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The *HSP*-Classes of Archimedean *l*-groups with Weak Unit

BERNHARD BANASCHEWSKI⁽¹⁾, ANTHONY HAGER⁽²⁾

To Mel Henriksen for his 80th Birthday

ABSTRACT. — W denotes the class of abstract algebras of the title (with homomorphisms preserving unit). The familiar H , S , and P from universal algebra are here meant in W . \mathbb{Z} and \mathbb{R} denote the integers and the reals, with unit 1, *qua* W -objects. V denotes a non-void finite set of positive integers. Let $\mathcal{G} \subseteq W$ be non-void and not $\{\{0\}\}$. We show

(1) $HSP\mathcal{G} = HSP(HS\mathcal{G} \cap S\mathbb{R})$, and

(2) $W \neq \mathcal{G} = HSP\mathcal{G}$ if and only if $\exists V(\mathcal{G} = HSP\{\frac{1}{v}\mathbb{Z} | v \in V\})$.

Our proofs are, for the most part, simple calculations. There is no real use of methods of universal algebra (e.g., free objects), and only one restricted use of representation theory (Yosida). Note that (1) implies the basic fact that $HSP\mathbb{R} = W$ (which can be proved in several ways). Note that (2) contrasts W with $\mathcal{C} =$ archimedean l -groups, and $\mathcal{C} =$ abelian l -groups, where $HSP\mathbb{Z} = \mathcal{C}$ in each case.

RÉSUMÉ. — Soit W la classe des algèbres abstraites décrit dans le titre de l'article (avec les homomorphismes qui préservent l'unité). On utilise dans W la notation H , S , et P , familière de l'algèbre universelle. On note par \mathbb{Z} et les nombres entiers et les nombres réels, avec l'unité 1, *qua* W -objets. On note par V un ensemble fini non-vide de nombres entiers. Soit $\mathcal{G} \subseteq W$ en ensemble non-vide, différent de $\{\{0\}\}$. Nous montrons que

(1) $HSP\mathcal{G} = HSP(HS\mathcal{G} \cap S)$, and

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(2) $W \neq \mathcal{G} = HSP\mathcal{G}$ if and only if $\exists V(\mathcal{G} = HSP\{\frac{1}{v}\mathbb{Z} | v \in V\})$.

Pour la plupart, nos démonstrations sont des calculs simples. On n'utilise pas de méthodes de l'algèbre universelle (par exemple, les objets libres); il n'y a qu'un seul endroit où on utilise, d'une façon très restreinte la théorie de représentations (Yosida). Nous signalons que (1) entraîne le fait basique $HSP = W$ (qui peut être démontré de plusieurs façons) et aussi que (2) souligne la différence entre W et $\mathcal{C} = l$ -groupes archimédien et $\mathcal{C} = l$ -groupes abéliens, où $HSP\mathbb{Z} = \mathcal{C}$ dans chaque cas.

1. Preliminaries

“ l -group” means “lattice-ordered group”, a set with group operation $+$ and compatible lattice order \leq . An “ l - homomorphism” is a group homomorphism which preserves the binary sup \vee and inf \wedge . [D] is a comprehensive reference. Let G be an l -group. G is archimedean if in $G, 0 \leq na \leq b \forall n \in \mathbb{N}$ implies $a = 0$. (This implies commutativity.) A (weak) unit of G is a positive element u for which $a \wedge u = 0$ implies $a = 0$.

W is the class of archimedean l -groups G with a designated unit e_G , and a W -homomorphism $G \xrightarrow{\varphi} H$ is a l -homomorphism with $\varphi(e_G) = e_H$ (analogous to homomorphisms of rings with identity). (Thus, W forms a category.)

Note that $\{0\} \in W$, and $\forall G \in W, G \xrightarrow{0} \{0\}$ is a W -homomorphism; $\{0\}$ is the terminal object of W . Any “ring of continuous functions” $C(X)$, with constant function 1 as unit, is in W , and all ring homomorphisms $C(X) \rightarrow C(X)$ are W -homomorphisms; see [GJ], p. 13. (In fact, conversely too; see [HR], p. 420.) W has products: $G = \prod_I G_i$ is the cartesian product with coordinate-wise $+$ and \leq , and $e_G = (e_{G_i})$. W has subobjects: “ G is a subobject of A ”, written $G \leq A$, means G is a sub- l -group of A and $e_G = e_A$ (the inclusion is a W -homomorphism). W has homomorphic images: For any: $\varphi : G \rightarrow A$ in W , $\text{Im}(\varphi) \in |W|$ and the corestriction $G \rightarrow \text{Im}(\varphi)$ is a surjective W -homomorphism. (The $\ker \varphi$, for $\varphi : G \rightarrow A$ in W , are exactly the convex sub- l -groups I of G for which G/I is archimedean with $e_{G/I}$ a unit in G/I . The requirement that G/I be archimedean is substantive; see [LZ], p. 427.)

