very irregular objects.

We note furthermore that our main result has been obtained independently by Kovchegov, Sheffield [11]. Their approach is quite different and makes use of Steiner trees to approximate connected compact sets.

In the next section we recall the definition and basic results of the percolation model. Then we define the measure $\mathcal{H}^1_\xi$ and the space $\overline{\mathcal{K}}_c$. Geometric results required about connected compact sets are given in Section 4. In Section 5 we introduce skeletons, and use them to approximate connected compact sets. The proof of the lower bound follows in Section 6. The coarse graining technique is given in Section 7, and the proof of the upper bound follows in Section 8.

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2. The model

We consider the site lattice $\mathbb{Z}^d$ where d is a fixed integer larger than or equal to two. We use the euclidian norm $|\,.\,|_2$ on $\mathbb{Z}^d$. We turn $\mathbb{Z}^d$ into a graph $\mathbb{L}^d$ by adding edges between all pairs x, y of points of $\mathbb{Z}^d$ such that $|x-y|_2 = 1$. The set of all edges is denoted by $\mathbb{E}^d$. A path in $(\mathbb{Z}^d, \mathbb{E}^d)$ is an alterning sequence $x_0, e_0, \ldots, e_{n-1}, x_n$ of distinct vertices x_i and edges e_i where e_i is the edge between x_i and x_{i+1}.

Let p be a parameter in $(0,1)$. The edges of $\mathbb{E}^d$ are open with probability p, and closed otherwise, independently from each others. We denote by P the product probability measure on the configuration space $\Omega = \{0,1\}^{\mathbb{E}^d}$. The measure P is the classic Bernoulli bond percolation measure. Two sites x and y are said *connected* if there is a path of open edges linking x to y. We note this event $\{x \leftrightarrow y\}$. A *cluster* is a connected component of the random graph.

The model exhibits a phase transition at a point p_c, called the *critical point:* for $p < p_c$ the clusters are finite and for $p > p_c$ there exists a unique infinite cluster. We work with a fixed value $p < p_c$.

The following properties describe the behaviour of the tail distribution of the law of a cluster (for a proof see [9]).

LEMMA 2.1. — *Let $p < p_c$ and let C be the cluster of the origin. There exists $a_0 > 0$ and $a_1 > 0$ such that for all n*

$$P(|C| \geqslant n) \leqslant \exp(-a_0 n), \tag{2.1}$$

$$P(\mathit{diam}\ C \geqslant n) \leqslant \exp(-a_1 n). \tag{2.2}$$

We briefly recall two fundamental correlation inequalities. To a configuration ω, we associate the set $K(\omega) = \{\, e \in \mathbb{E}^2 : \omega(e) = 1 \,\}$. Let A and B be two events. The disjoint occurrence $A \circ B$ of A and B is the event

$$\left\{ \begin{array}{l} \omega \text{ such that there exists a subset } H \text{ of } K(\omega) \text{ such that if} \\ \omega', \omega'' \text{ are the configurations determined by } K(\omega') = H \\ \text{and } K(\omega'') = K(\omega) \setminus H, \text{ then } \omega' \in A \text{ and } \omega'' \in B\ . \end{array} \right\}$$

There is a natural order on Ω defined by the relation: $\omega_1 \leqslant \omega_2$ if and only if all open edges in ω_1 are open in ω_2. An event is said to be increasing (respectively decreasing) if its characteristic function is non decreasing (respectively non increasing) with respect to this partial order.

Suppose A and B are both increasing (or both decreasing). The Harris–FKG inequality [7, 10] says that $P(A \cap B) \geqslant P(A)P(B)$. The van den Berg–Kesten inequality [1] says that $P(A \circ B) \leqslant P(A)P(B)$.

