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INTERPOLATION BY BOUNDED FUNCTIONS

par **W. HAYMAN.**

1. Let D be a domain in the plane or more generally a Riemann surface, which admits bounded analytic functions. In a recent lecture R. C. Buck raised the following problem. Do there exist infinite sequences z_n in D , such that an arbitrary bounded sequence ω_n can be interpolated at z_n by a function $f(z)$ regular and bounded in D , and if so does every sequence z_n , which approaches the boundary of D sufficiently rapidly have this property? Although the existence and uniqueness problem for fixed sequences ω_n and z_n has been extensively treated by Pick, Schur, Grunsky, Carathéodory, Denjoy, Nevanlinna and others⁽¹⁾, Buck's questions does not seem answerable by the classical methods.

We shall in this paper supply an affirmative answer to both problems in case D is the unit circle. A sequence $z_n, n = 1, 2, \dots$ will be called a *universal interpolation sequence*, (u.i.s.) if

$$|z_n| < 1, \quad n = 1, 2, \dots$$

and given any complex sequence ω_n satisfying

$$|\omega_n| \leq 1, \quad n = 1, 2, \dots$$

we can find $f(z)$ regular and bounded in $|z| < 1$ and such that

$$f(z_n) = \omega_n. \quad (1. 1)$$

⁽¹⁾ See e. g. R. Nevanlinna, Über beschränkte analytische Funktionen, *Annales Acad. Sci. Fenn.* 32, nr. 7 (1929), for a good account of the problem.

The conditions evidently imply that the z_n are distinct and have no limit point in $|z| < 1$. We write

$$r_{m,n} = \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right|.$$

We shall denote by C, C_1, C_2, \dots positive constants independent of m, n not necessarily the same each time. The letter A will denote positive absolute constants and $A(\epsilon)$ constants depending only on ϵ . Our main result can now be stated as follows.

THEOREM 1. — *A necessary condition for a sequence z_n to be a u.i.s. is that*

$$\prod_n = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} r_{m,n} \geq C_1, \quad \text{all } n. \quad (1.2)$$

A sufficient condition is that there exists $\lambda < 1$ and $C_2 > 0$ so that

$$\prod_n(\lambda) = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} [1 - (1 - r_{m,n})^\lambda] \geq C_2, \quad \text{all } n. \quad (1.3)$$

We note that (1.3) reduces to (1.2) if we put $\lambda = 1$. Thus the necessary and sufficient conditions are not too far apart. It seems quite possible that (1.2) is in fact sufficient as well as necessary, but I have been unable to prove this.

From Theorem 1 we shall be able to deduce

THEOREM 2. — *A sufficient condition for a sequence of distinct numbers z_n in $|z| < 1$ to be a u.i.s. is that*

$$\lim_{n \rightarrow \infty} \frac{1 - |z_{n+1}|}{1 - |z_n|} < 1. \quad (1.4)$$

If z_n is positive increasing, the condition is also necessary.

2. PROOF OF THEOREM 1, NECESSITY. — Suppose that z_n is a u.i.s. and that (1.2) is false. Then we can find an increasing sequence of integers $n_p, p = 1, 2, \dots$, such that

$$\prod_{n_p} \rightarrow 0, \quad \text{as } p \rightarrow \infty. \quad (2.1)$$

Since $\{z_n\}$ is a u.i.s. $\{z_n\}$ has no limit point in $|z| < 1$ and so

$$r_{m,n} \rightarrow 1, \quad \text{as } m \rightarrow \infty \text{ for fixed } n.$$

By choosing a subsequence of our sequence n_p if necessary, we may therefore suppose in addition to (2. 1) that, given n_1, n_2, \dots, n_{p-1} ; n_p is chosen so large that

$$r_{n_p, n_k} > \exp [-2^{-(p-k)}], \quad k = 1, 2, \dots, p-1.$$

We deduce that

$$\begin{aligned} Q_k &= \prod_{\substack{p=1 \\ p \neq k}}^{\infty} r_{n_p, n_k} > \exp \left[- \left(\sum_{\substack{p=1 \\ p \neq k}}^{\infty} 2^{-|p-k|} \right) \right] \\ &> \exp \left[-2 \sum_{t=1}^{\infty} 2^{-t} \right] = e^{-2}. \end{aligned} \tag{2. 2}$$