Henceforth, W -homomorphisms are called simply “morphisms.” All classes $\mathcal{G} \subseteq W$ are supposed isomorphism-closed, non-void, and not $\{\{0\}\}$. Let $\mathcal{G} \subseteq W$. $P\mathcal{G}$ (respectively, $S\mathcal{G}$; $H\mathcal{G}$) consists of all products (respectively, subobjects; morphic images) of objects from \mathcal{G} (all meant in W). $P\{G\}$ is abbreviated to PG , etc... \mathcal{G} is a P (respectively, S ; H) class if $\mathcal{G} = P\mathcal{G}$ (respectively, $S\mathcal{G}$; $H\mathcal{G}$). Any $HSP\mathcal{G}$ is an H -, S -, P - class, and is called

an *HSP*-class. (Note that $\{\{0\}\}$ is an *HSP*-class. We do not mention this further.)

2. Subobjects of \mathbb{R} Generate

Let $\mathcal{G} \subseteq W$; we exclude \emptyset and $\{\{0\}\}$. Let $\mathcal{K}(\mathcal{G}) = \text{HSG} \cap \text{SR}$. For $G \in W$, let $G^* = \{g \in G \mid \exists n(|g| \leq ne_G)\}$ (the subobject of bounded “functions”, in allusion to Yosida Representation (see [HR]); notation follows [GJ]). For $K \in \text{SR}$, we have $K = K^*$. For any morphism $G \xrightarrow{\varphi} L$, we have $G^* \xrightarrow{\varphi^*} L^*$ given by $\varphi^*(g) = \varphi(g)$ (since $\varphi(e_G) = e_L$, and then $\varphi(G^*) \subseteq L^*$): “ $(\)^*$ is a functor”.

THEOREM 2.1. — $\text{HSPG} = \text{HSPK}(\mathcal{G})$.

Proof. — The containment \supseteq is clear. For \subseteq we claim, and prove below, that for $G \neq \{0\}$,

- (a) $\mathcal{K}(G) = \mathcal{K}(G^*)$,
- (b) $G = G^* \Rightarrow G \in \text{SP}(HG \cap \text{SR}) \quad (\subseteq \text{SPK}(G))$,
- (c) $G \in \text{HS}(G^*)^\omega \quad ((\)^\omega \text{ the countable power})$.

And thus $G \in \text{HSPK}(G)$. \square

Now, for the general $\text{HSPG} \subseteq \text{HSPK}(\mathcal{G})$: Let $A \in \text{HSPG}$. Then $A \in \text{HSG}$ for $G \in \mathcal{P}\mathcal{G}$, so $\mathcal{K}(A) \subseteq \mathcal{K}(G)$, and $\text{HSPK}(A) \subseteq \text{HSPK}(G) \subseteq \text{HSPK}(\mathcal{G})$. By the above, $A \in \text{HSPK}(A)$, so $A \in \text{HSPK}(\mathcal{G})$.

We prove the claims (a), (b), (c).

- (a) $\mathcal{K}(G) \supseteq \mathcal{K}(G^*)$ since $G^* \in \text{SG}$.

$\mathcal{K}(G) \subseteq \mathcal{K}(G^*)$: If $K \in \mathcal{K}(G)$, as $G \geq B \xrightarrow{\varphi} K$, we have $G^* \geq B^* \xrightarrow{\varphi^*} K^* = K$. (φ^* is onto K since φ is onto: if $\varphi(b) = k$, choose $n \in \mathbb{N}$ with $|k| \leq n$, and let $c = (b \wedge ne_G) \vee (-ne_G)$. Then, $c \in B^*$ and $\varphi(c) = k$.)

(b). (This is a version of the classical Yosida Theorem for strong unit [Y]. This is the only place in the paper where the word “ideal” is used (see 5.5, *en passant*). One might also hold the view that this is the only place in the paper where actual Algebra occurs.)