For x, y two sites we consider $\{x \leftrightarrow y\}$ the event that x and y are connected. In the subcritical regime the probability of this event decreases exponentially: for any x in $\mathbb{R}^d$, we denote by $\lfloor x \rfloor$ the site of $\mathbb{Z}^d$ whose coordinates are the integer part of those of x. Then

PROPOSITION 2.2. — *The limit*

$$\xi(x) = -\lim_{N \to \infty} \frac{1}{N} \ln P(0 \leftrightarrow \lfloor Nx \rfloor)$$

exists and is > 0, *see* [9, section 6.2]. The function ξ thus obtained is a norm on $\mathbb{R}^d$.

In addition for every site x in $\mathbb{Z}^d$, we have

$$P(0 \leftrightarrow x) \leqslant \exp(-\xi(x)). \tag{2.3}$$

Since ξ is a norm there exists a positive constant $a_2 > 0$ such that for all x in $\mathbb{R}^d$,

$$a_2 |x|_2 \leqslant \xi(x). \tag{2.4}$$

3. The $\mathcal{H}^1_\xi$ measure and the space of the large deviation principle

With the norm ξ, we construct the one-dimensional Hausdorff measure $\mathcal{H}^1_\xi$. If U is a non-empty subset of $\mathbb{R}^d$ we define the ξ-*diameter* of U as $\xi(U) = \sup\{\xi(x-y) : x, y \in U\}$. If $E \subset \cup_{i \in I} U_i$ and $\xi(U_i) < \delta$ for each i, we say that $\{U_i\}_{i\in I}$ is a δ-cover of E. For every subset E of $\mathbb{R}^d$, and every real $\delta > 0$ we define

$$\mathcal{H}^1_{\xi,\delta}(E) = \inf \sum_{i=1}^{\infty} \xi(U_i),$$

where the infimum is taken over all countable δ-covers of E. Then we define the *one-dimensional Hausdorff measure* of E as

$$\mathcal{H}^1_\xi(E) = \lim_{\delta\to 0} \mathcal{H}^1_{\xi,\delta}(E).$$

For a study of the Hausdorff measure, see e.g. [6].

We denote by $\mathcal{K}$ the collection of all compact sets of $\mathbb{R}^d$. The Euclidian distance between a point and a set E is

$$d(x, E) = \inf\{|x - y|_2 : y \in E\}.$$

We endow $\mathcal{K}$ with the Hausdorff metric $\overline{D}_H$:

$$\forall K_1, K_2 \in \mathcal{K},\ \overline{D}_H(K_1, K_2) = \max\{\max_{x_1\in K_1} d(x_1, K_2),\ \max_{x_2\in K_2} d(x_2, K_1)\}$$

Let $\mathcal{K}_c$ be the subset of $\mathcal{K}$ consisting of connected sets. An element of $\mathcal{K}_c$ is called a continuum. We define an equivalence on $\mathcal{K}_c$ by: K_1 is equivalent to K_2 if and only if K_1 is a translate of K_2. We denote by $\overline{\mathcal{K}}_c$ the quotient set of classes of $\mathcal{K}_c$ associated to this relation, and by $\overline{D}_H$ the resulting quotient metric:

$$\overline{D}_H(\overline{K}_1, \overline{K}_2) = \inf_{x_1,x_2\in\mathbb{R}^d} D_H(K_1 + x_1, K_2 + x_2) = D_H(K_1, \overline{K}_2).$$

We finally define the Hausdorff measure on $\overline{\mathcal{K}}_c$ by

$$\forall \overline{K} \in \overline{\mathcal{K}}_c\ \mathcal{H}^1_\xi(\overline{K}) = \mathcal{H}^1_\xi(K),$$

which makes sense since $\mathcal{H}^1_\xi$ is invariant by translation on $\mathcal{K}_c$.

Now we state an essential property required by the large deviation principle.