Suppose then that our sequence n_k satisfies (2. 1) and (2. 2). We choose ω_n so that

$$\begin{aligned} \omega_{n_p} &= 1, & p &= 1, 2, \dots, \\ \omega_n &= 0, & \text{if } n &\neq n_p \text{ for any } p, \end{aligned}$$

and suppose that there exists $f(z)$ regular in $|z| < 1$ and satisfying (1. 1) and $|f(z)| < M$ there. Let N be a positive integer and set

$$\varphi(z) = f(z) \prod_{n=1}^N \left| \frac{1 - \bar{z}_n z}{z_n - z} \right|$$

where the prime denotes a product over integers not belonging to the sequence n_p . Then $\varphi(z)$ is regular in $|z| < 1$ and

$$\overline{\lim}_{|z| \rightarrow 1} |\varphi(z)| \leq M.$$

Thus the maximum modulus principle gives $|\varphi(z)| \leq M$ in $|z| < 1$, and so

$$|f(z)| \leq M \prod_{n=1}^N \left| \frac{z - z_n}{1 - \bar{z}_n z} \right|$$

Setting $z = z_{n_k}$ for a fixed k and making $N \rightarrow \infty$ we deduce

$$1 \leq M \prod_{n=1}^{\infty} r_{n, n_k} = M \frac{\Pi_{n_k}}{Q_k} \leq M e^2 \Pi_{n_k}.$$

This contradicts (2. 1) and so proves the necessity part of Theorem 1.

3. PROOF OF THEOREM 1, SUFFICIENCY. — Let z_n be a sequence of points in $|z| < 1$ satisfying (1. 3), or more gene-

rally (1. 2) and suppose that we can find a sequence of functions $f_n(z)$ regular in $|z| < 1$ and satisfying

$$|f_n(z_n)| \geq C', \quad \text{all } n \quad (3. 1)$$

and

$$\sum_{n=1}^{\infty} |f_n(z)| \leq C'', \quad |z| < 1. \quad (3. 2)$$

We write

$$g_n(z) = f_n(z) \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left\{ \frac{z_m - z}{1 - \bar{z}_m z} \cdot \frac{\bar{z}_m}{|z_m|} \right\}. \quad (3. 3)$$

Then the condition (1. 2) implies that $g_n(z)$ is regular in $|z| < 1$,

$$g_n(z_m) = 0, \quad m \neq n,$$

and

$$|g_n(z_n)| = |f_n(z_n)| \Pi_n \geq C.$$

We now put

$$h_n(z) = \frac{g_n(z)}{g_n(z_n)}. \quad (3. 4)$$

Then we have for $|z| < 1$

$$|h_n(z)| \leq \frac{|g_n(z)|}{C} \leq \frac{|f_n(z)|}{C},$$

and so by (3. 2)

$$\sum_{n=1}^{\infty} |h_n(z)| \leq \frac{C''}{C'}, \quad |z| < 1. \quad (3. 5)$$

Also by (3. 3) and (3. 4) we have

$$h_n(z_n) = 1, \quad h_n(z_m) = 0, \quad n \neq m. \quad (3. 6)$$

Thus if ω_n is any bounded sequence we set

$$f(z) = \sum_{n=1}^{\infty} \omega_n h_n(z).$$

It now follows from (3. 6) that $f(z)$ satisfies (1. 1) and from (3. 5) that $f(z)$ is bounded in $|z| < 1$.

In order to complete the proof of Theorem 1 it therefore remains only to construct the sequence $f_n(z)$ satisfying (3. 1) and (3. 2), given a sequence z_n satisfying (1. 3) and this we proceed to do.