In an abelian *l*-group, “ideal” means “convex sub-*l*-group,” and ideals are kernels of *l*-homomorphisms. The ideal P in G is prime if and only if G/P is

totally ordered; a maximal ideal is prime. If M is a maximal ideal in G , then G/M has no proper ideals, and thus is archimedean (since “archimedean” means “no infinitesimals” and the infinitesimals form an ideal). Hölder’s Theorem: L is archimedean and totally ordered if and only if there is an l -embedding $L \xrightarrow{\varphi} \mathbb{R}$. If $0 \neq r \in \mathbb{R}^+$, then $x \mapsto rx$ defines an l -automorphism of \mathbb{R} . Thus:

If $G = G^*$ in W and M is a proper maximal ideal, then $e_G \notin M$ and we have a W -embedding $\alpha_M : G/M \hookrightarrow \mathbb{R}$, $\alpha_M = a \circ \varphi$, where $G/M \xrightarrow{\varphi} \mathbb{R}$ is any Hölder map and a is the automorphism of \mathbb{R} , $x \mapsto (1/\varphi(e_G + M))x$. So $G/M \in HG \cap S\mathbb{R}$.

If $G = G^*$ in W then, $\forall g \neq 0 \exists M (g \notin M)$ (by Zorn’s lemma; see [HR], p. 415). Thus the “reduced product” $\langle \alpha_M \rangle : G \rightarrow \prod \{G/M \mid M \text{ maximal}\}$ is a W -embedding, and $G \in SP(HG \cap S\mathbb{R})$. (See the early parts of [D] for the basic facts quoted above.)

(c). (This proof uses a sequential convergence in W introduced in [BH], injected into an argument from [HIJ], which originally used pointwise convergence on the Yosida space.)

Fix $G \neq \{0\}$ in W . In the following, $a \leq \epsilon$ in G means $\epsilon = 1/k$ for $k \in \mathbb{N}$ and $ka \leq e_G$; and, for $p \in \mathbb{N}$, $pe_G \in G$ is just called p .

DEFINITION 2.2. — ([BH], 2.2.4) In G : let $r \geq 1$.
 $g_n \xrightarrow{r} g$ means $[\forall \epsilon \forall p \in \mathbb{N} \exists n_0 (n \geq n_0 \Rightarrow (p - r)^+ \wedge |g_n - g| \leq \epsilon)]$ and
 $g_n \rightarrow g$ means $[\exists r (g_n \xrightarrow{r} g)]$.

(Intuition: View G as an l -group of continuous extended real-valued functions on a compact space X , e.g., in its Yosida representation - see [D] or [HR]. Then, $g_n \xrightarrow{r} g$ means, for the functions, g_n converges uniformly to g on each subset of X on which r is bounded - or, on each compact subset of $r^{-1}(\mathbb{R})$.)

The convergence has the following basic features gleaned from [BH].

- (i) $g_n \rightarrow g$ and $g_n \rightarrow g' \Rightarrow g' = g$.
- (ii) $g_n \rightarrow g$ and $h_n \rightarrow h \Rightarrow g_n \otimes h_n \rightarrow g \otimes h$ for $\otimes = +, -, \vee, \wedge$.
- (iii) For each g , $(g \wedge n) \vee (-n) \rightarrow g$ (indeed, \xrightarrow{r} for $r = |g| \vee 1$).

(Here, (i) and (ii) are parts of [BH], 2.2.5, (iii) follows from [BH], 2.1.8. The proofs are not difficult but use “archimedean” several times.)

We now show that $G \in HS(G^*)^\omega$. Define $B \subseteq (G^*)^\omega$ as: $b = (b_n) \in B$ means $\exists g \in G (b_n \rightarrow g)$. Given $b \in B$, this g is unique by (i); call it $\lambda(b)$. We have a function $B \xrightarrow{\lambda} G$. By (ii), $B \leq (G^*)^\omega$ (i.e., B is a W -subobject) and λ is a morphism. By (iii), λ is onto. \square

COROLLARY 2.3. — $W = HSP\mathbb{R}$.

Proof. — By 2.1 any $G \in HSPK(G)$, while $K(G) \subseteq HSP\mathbb{R}$. \square

There are other quite different proofs of 2.3. See 5.1.

3. Generating \mathbb{R} from $S\mathbb{R}$.

Let K be a subgroup of \mathbb{R} . It is elementary that : Topologically, K is either dense or discrete, in the latter case cyclic. If $1 \in K$ (i.e., $K \leq \mathbb{R}$), then K is either dense or of the form $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N} - \{0\}$.