PROPOSITION 3.1. — *The measure $\mathcal{H}^1_\xi$ is a good rate function on the space $\overline{\mathcal{K}}_c$.*

Proof. — The lower semicontinuity is due to Golab and the proof can be found in [6, p 39]. We follow now the proof of the proposition 5 in [3]. Let $t > 0$ and let $(\overline{K}_n, n \in \mathbb{N})$ be a sequence in $\overline{\mathcal{K}}_c$ such that $\mathcal{H}^1_\xi(\overline{K}_n) \leqslant t$ for all

n in $\mathbb{N}$. For each n we can assume that the origin belongs to K_n. Since the diameter of an element of $\mathcal{K}_c$ is bounded by a constant time its $\mathcal{H}^1_\xi$-measure, there exists a bounded set B such that

$$K \in \mathcal{K}_c,\ 0 \in K,\ \mathcal{H}^1_\xi(K) \leqslant t \Rightarrow K \subset B.$$

Thus, the sets K_n are subsets of B. For every compact set K_0 the subset $\{K \in \mathcal{K} : K \subset K_0\}$ is itself compact with respect to the metric D_H [2]. Hence $(K_n)_{n\in\mathbb{N}}$ admits a subsequence converging for the metric D_H; the same subsequence of $(\overline{K}_n)_{n\in\mathbb{N}}$ converges for the metric $\overline{D}_H$. □

4. Curves and continua

A *curve* is a continuous injection $\Gamma : [a,b] \to \mathbb{R}^d$, where $[a,b] \subset \mathbb{R}$ is a closed interval. We write also Γ for the image $\Gamma([a,b])$. We call $\Gamma(a)$ the *first point* of the curve and $\Gamma(b)$ its *last point.* Any curve is a continuum. We say that a curve is *rectifiable* if its $\mathcal{H}^1_\xi$-measure is finite.

We state a simple lemma:

LEMMA 4.1. — *For each curve $\Gamma : [a,b] \to \mathbb{R}^d$,*

$$\mathcal{H}^1_\xi(\Gamma) \geqslant \mathcal{H}^1_\xi([\psi(a),\psi(b)]) = \xi(\psi(a) - \psi(b)).$$

Next, we associate to a continuum a finite family of curves in two different manners. With the first one, we shall prove the lower bound, and with the second one, we shall prove the upper bound.

DEFINITION 4.2. — *A family of curves $\{\gamma_i\}_{i\in I}$ is said hardly disjoint if for all $i \neq j$, the curve γ_j can intersect γ_i only on one of the endpoints of γ_i.*

PROPOSITION 4.3. — *Let Γ be a continuum with $\mathcal{H}^1_\xi(\Gamma) < \infty$. Then for all parameter $\delta > 0$, there exists a finite family $\{\Gamma_i\}_{i\in I}$ of rectifiable curves included in Γ such that $D_H(\Gamma, \cup_{i\in I}\Gamma_i) < \delta$, $\cup_{i\in I}\Gamma_i$ is connected and the family $\{\Gamma_i\}_{i\in I}$ is hardly disjoint.*

Furthermore, there exists a deterministic way to choose the Γ_i's such that if Γ' is a translate of Γ, the resultant Γ'_i's are the translates of the Γ_i's by the same vector.

PROPOSITION 4.4. — *Let Γ be a continuum with $\mathcal{H}^1_\xi(\Gamma) < \infty$. Then for all parameter $\delta > 0$, there exists a finite family $\{\Gamma_i\}_{i\in I}$ of rectifiable curves included in Γ such that $D_H(\Gamma, \cup_{i\in I}\Gamma_i) < \delta$, with the following properties: the euclidian diameter of Γ_i is larger than δ for all i in I, $\cup_{i=1}^{l}\Gamma_i$ is connected for all $l \geqslant 1$, and the first point of Γ_l is in $\cup_{k<l}\Gamma_k$.*

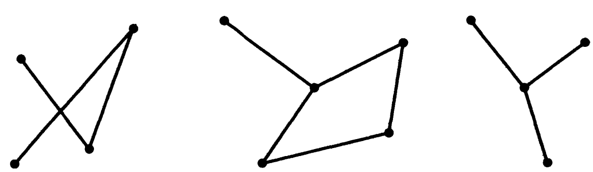
Propositions 4.3 and 4.4 are corollaries of lemma 3.13 of [6] in which we have stated the additional facts coming from the proof. □

We often think of $\mathbb{L}^d$ as embedded in $\mathbb{R}^d$, the edges $\{x,y\}$ being straight line segments $[x,y]$. An *animal* is a finite connected subgraph of $\mathbb{L}^d$ containing the origin. The Hausdorff distance between an animal and its corresponding cluster is $\frac{1}{2}$. So, to prove the large deviation principle we shall consider the animal of the origin instead of the cluster. The point is that an animal is a continuum. Hence we shall be able to apply Propositions 4.3 and 4.4 to an animal.