3.1. In order to construct our sequence $f_n(z)$ we shall construct functions $U_n(z)$ positive and harmonic in $|z| < 1$ and such that for some positive ε

$$U_n(z_n) \leq C, \tag{3.7}$$

$$\max\{U_n(z), U_m(z)\} \geq (1 - r_{m,n})^{-\varepsilon}, \quad m \neq n, \quad |z| < 1. \tag{3.8}$$

We then define $f_n(z)$ by the equation

$$|f_n(z)| = e^{-U_n(z)}.$$

Then (3.7) shows that (3.1) holds. Also (3.8) shows that

$$\min\{|f_m(z)|, |f_n(z)|\} \leq \exp(1 - r_{m,n})^{-\varepsilon}, \quad |z| < 1, \quad m \neq n.$$

For any z in $|z| < 1$ let

$$t(z) = \sup_m |f_m(z)| = f_M(z),$$

say. Then if

$$\exp[-(1 - r_{M,n})^{-\varepsilon}] < t(z), \tag{3.9}$$

we have

$$|f_n(z)| \leq \exp[-(1 - r_{M,n})^{-\varepsilon}]. \tag{3.10}$$

Now if $N = N(r)$ is the total number of indices n for which $r_{M,n} \leq r$ it follows from (1.3) that

$$[1 - (1 - r)^\lambda]^N \geq C,$$

and hence

$$N(r) \leq C(1 - r)^{-\lambda}.$$

We choose r so that

$$\exp[-(1 - r)^{-\varepsilon}] = t(z), \quad (1 - r)^{-\varepsilon} = \log[1/t(z)].$$

Thus in this case

$$N \leq C \{\log[1/t(z)]\}^{\lambda/\varepsilon}. \tag{3.11}$$

We see that the number N of indices n for which (3.9) is false satisfies (3.11) for any z in $|z| < 1$. For all other values of n we have (3.10). Thus

$$\begin{aligned} \sum_{n=1}^{\infty} |f_n(z)| &\leq Nt(z) + \sum_{\substack{n=1 \\ n \neq M}}^{\infty} \exp[-(1 - r_{M,n})^{-\varepsilon}] \\ &\leq Ct(z) \{\log[1/t(z)]\}^{\lambda/\varepsilon} + A(\varepsilon) \sum_{\substack{n=1 \\ n \neq M}}^{\infty} (1 - r_{M,n}) \leq C, \end{aligned}$$

in view of (1. 3). This yields (3. 2). Thus our problem of constructing the regular functions $f_n(z)$ is reduced to the construction of the positive harmonic functions $U_n(z)$ satisfying (3. 7) and (3. 8).

4. CONSTRUCTION OF THE FUNCTIONS $U_n(z)$. — For any pair of points z, z' in the unit circle we set

$$r(z, z') = \left| \frac{z - z'}{1 - \bar{z}z'} \right|.$$

We shall need a number of lemmas.

LEMMA 1. — Given $\varepsilon > 0$ and ρ such that $0 < \rho < 1$, there exists $u(z)$ harmonic and positive in $|z| < 1$ and such that $u(\rho) = 1$,

$$u(z) > \sin\left(\frac{\pi}{2}\varepsilon\right) \left\{ \frac{1+\rho}{1-\rho} \cdot \left| \frac{1-z}{1+z} \right| \right\}^{1-\varepsilon}, \quad |z| < 1.$$

Choose

$$u = \Re \left\{ \frac{1+\rho}{1-\rho} \cdot \frac{1-z}{1+z} \right\}^{1-\varepsilon},$$

and write

$$\frac{1-z}{1+z} = Te^{i\varphi}, \quad u = \left\{ \frac{1+\rho}{1-\rho} T \right\}^{1-\varepsilon} \cos[(1-\varepsilon)\varphi].$$

Then $|\varphi| < \frac{\pi}{2}$ and so

$$\cos[(1-\varepsilon)\varphi] \geq \cos\left[(1-\varepsilon)\frac{\pi}{2}\right] = \sin\left(\frac{\pi}{2}\varepsilon\right).$$

and this proves the Lemma.

We have next

LEMMA 2. — Let D be a subdomain of $|z| < 1$ bounded by an arc of a circle orthogonal to $|z| = 1$ and an arc of $|z| = 1$. Let z_0 be a point of $|z| < 1$ outside D and such that for every z in D we have $r(z, z_0) \geq r_0$.