In the proof of 3.1 (3) below and later, we use the further easy observations: $\frac{1}{m}\mathbb{Z} \leq \frac{1}{n}\mathbb{Z}$ if and only if $m|n$; $S_n^1\mathbb{Z} = \{\frac{1}{v}\mathbb{Z} : v \in \text{Div } n\}$. ($m|n$ is “ m divides n ”. $\text{Div } n$ is the set of divisors of n .)

Consider finite $V \subseteq \mathbb{N} - \{0\}$, put $\mathcal{K}_V = \{\frac{1}{v}\mathbb{Z} : v \in V\} (\subseteq S\mathbb{R})$ and $B(V) = \Pi\mathcal{K}_V (\in W)$. Any finite $\mathcal{K} \subseteq S\mathbb{R}$ consisting of discrete subgroups is a \mathcal{K}_V . Note that $SK_V = \mathcal{K}_{\text{Div } V}$ ($\text{Div } V = \cup\{\text{Div } v | v \in V\}$).

The following does not use 2.1. (Of course though, knowing [2.3: $HSP\mathbb{R} = W$], the point is partly which $\mathcal{K} \subseteq S\mathbb{R}$ do / do not have $HSPK$ proper; we consider this explicitly in the next section.)

THEOREM 3.1. — *Let $K \in S\mathbb{R}$ be dense. Let $\mathcal{K} \subseteq S\mathbb{R}$ consist of discrete subgroups.*

- (1) $\mathbb{R} \in HS(K^w)$
- (2) *If \mathcal{K} is infinite, then $\mathbb{R} \in HS(\prod \mathcal{K})$.*
- (3) *If \mathcal{K} is finite, then $\mathcal{K} = \mathcal{K}_V$ for some V , $(HSPK) \cap S\mathbb{R} = SK$, and $\mathbb{R} \notin HSPK$.*

Proof. — For (1) and (2), consider $\{K_n\}_{\mathbb{N}} \subseteq S\mathbb{R}$ and $P \equiv \prod_{\mathbb{N}} K_n$. For (1), we take $K_n = K \forall n$, and for (2), we take $k_0 < k_1 < \dots$ and $K_n = \frac{1}{k_n}\mathbb{Z}$. Then let $A \equiv \{a \in P \mid \text{The sequence } (a(n)) \text{ converges in } \mathbb{R}\}$, and define $\lambda : A \rightarrow \mathbb{R}$ by $\lambda(a) = \lim a(n)$. Then $A \in W$ and λ is a morphism, as is

checked easily. We claim λ is onto in either case : Let $\gamma \in \mathbb{R}$. For (1), $\forall n$ choose $a(n) \in K$ with $|a(n) - \gamma| \leq 1/n$, by density. For (2), $\forall n$ choose the least m with $\gamma - 1/k_n \leq m/k_n$, then $m/k_n \leq \gamma + 1/k_n$; define $a(n) = m/k_n$.

(3) We noted above that $\mathcal{K} = \mathcal{K}_V$ for some V and $S\mathcal{K} = \mathcal{K}_{\text{Div } V}$. Clearly $\mathbb{R} \notin \mathcal{K}_{\text{Div } V}$. We show $HSP\mathcal{K}_V \cap S\mathbb{R} = S\mathcal{K}_V$. The inclusion \supseteq is clear. For \subseteq :

$HSP\mathcal{K}_V = HSPB(V)$ of course, so consider $B(V)^\Gamma \geq A \xrightarrow{\varphi} \mathbb{R}$.

We are to show $\varphi(A) \subseteq \text{some } \frac{1}{v}\mathbb{Z}$. Now $\mathbb{R}^V \geq \{f \in \mathbb{R}^V \mid f(v) \in \frac{1}{v}\mathbb{Z} \forall v\} = B(V)$. Let $X = V \times \Gamma = \dot{\bigcup}_v \pi^{-1}(v)$ (π the first projection, $\dot{\bigcup}$ denoting disjoint union). So

$$\mathbb{R}^X \geq \{f \in \mathbb{R}^X \mid f(v, \gamma) \in \frac{1}{v}\mathbb{Z} \forall (v, \gamma)\} = B(V)^\Gamma$$

and

$$f \in B(V)^\Gamma \Rightarrow f(\pi^{-1}(v)) \subseteq \frac{1}{v}\mathbb{Z} \forall v,$$

and if f is bounded, then $f(\pi^{-1}(v))$ is finite, thus closed. Now

$$(\mathbb{R}^X)^* \geq (B(V)^\Gamma)^* \geq A^* \xrightarrow{\varphi^*} \mathbb{R},$$

with $\varphi^*(A^*) = \varphi(A)$ (by properties of $()^*$ as in (a) of the proof of 2.1).