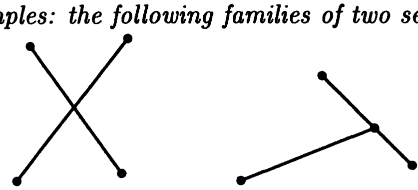
5. The skeletons

DEFINITION 5.1. — *A skeleton S is a finite family of segments that are linked by their endpoints. We denote by $E(S)$ the set of the vertices of the segments of S and by* $\operatorname{card} S$ *the cardinal of $E(S)$. We define $\mathcal{HS}^1_\xi(S)$ as the sum of the ξ-length of the segments of S. A point is also considered as a skeleton.*

Examples:



Counter-examples: the following families of two segments are not skeletons



Sometimes a skeleton S is simply understood as the union of its segments, and so is a compact connected subset of $\mathbb{R}^d$. This is the case when we write $\mathcal{H}^1_\xi(S)$. We always have

$$\mathcal{H}^1_\xi(S) \leqslant \mathcal{HS}^1_\xi(S). \tag{5.1}$$

If S_1 and S_2 are two skeletons which have a vertex in common, then $S = S_1 \cup S_2$ is also a skeleton, and

$$\mathcal{HS}^1_\xi(S) = \mathcal{HS}^1_\xi(S_1) + \mathcal{HS}^1_\xi(S_2). \tag{5.2}$$

LEMMA 5.2. — *For every* Γ *continuum with* $\mathcal{H}^1_\xi(\Gamma) < \infty$, *for all* $\delta > 0$, *there exists a skeleton* S *such that*

$$D_H(S,\Gamma) < \delta,\ \mathcal{HS}^1_\xi(S) \leqslant \mathcal{H}^1_\xi(\Gamma).$$

The skeleton S *is said to* δ*-approximate* Γ.

Proof. — Let Γ be a continuum with $\mathcal{H}^1_\xi(\Gamma) < \infty$. Let $\{\Gamma_k\}_{k\in I}$ be the sequence of rectifiable curves coming from Proposition 4.3 with parameter $\delta/2$. Consider Γ_1. We take $t_0 = 0$, $x_0 = \Gamma_1(0)$ and for $n \geqslant 0$

$$t_{n+1} = \inf\{t > t_n : |\Gamma_1(t) - \Gamma_1(t_n)| \geqslant \delta/2\}.$$

If t_{n+1} is finite then $x_{n+1} = \Gamma_1(t_{n+1})$. Otherwise, we take for x_{n+1} the last point of Γ_1 if it is different from x_n, and we stop the sequence of the x_i's. Since Γ_1 is rectifiable and because of Lemma 4.1, this sequence is finite. We call S_1 the family of the segments $[x_i, x_{i+1}]$ for $i = 0$ to $n-1$. By construction S_1 is a skeleton, the endpoints of Γ_1 are vertices of S_1 and S_1 $\delta/2$-approximates Γ_1. We construct in the same way the other S_i's for i in I. By assumption, the Γ_i's are connected by their endpoints. Since these endpoints are vertices of S_i's, the union of the S_i's denoted by S is also a skeleton. We control the $\mathcal{HS}^1_\xi$ measure of S by

$$\mathcal{HS}^1_\xi(S) = \sum_{i\in I}\mathcal{HS}^1_\xi(S_i) \leqslant \sum_{i\in I}\mathcal{H}^1_\xi(\Gamma_i) \leqslant \mathcal{H}^1_\xi(\Gamma),$$

where we use (5.2) and Lemma 4.1. The Hausdorff distance between S and Γ is controlled by

$$D_H(S,\Gamma) < D_H(S, \cup_{i\in I}\Gamma_i) + \delta/2 < \sup_{i\in I} D_H(S_i,\Gamma_i) + \delta/2 < \delta. \qquad \square$$

Remark 5.3. — If Γ' is the image of Γ by a translation of vector $\vec{u}$, then the skeleton S' constructed as above from Γ' is the image by the same translation of the skeleton S constructed from Γ.