Then, given $\varepsilon > 0$, we can find $v(z)$ harmonic and positive in $|z| < 1$ and such that $v(z_0) = 1$ and

$$v(z) \geq \sin\left(\frac{\pi}{2}\varepsilon\right) \left(\frac{1+r_0}{1-r_0} \right)^{1-\varepsilon} \text{ in } D.$$

We may suppose without loss in generality that z_0 is the origin and that D is bisected by the positive real axis, since these results may be achieved by a conformal map of $|z| < 1$ onto itself, which leaves $r(z, z_0)$ invariant. It now follows that D is the domain given by

$$\left| \frac{1+z}{1-z} \right| \geq R, \quad \text{where} \quad R \geq \frac{1+r_0}{1-r_0}.$$

We now set

$$\nu(z) = \Re \left(\frac{1+z}{1-z} \right)^{1-\varepsilon},$$

and note as in Lemma 1, that

$$\nu(z) \geq \sin \left(\frac{\pi}{2} \varepsilon \right) \left| \frac{1+z}{1-z} \right|^{1-\varepsilon} \geq \sin \left(\frac{\pi}{2} \varepsilon \right) \left(\frac{1+r_0}{1-r_0} \right)^{1-\varepsilon}$$

for z in D , and this proves the Lemma.

4. 1. In order to make use of Lemmas 1 and 2 in our construction we need some inequalities for $r(z, z')$.

LEMMA 3. — Suppose that z_1, z_2, z_3, z_4 are points in $|z| < 1$ and that $0 < z_2 \leq z_4 < 1$. Suppose further that

$$2 \left| \frac{1+z_1}{1-z_1} \right| \leq \frac{1+z_2}{1-z_2} \leq \left| \frac{1+z_3}{1-z_3} \right|.$$

Then we have

$$1 - r(z_1, z_3) \leq A \frac{1 - r(z_1, z_4)}{1 - r(z_2, z_4)}.$$

Write

$$\begin{aligned} Z_1 &= \frac{1+z_1}{1-z_1} = R_1 e^{i\varphi_1}, & \frac{1+z_2}{1-z_2} &= R_2, \\ Z_3 &= \frac{1+z_3}{1-z_3} = R_3 e^{i\varphi_3}, & \frac{1+z_4}{1-z_4} &= R_4, \end{aligned}$$

where by hypothesis $2R_1 \leq R_2 \leq R_3, R_2 \leq R_4$ and also

$|\varphi_1| < \frac{\pi}{2}, |\varphi_3| < \frac{\pi}{2}$. Then

$$z_1 = \frac{Z_1 - 1}{Z_1 + 1}, \quad z_3 = \frac{Z_3 - 1}{Z_3 + 1},$$

and

$$r(z_1, z_3) = \left| \frac{z_1 - z_3}{1 - \bar{z}_1 z_3} \right| = \left| \frac{Z_1 - Z_3}{\bar{Z}_1 + Z_3} \right|.$$

Also

$$1 - r(z_1, z_3)^2 = \frac{|\bar{Z}_1 + Z_3|^2 - |Z_1 - Z_3|^2}{|\bar{Z}_1 + Z_3|^2} = \frac{4R_1 R_3 \cos \varphi_1 \cos \varphi_3}{|\bar{Z}_1 + Z_3|^2}.$$