Simplify this notation: $B = A^*, \psi = \varphi^*$, so $(\mathbb{R}^X)^* \geq B \xrightarrow{\psi} \mathbb{R}$. For $f \in \mathbb{R}^X$, let $Z(f) = \{x \mid f(x) = 0\}$. Note that $Z(b) = \emptyset$ implies b is bounded away from 0 (since V is finite), so $1 \leq n|b|$ for some $n \in \mathbb{N}$, thus $\psi(b) \neq 0$. Therefore, $\mathcal{Z}(\psi) = \{Z(b) \mid b \in \ker \psi\}$ is a proper filter base on X , so there is an ultrafilter $\mathcal{U} \supseteq \mathcal{Z}(\psi)$, and there is a unique v with $\pi^{-1}(v) \in \mathcal{U}$ (since $X = \dot{\bigcup}_v \pi^{-1}(v)$).

Whenever \mathcal{F} is an ultrafilter on a set Y , for each $f \in (\mathbb{R}^Y)^*$, “ f converges along \mathcal{F} ”, i.e., $\cap \{\overline{f(F)} \mid F \in \mathcal{F}\}$ is a singleton, whose element we denote $e_{\mathcal{F}}(f)$. This defines a function $e_{\mathcal{F}} : (\mathbb{R}^Y)^* \rightarrow \mathbb{R}$, which is easily seen to be a W -morphism with $\ker e_{\mathcal{F}} \supseteq \{f \mid \mathcal{Z}(f) \in \mathcal{F}\}$. (These assertions can be verified directly, but their truth is transparent upon recognizing $e_{\mathcal{F}}(f)$ as $\beta f(\mathcal{F})$, which means: Discrete Y has its Čech-Stone compactification βY , of which \mathcal{F} “is” a point, and each $f \in (\mathbb{R}^Y)^*$ has its extension $\beta f \in C(\beta Y)$. Then $\beta f(\mathcal{F}) = e_{\mathcal{F}}(f)$. See [GJ], 3.17, 4.12, 6.24.)

Applying this to $(Y, \mathcal{F}) = (X, \mathcal{U})$ with the $\pi^{-1}(v) \in \mathcal{U} \supseteq \mathcal{Z}(\psi)$, we see that $\ker e_{\mathcal{U}} \supseteq \ker \psi$, and that for $b \in B$ (or even $(B(V)^{\Gamma})^*$), $b(\pi^{-1}(v)) = \overline{b(\pi^{-1}(v))} \supseteq \{e_{\mathcal{U}}(b)\}$ so $e_{\mathcal{U}}(b) \in \frac{1}{v}\mathbb{Z}$.

Thus we have the commuting square in W , where α is defined by $\alpha(\psi(b)) = e_{\mathcal{U}}(b)$.

$$\begin{array}{ccc} B & \xrightarrow{\leq} & (\mathbb{R}^X)^* \\ \downarrow \psi & & \downarrow e_{\mathcal{U}} \\ \psi(B) & \xrightarrow{\alpha} & \mathbb{R} \end{array}$$

However, α is just the inclusion $\psi(B) \hookrightarrow \mathbb{R}$ (by Hion's Theorem, [D], p. 147), so $\psi(B) = e_{\mathcal{U}}(B) \subseteq \frac{1}{v}\mathbb{Z}$. \square

Regarding 3.1(1): J. Madden first pointed out to us that $\mathbb{R} \in HSP\mathbb{Q}$, using a representation of free objects in *HSPQ*. The referee points out that the corresponding fact in *f*-rings is noted in [HI], p. 541-542, and that [W], p. III-44 contains “the sequence proof” of a statement like 3.1(1) (we don't know the context; we have not seen [W]).

4. The *HSP*-classes

We combine sections 2 and 3 as follows. 4.1: 2.3 says $HSP\mathcal{G} = W$ if and only if $\mathbb{R} \in HSP\mathcal{G}$; we take a closer look at that using 3.1. 4.2: 3.1 and 2.3 show that the proper *HSP*-classes are the $HSPK_V$; we interpret “ $G \in HSPK_V$.”