6. The lower bound

We prove in this section the lower bound stated in Theorem 1.1. By a standard argument [5], it is equivalent to prove that for all $\delta > 0$, all Γ in $\mathcal{K}_c$,

$$\liminf_{N\to\infty} \frac{1}{N} \ln P(\overline{D}_H(\overline{C}/N, \overline{\Gamma}) < \delta) \geqslant -\mathcal{H}^1_\xi(\Gamma).$$

We introduce two notations. The r-neighbourhood of a set E is the set

$$\mathcal{V}(E,r) = \{x \in \mathbb{R}^d : d(x,E) < r\}.$$

Let E_1, E_2 be two subsets of $\mathbb{R}^d$. We define

$$e(E_1, E_2) = \inf\{r > 0 : E_2 \subset \mathcal{V}(E_1, r)\}.$$

We now take Γ in $\overline{\Gamma}$ such that the origin is a vertex of the skeleton S constructed from Γ, as described in the proof of Lemma 5.2. This can be done because of the previous remark. First observe that

$$\begin{aligned} P(\overline{D}_H(\overline{C}/N, \overline{\Gamma}) < \delta)) &\geqslant P(D_H(C/N, \Gamma) < \delta)) \\ &\geqslant P(\{e(C/N, \Gamma) < \delta/2\} \cap \{e(\Gamma, C/N) < \delta\}). \end{aligned}$$

We let

$$\begin{aligned} G(N, \delta/2, \Gamma) = \{\exists \text{ a connected set } C' &\text{ of the percolation process,} \\ &\text{containing } 0, \text{ such that } D_H(C'/N, \Gamma) < \delta/2\}. \end{aligned}$$

We have $G(N, \delta/2, \Gamma) \subset \{e(C/N, \Gamma) < \delta/2\}$. So

$$\begin{aligned} P(D_H(\overline{C}/N, \overline{\Gamma}) < \delta) &\geqslant P(G(N, \delta/2, \Gamma) \cap \{e(\Gamma, C/N) < \delta\}) \\ &\geqslant P(G(N, \delta/2, \Gamma)) \times \\ &\qquad P(e(\Gamma, C/N) < \delta \,|\, G(N, \delta/2, \Gamma)). \qquad (6.1) \end{aligned}$$

We study the first term of the product. Let r be positive and let x and y be two sites. The event that there exists an open path from x to y whose Hausdorff distance to the segment $[x, y]$ is less than r is denoted by $x \overset{r}{\longleftrightarrow} y$. We restate lemma 8 in Section 5 of [3]:

LEMMA 6.1. — *Let $\phi(n)$ be a function such that $\lim_{n\to\infty} \phi(n) = \infty$. For every point x, we have*

$$\lim_{n\to\infty} \frac{1}{n} P(0 \overset{\phi(n)}{\longleftrightarrow} \lfloor nx \rfloor) = -\xi(x).$$

Take the skeleton S which $\delta/4$-approximates Γ, as in Lemma 5.2. We have carefully chosen Γ such that the origin is a vertex of S. We label $x_1, \ldots, x_n$ the vertices of S. We note $i \sim j$ if $[x_i, x_j]$ is a segment of S. Then

$$\begin{aligned} P(G(N, \delta/2, \Gamma)) &\geqslant P(G(N, \delta/4, S)) \\ &\geqslant P(\lfloor Nx_i \rfloor \overset{N\delta/4}{\longleftrightarrow} \lfloor Nx_j \rfloor, \forall i < j \text{ such that } i \sim j). \end{aligned}$$