PROPOSITION 4.1. — *For $\mathcal{G} \subseteq W$, the following are equivalent.*

- (1) $\mathbb{R} \in HSP\mathcal{G}$.
- (2) $\mathcal{K}(\mathcal{G})$ contains either a dense member or infinitely many discrete members of $S\mathbb{R}$.
- (3) There is $\{G_n | n \in \mathbb{N}\} \subseteq \mathcal{G}$ for which $\mathbb{R} \in HS(\Pi G_n)$.

Proof. — (3) \Rightarrow (1) is obvious.

(1) \Rightarrow (2). If (2) fails, then 3.1 (3) says $\mathbb{R} \notin HSPK(\mathcal{G})$. But 2.1 says $HSP\mathcal{G} \subseteq HSPK(\mathcal{G})$.

(2) \Rightarrow (3) Let $K \in HSG \cap S\mathbb{R}$ be dense. We have $\mathcal{G} \ni G \geq A \rightarrow K$. Now, $\mathbb{R} \in HS(K^\omega)$, by 3.1 (1), as $K^\omega \geq B \rightarrow \mathbb{R}$. Combining these, $G^\omega \geq A^\omega \rightarrow K^\omega \geq B \rightarrow \mathbb{R}$. So $\mathbb{R} \in H(SH)S(G^\omega) \subseteq H(HS)S(G^\omega) = HS(G^\omega)$ (since $SH \leq HS$).

Let $\{K_n\}_{\mathbb{N}} \subseteq HSG \cap S\mathbb{R}$ be distinct, discrete. We have $\forall n G_n \geq A_n \rightarrow K_n$. Now, $\mathbb{R} \in HS(\Pi K_n)$, by 3.2 (2), as $\Pi K_n \geq B \rightarrow \mathbb{R}$. Combining these, $\Pi G_n \geq \Pi A_n \rightarrow \Pi K_n \geq B \rightarrow \mathbb{R}$, so $\mathbb{R} \in \dots HS(\Pi G_n)$ as previously. \square

We consider the $HSPK_V$'s (the proper HSP -classes). We noted in and before 3.1 that $\mathcal{K}(\mathcal{K}_V) \equiv HSPK_V \cap S\mathbb{R} = SK_V = \mathcal{K}_{\text{Div } V}$, so using 2.1, $HSPK_V = HSPK_{\text{Div } V}$. A little further thought shows $HSPK_U \subseteq HSPK_V$ if and only if $SK_U \subseteq SK_V$ if and only if $U \subseteq \text{Div } V$. In particular, $\frac{1}{6}\mathbb{Z} \notin HSP\{\frac{1}{2}\mathbb{Z}, \frac{1}{3}\mathbb{Z}\}$.

For $G \in W$, let $V(G) = \{n|\frac{1}{n}\mathbb{Z} \in \mathcal{K}(G)\}$. This may be \emptyset , or infinite (in which case our reserving “ V ” for finite sets is abused - continuing the possibility of such abuse) $\mathcal{K}_{V(G)} = \{\frac{1}{n}\mathbb{Z}|n \in V(G)\} \subseteq \mathcal{K}(G)$, with equality if and only if all members of $\mathcal{K}(G)$ are discrete.

PROPOSITION 4.2. — *For finite $V \subseteq \mathbb{N} - \{0\}$, and $G \in W$, these are equivalent.*

- (1) $G \in HSPK_V$.
- (2) $\mathcal{K}(G) \subseteq \mathcal{K}_{\text{Div } V}$.
- (3) $\mathcal{K}(G) = \mathcal{K}_{V(G)}$ and $V(G) \subseteq \text{Div } V$.

Proof. — (1) \Leftrightarrow (2). $G \in HSPK_V$ if and only if $HSPG \subseteq HSPK_V$ if and only if $\mathcal{K}(G) \subseteq \mathcal{K}(HSPK_V) = \mathcal{K}_{\text{Div } V}$ using 2.1 and 3.1.

(2) \Rightarrow (3). If $\mathcal{K}(G) \subseteq \mathcal{K}_{\text{Div } V}$, then all members of $\mathcal{K}(G)$ are discrete. So $\mathcal{K}_{V(G)} = \mathcal{K}(G) \subseteq \mathcal{K}_{\text{Div } V}$ and $V(G) \subseteq \text{Div } V$.

(3) \Rightarrow (1). $\mathcal{K}(G) = \mathcal{K}_{V(G)}$ implies $G \in HSPK_{V(G)}$ (by 2.1), and $V(G) \subseteq \text{Div } V$ implies $HSPK_{V(G)} \subseteq HSPK_V$ (above). \square

Let us be more explicit about the association $V \mapsto HSPK_V$.

Let \mathbb{D} be the collection of all finite $V \subseteq \mathbb{N} - \{0\}$ for which $V = \text{Div } V$, together with $\mathbb{N} - \{0\}$, partially ordered by inclusion. Let \mathbb{C} be the collection of all HSP -classes in W , partially ordered by inclusion. Both \mathbb{D} and \mathbb{C} are complete lattices in which inf is intersection, and the sup of any infinite subset is the top. Note $\emptyset \in \mathbb{D}$, and $\mathcal{K}_\emptyset = \{\{0\}\} \in \mathbb{C}$.

COROLLARY 4.3. — \mathbb{D} and \mathbb{C} are lattice-isomorphic by $V \mapsto HSPK_V$.

See 5.6 below for a discussion of further ramifications of 2.1, 4.2, and 4.3.

5. Assorted Remarks

5.1. $W = HSP\mathbb{R}$. This fact (2.3) can be obtained in other ways:

[H], 7.1 uses full Yosida Representation to show that the free W -object on the set I is the l -subgroup of $C(\mathbb{R}^I)$ generated by 1 and the projections $\pi_i : \mathbb{R}^I \rightarrow \mathbb{R}$. 2.3 follows. J. Madden pointed out (see [H]) a more general algebraic route to this description of the free objects.

Another route to 2.3 is this. (a) If A is countable, then A embeds in some $C(X)$. From Yosida Representation and the Baire Category Theorem, we can use $X = \bigcap_A a^{-1}\mathbb{R}$. On the other hand, there is a “point-free” argument using as X the frame spectrum of the frame of W -kernels; it is shown that the A -uniformity on the latter is complete [B1], with a countable base, thus spatial [I]. (b) In a *SP*-class of abstract algebras, if $\mathcal{D} \subseteq \mathcal{E}$ are *HSP*-classes, and each finitely generated member of \mathcal{E} belongs to \mathcal{D} , then $\mathcal{D} = \mathcal{E}$. (See [P], p. 134.) 2.3 follows from (a) and (b).

5.3. $\mathcal{K}(HSP\mathcal{G})$. The question “ $\mathcal{K}(HSP\mathcal{G}) = \mathcal{K}(\mathcal{G})$?” is invited by 2.1. This is so for $\mathcal{G} = \mathcal{K}_V$ (by 3.1(3) and the definition of \mathcal{K}), and of course it is so if $\mathcal{K}(\mathcal{G}) = S\mathbb{R}$. In all other cases, it is Not so, by 3.1. For example, $G = \mathbb{Z} + \sqrt{2}\mathbb{Z}$ is dense, and $\frac{1}{n}\mathbb{Z} \in \mathcal{K}(G)$ if and only if $n = 1$.

Related to that, the question “ $\mathcal{K}_V \subseteq HSP\mathcal{G}$?” is invited as a “dual” to 4.2. It is not hard to see that this is so if and only if $\mathcal{K}(\mathcal{G}) = S\mathbb{R}$ or $\mathcal{K}(\mathcal{G})$ is an SK_U with $V \subseteq \text{Div } U$.

5.4. *HSPZ*. In W , this is the least *HSP*-class (besides $\{\{0\}\}$). The members, and the category, are examined in [HM].

Consider the class *lAb* of lattice-ordered abelian groups, and now H, S, P meant in *lAb*. In [D], § 52, it is shown that for each set I , a certain subobject $F(I)$ of \mathbb{Z}^I is the free *lAb*-object on I . Thus $HSP\mathbb{Z} = lAb$, contrasting with W .

Consider the class Arch of archimedean *l*-groups (no designated unit; perhaps no units at all), and now H, S, P meant in Arch. The above $F(I) \in SP\mathbb{Z}$ in Arch (the *lAb* S and P preserve Arch, thus applied in Arch are the Arch S and P ; not so for H). So $HSP\mathbb{Z} = \text{Arch}$, contrasting with W .

5.5. Subdirectly irreducible. Let \mathbb{A} be, say, an SP -class of abelian groups. In \mathbb{A} ; a subdirect representation of A is an embedding $A \subseteq \prod A_i$ with $\pi_i(A) = A_i$ for each i . A is subdirectly irreducible if each subdirect representation $A \subseteq \prod A_i$ has some $\pi_i|_A$ one-to-one, equivalently, A has a smallest \mathbb{A} -ideal $\neq (0)$. Let $si\mathbb{A} = \{A \mid A \text{ is subdirectly irreducible}\}$.