The fact that the origin is a vertex of S is used in the last inequality. Since the events last considered are increasing, the FKG inequality leads to

$$P(G(N, \delta/2, \Gamma)) \geqslant \prod_{i<j, i\sim j} P(\lfloor Nx_i \rfloor \overset{N\delta/4}{\longleftrightarrow} \lfloor Nx_j \rfloor).$$

But by Lemma 6.1

$$\begin{aligned}\lim \frac{1}{N} \ln \prod_{i<j, i\sim j} P(\lfloor Nx_i \rfloor \overset{N\delta/4}{\longleftrightarrow} \lfloor Nx_j \rfloor) &= -\sum_{i<j, i\sim j} \mathcal{H}^1_\xi([x_i, x_j]) \\ &= -\mathcal{HS}^1_\xi(S).\end{aligned}$$

Hence

$$\liminf \frac{1}{N} \ln P(G(N, \delta/2, \Gamma)) \geqslant -\mathcal{HS}^1_\xi(S) \geqslant -\mathcal{H}^1_\xi(\Gamma). \tag{6.2}$$

Now we analyze the second term $P(e(\Gamma, C/N) < \delta \big| G(N, \delta/2, \Gamma))$ of the product in (6.1). First observe that the event

$$\{e(\Gamma, C/N) \geqslant \delta\} \cap G(N, \delta/2, \Gamma)$$

is included in

$$\begin{aligned}\{ &\exists \text{ an open path of length } \geqslant N\delta/2 \text{ lying in} \\ &\mathcal{V}(N\Gamma, N\delta)) \setminus \mathcal{V}(N\Gamma, N\delta/2)\} \cap G(N, \delta/2, \Gamma).\end{aligned}$$

The two events appearing in the last intersection are independent, since they depend on disjoint sets of bonds. So

$$\begin{aligned}&P(e(\Gamma, C/N) \geqslant \delta \big| G(N, \delta/2, \Gamma)) \\ &\qquad \leqslant \ P(\exists \text{ an open path of length } \geqslant N\delta/2 \text{ lying in } \mathcal{V}(N\Gamma, N\delta)) \\ &\qquad \leqslant \ c_1(\mathcal{H}^1_\xi(\Gamma) + \delta)\delta^{d-1} N^d \exp(-a_0 N\delta/2),\end{aligned}$$

for a certain constant $c_1 > 0$. In the last inequality, we use (2.1) and a bound of the cardinality of $\mathcal{V}(N\Gamma, N\delta) \cap \mathbb{Z}^d$. The member on the RHS tends to 0 as N tends to infinity. Hence

$$\lim_{N\to\infty} P(e(\Gamma, C/N) < \delta \big| G(N, \delta/2, \Gamma)) = 1. \tag{6.3}$$

By limits (6.2) and (6.3), the inequatity (6.1) yields to the lower bound. □

7. Coarse graining

Now we associate a skeleton to an animal. By a counting argument it will yield to the desired upper bound.

DEFINITION 7.1. — *Let $S = \{T_i\}_{i \in I}$ be a skeleton, and let C be an animal. We say that S fits C if $E(S)$ is included in the set of vertices of C, if for all i in I there exists a curve γ_i such that γ_i is included in C and has the same endpoints than T_i, and if the family $\{\gamma_i\}_{i \in I}$ is hardly disjoint.*

LEMMA 7.2. — *Let $s > 4$. For all animal C with diam$(C) > s$, there exists a skeleton S such that $\mathcal{HS}_\xi^1(S) \geqslant a_2(s/8)$card$S$, $D_H(C, S) < s$, and the skeleton S fits the animal C.*

Such a skeleton is said to be *s-compatible* with the animal C.