If \mathbb{A} is a variety, then (Birkoff; See [P]) : $A \in SP(HA \cap si\mathbb{A}) \forall A; \mathbb{A} = HSP(si\mathbb{A})$. For W , $siW = S\mathbb{R}$ (proof below). Thus, by 2.1, $W = HSP(siW)$ (like a variety), but many $A \notin SP(HA \cap S\mathbb{R})$ (unlike a variety). (Those $A \in SP(HA \cap S\mathbb{R})$) may be called real-semi-simple. Some A have $HA \cap S\mathbb{R} = \emptyset$, e.g., Lebesgue measurable functions on $[0, 1]$ mod null functions; also the real-valued functions on any countably generated Boolean frame [B2].)

($siW = S\mathbb{R}$: If $K \in S\mathbb{R}$, the only ideals are (0) and K , so $K \in siW$. Conversely, let $G \in siW$. Then, G is totally ordered (because any $a^{\perp\perp}$ is a W -ideal and if G is not totally ordered, there are disjoint $a, b > 0$, thus $a^{\perp\perp} \cap b^{\perp\perp} = (0)$. Thus there is no smallest W -ideal). Now $G \in S\mathbb{R}$, by Hölder's Theorem.)

5.6. With strong unit. For $G \in W$, if $G = G^*$, the unit e_G is called “strong”. For $\mathcal{G} \subseteq W$ let $\mathcal{G}^* = \{G \in \mathcal{G} \mid G = G^*\}$. [HK] studies in some detail the class of l -groups called “uniformly hyperarchimedean”, which class turns out to be $\cup\{(HSPK_V)^* \mid \text{finite } V \subseteq \mathbb{N} - \{0\}\}$. We sketch the connection. (Present notation differs from [HK].)

As a class of algebras, W^* has its H and S inherited from W , but its P comes from the W^* -product $(\prod G_i)^*$, \prod being the W - product.

For brevity, we use generic notation \mathcal{G} for a subclass of W , \mathcal{S} for a subclass of W^* . As usual, V is a finite subset of $\mathbb{N} - \{0\}$. 2.1 and 3.1 (and remarks in section 4) easily yield the following.

- (a) The HSP -classes of W and W^* are in bijective correspondence via $\mathcal{G} \mapsto \mathcal{G}^*$; $HSPS \leftarrow \mathcal{S}(HSP \text{ in } W)$. In W^* , $HSP\mathbb{R} = W^*$. \mathcal{S} is proper HSP in W^* if and only if $\exists V(\mathcal{S} = HSPK_V \text{ in } W^*)$.

Connecting with [HK]: An l -group is called hyperarchimedean if each of its l -group quotients is archimedean. In such, any weak unit is strong. HA denotes $\{G \in W \mid G \text{ is hyperarchimedean}\} = \{G \in W^* \mid G \text{ is hyperarchimedean}\}$. \mathcal{L}_1 is the class of abelian l -groups with strong unit and unit-preserving homomorphisms; \mathcal{L}_1 has its H , S , and P (the product being the l -ideal in $\prod G_i$ generated by (e_i) , e_i being the strong unit of G_i and $\prod G_i$ the l -group product). Note $W^* = \mathcal{L}_1 \cap W$.

- (b) ([HK],4.4(a)) The following are equivalent. \mathcal{S} is *HSP* in \mathcal{L}_1 and $\mathcal{S} \subseteq HA$; \mathcal{S} is proper *HSP* in W^* (i.e., (a) above); $\exists V(\mathcal{S} = HSPK_V$ in \mathcal{L}_1).

The most interesting part of the proof of (b) is : $S \in W^* - HA \Rightarrow \mathbb{R} \in HSPS$ in W^* . The present 3.1(2) is a version of that.

Also note that the present 4.2 is the correspondent in W of ([HK], 4.4(b)).

Finally, we record briefly the connection with *MV*-algebras. This is discussed further in [HK]. The theory of *MV*-algebras is exposed in [COM].

There is the Chang-Mundici categorical equivalence between \mathcal{L}_1 and *MV*, the latter being a variety, in consequence of which the *HSP* classes in \mathcal{L}_1 and the subvarieties of *MV* correspond. The latter have been completely classified (with equations) by Komori. The correspondent of the $\mathcal{S} = HSPK_V$ in (b) above is called $\mathcal{C}(\{L_v\}_V)$ in [COM], p. 169.

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