Proof. — We recall that an animal is also a continuum. Let $\{\Gamma_k\}_{k \in I}$ be a sequence of rectifiable curves as in Proposition 4.4 with parameter $s/2$. Consider for example Γ_1. We take $x_0 = \Gamma_1(0)$ and $t_0 = 0$. For $n \geqslant 0$, let

$$t_{n+1} = \inf\{t > t_n : \Gamma_1(t) \in \mathbb{Z}^d,\ |\Gamma_1(t) - \Gamma_1(t_n)|_2 \geqslant s/4\}.$$

If t_{n+1} is finite, then $x_{n+1} = \Gamma_1(t_{n+1})$. Otherwise, we erase x_n, we put x_n the last point of Γ_1 and we stop the sequence. Note that t_1 cannot be infinite.

We call S_1' the family of the segments $[x_j, x_{j+1}]$. The set S_1' is a skeleton, and is called the *s-skeleton* of Γ_1. For the other i's in I we construct S_i' the s-skeleton of Γ_i in the same way. For each i in I we have

$$\mathcal{HS}_\xi^1(S_i') \geqslant (\operatorname{card} S_i' - 1)a_2(s/4).$$

Since the euclidian diameter of Γ_i is larger than s for each i in I, we have card $S_i' \geqslant 2$. Since $s > 4$, it follows that $\mathcal{HS}_\xi^1(S_i') \geqslant a_2(s/8)\operatorname{card} S_i'$, for each i in I.

We now refine the skeleton S_i' into another skeleton S_i. For each $j > i$ such that the first point of Γ_j, say z, is in Γ_i but is not a vertex of S_i', we take the segment of S_i' whose endpoints x and y surround z on Γ_i. We replace in S_i' the segment $[x, y]$ by the two segments $[x, z]$ and $[z, y]$. When we have done this for all j we rename S_i' by S_i. The set S_i is always a skeleton which satisfies $D_H(S_i, \Gamma_i) < s/2$. By triangular inequality, $\mathcal{HS}_\xi^1(S_i) \geqslant \mathcal{HS}_\xi^1(S_i')$. We denote by S the concatenation of the S_i's. By induction, S is a skeleton. Furthermore, each vertex of S is a vertex of S_i' for a certain i.

Now we check that S fulfills the good properties. We have

$$\mathcal{HS}_\xi^1(S) = \sum_{i \in I} \mathcal{HS}_\xi^1(S_i) \geqslant \sum_{i \in I} a_2(s/8)\operatorname{card} S_i' \geqslant a_2(s/8)\operatorname{card} S,$$

and

$$D_H(S,\Gamma) < \sup_{i\in I} D_H(S_i,\Gamma_i) + s/2 < s. \qquad \square$$

The next statement gives the interest of such a construction. For a given skeleton S we let $\mathcal{A}(\mathcal{S})$ be the event that S is s-compatible with an animal.

LEMMA 7.3. — *For all scales $s > 4$,*

$$P(\mathcal{A}(S)) \leqslant \exp\{-\mathcal{HS}^1_\xi(S)\}.$$

Proof. — If S is compatible with an animal, we have the disjoint occurrences of the events $\{x_i \leftrightarrow x_j\}$ for all $i < j$ such that $[x_i, x_j]$ is a segment of S. The BK inequality implies

$$P(\mathcal{A}(S)) \leqslant \prod_{\substack{i<j \\ [x_i,x_j] \text{ is a segment of } S}} P(x_i \leftrightarrow x_j).$$

The last sentence of Proposition 2.2 yields to the desired result. $\square$

8. The upper bound

We prove here the upper bound stated in Theorem 1.1. Consider the animal C containing the origin. Let $\Phi_H(u) = \{\overline{K} \in \overline{\mathcal{K}}_c : \mathcal{H}^1_\xi(\overline{K}) \leqslant u\}$. We prove that $\forall\, u \geqslant 0$, $\forall\, \delta > 0$, $\forall\, \alpha > 0$, $\exists\, N_0$ such that $\forall\, N \geqslant N_0$,

$$P(D_H(C/N, \Phi_H(u)) \geqslant \delta) \leqslant \exp -Nu(1-\alpha).$$

This is the Freidlin-Wentzell presentation of the upper bond of our large deviation principle, see [8].

Let c be a positive constant to be chosen later, and take $s = 8c \ln N$. For N large enough, $D_H(C/N, \Phi_H(u)) \geqslant \delta$ implies diam $C > s$. By Lemma 7.2, we can take S a skeleton that s-approximates C. We have $D_H(C/N, S) \leqslant 8c\ln N/N$, so for N large enough,

$$P(D_H(C/N, \Phi_H(u)) \geqslant \delta) \leqslant P(D_H(S/N, \Phi_H(u)) \geqslant \delta/2).$$

Since S is an element of $\mathcal{K}_c$, the inequality $D_H(S/N, \Phi_H(u)) \geqslant \delta/2$ implies that $\mathcal{H}^1_\xi(S) \geqslant uN$ and so $\mathcal{HS}^1_\xi(S) \geqslant uN$ by (5.1).

Let a be such that $a > u/a_1$. We have

$$\begin{aligned}&P(\mathcal{HS}^1_\xi(S) \geqslant uN)\\ &\qquad \leqslant P(\mathcal{HS}^1_\xi(S) \geqslant uN,\ \text{diam}\ C \leqslant aN) + P(\text{diam}\ C > aN).\end{aligned}$$

But $P(\text{diam}\ C > aN) < \exp -a_1 aN$ by inequality (2.2). Since $a > u/a_1$, we have $P(\text{diam}\ C > aN) < \exp -uN$.

We estimate now the term $P(\mathcal{HS}^1_\xi(S) \geqslant uN,\ \text{diam}\ C \leqslant aN)$. Let $\mathcal{A}(n,u,a,N)$ be the set of skeletons T such that $\mathcal{HS}^1_\xi(T) \geqslant uN$, $E(T)$ is included in $\mathbb{Z}^d$, $\text{card}\, T = n$, and there exists a connected set of sites containing the origin of diameter less than aN that is s-compatible with the skeleton T. We have

$$P(\mathcal{HS}^1_\xi(S) \geqslant uN,\ \text{diam}\ C \leqslant aN) \leqslant \sum_n \sum_{T\in\mathcal{A}(n,u,a,N)} P(S=T).$$

The number of skeletons we can construct from n points is bounded by $(n^n)^2$. Take a skeleton in $\mathcal{A}(n,u,a,N)$. All its vertices are in a box centered at 0, of side length $2(aN + c\ln N)$. So the cardinal of $\mathcal{A}(n,u,a,N)$ is less than $2^{dn}(aN + c\ln N)^{dn}(n^n)^2 \leqslant \exp a_3 n \ln N$, for a certain constant $a_3 > 0$. Take $b > 0$ a constant such that $a_3 - a_2 b < 0$. We assume now that $c > b$. We have

$$\begin{aligned}\mathcal{HS}^1_\xi(T) &= \mathcal{HS}^1_\xi(T)(1-b/c) + b/c\mathcal{HS}^1_\xi(T)\\ &\geqslant uN(1-b/c) + a_2 bn \ln N\end{aligned}$$

because $\mathcal{HS}^1_\xi(T) \geqslant a_2(s/8)\text{card}\, T$. Then for N large enough by Lemma 7.3

$$\begin{aligned}P(\mathcal{HS}^1_\xi(S) &\geqslant uN,\ \text{diam}\ C \leqslant aN)\\ &\leqslant \sum_n \sum_{T\in\mathcal{A}(n,u,a,N)} \exp -\mathcal{HS}^1_\xi(T)\\ &\leqslant \sum_n \sum_{T\in\mathcal{A}(n,u,a,N)} \exp(-uN(1-b/c) - a_2 bn\ln N)\\ &\leqslant \exp(-uN(1-b/c)) \sum_n \exp((a_3 - a_2 b)n \ln N)\\ &\leqslant \exp -uN(1-a_4/c)\end{aligned}$$

for any $a_4 > b$ and N large enough. We take c such that $a_4/c < \alpha$ and this concludes the proof. $\square$

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