Similarly

$$1 - r(z_1, z_2)^2 = \frac{4R_1 R_2 \cos \varphi_1}{|\bar{Z}_1 + R_2|^2},$$

$$1 - r(z_2, z_4)^2 = \frac{4R_2 R_4}{(R_2 + R_4)^2},$$

$$1 - r(z_1, z_4)^2 = \frac{4R_1 R_4 \cos \varphi_1}{|\bar{z}_1 + R_4|^2}.$$

Now we have by hypothesis $2R_1 \leq R_2 \leq R_3$, $R_2 \leq R_4$, and so

$$\frac{1}{4} R_2^2 \leq \frac{1}{4} R_3^2 \leq |\bar{Z}_1 + Z_3|^2,$$

$$R_4^2 \leq (R_2 + R_4)^2,$$

$$|\bar{Z}_1 + R_4|^2 \leq \frac{9}{4} R_4^2.$$

Thus

$$\begin{aligned} \frac{1 - r(z_1, z_4)}{1 - r(z_2, z_4)} &\geq \frac{1}{2} \frac{1 - r(z_1, z_4)^2}{1 - r(z_2, z_4)^2} \geq \frac{4R_1 R_4 \cos \varphi_1}{2 \cdot \frac{9}{4} R_4^2} \cdot \frac{R_4^2}{4R_2 R_4} \\ &= \frac{2}{9} \frac{R_1 \cos \varphi_1}{R_2} \geq \frac{2}{9} \frac{R_1 R_3 \cos \varphi_1}{R_3^2} \geq \frac{1}{18} \frac{R_1 R_3 \cos \varphi_1}{|\bar{Z}_1 + Z_3|^2} \\ &\geq \frac{1}{72} [1 - r(z_1, z_3)^2] \geq \frac{1}{72} [1 - r(z_1, z_3)]. \end{aligned}$$

This proves the Lemma.

4. 2. The key result in our construction is

LEMMA 4. — Suppose that ρ , $u(z)$ are defined as in Lemma 1, that $0 < \lambda < 1$, and $\varepsilon = \frac{1}{4}(1 - \lambda)$, further that $z' = \rho' e^{i\vartheta}$ where $0 \leq \rho' \leq \rho$. Let

$$r = r(z', \rho) = \left| \frac{z' - \rho}{1 - \rho z'} \right|.$$

Then there exists $\nu(z)$, positive and harmonic in $|z| < 1$, and such that $\nu(z') = (1 - r)^\lambda$ and

$$u(z) + \nu(z) \geq A(\varepsilon)(1 - r)^{-\varepsilon}, \quad |z| < 1.$$

We distinguish two cases. Suppose first that

$$2 \left| \frac{1 + z'}{1 - z'} \right| \leq \frac{1 + t}{1 - t}, \tag{4. 1}$$

where

$$(1 - t) = (1 - \rho)(1 - r)^{-2\varepsilon}. \tag{4. 2}$$

Let D be the set given by

$$\left| \frac{1 + z}{1 - z} \right| \geq \frac{1 + t}{1 - t}.$$

Then if z lies in D we have by Lemma 3, with z_1, z_2, z_3, z_4 replaced by z', t, z, ρ

$$\begin{aligned} 1 - r(z, z') &\leq A \frac{1 - r(z', \rho)}{1 - r(t, \rho)} \\ &= \frac{A(1 - r)(1 - t\rho)}{(1 - \rho)(1 + t)} \leq \frac{A(1 - r)(1 - t^2)}{(1 - \rho)(1 + t)} = A(1 - r)^{1 - 2\varepsilon}. \end{aligned}$$

Hence by Lemma 2 we can construct a positive harmonic function $\nu_1(z)$ such that $\nu_1(z') = 1$, and for all z in D

$$\nu_1(z) \geq A(\varepsilon)(1 - r)^{-(1 - \varepsilon)(1 - 2\varepsilon)} \geq A(\varepsilon)(1 - r)^{-1 + 3\varepsilon}.$$

Also outside D we have by Lemma 1 and (4. 2)

$$\begin{aligned} u(z) &\geq A(\varepsilon) \left\{ \frac{1 - t}{1 + t} \cdot \frac{1 + \rho}{1 - \rho} \right\}^{1 - \varepsilon} \geq A(\varepsilon)(1 - r)^{-2\varepsilon(1 - \varepsilon)} \\ &\geq A(\varepsilon)(1 - r)^{-\varepsilon}, \end{aligned} \tag{4, 3}$$

since $\varepsilon \leq \frac{1}{2}$.

Choose now

$$\nu(z) = (1 - r)^\lambda \nu_1(z) = (1 - r)^{1 - 4\varepsilon} \nu_1(z).$$

Then

$$\nu(z') = (1 - r)^\lambda,$$

and in D we have

$$\nu(z) \geq A(\varepsilon)(1 - r)^{\lambda - 1 + 3\varepsilon} = A(\varepsilon)(1 - r)^{-\varepsilon},$$

while outside D (4.3) holds. Thus Lemma 4 is proved in this case.

We next consider the case in which (4.1) is false. Suppose first that

$$1 - \rho' \leq |\varphi| \leq \pi.$$

In this case

$$\begin{aligned} 1 - r^2 &= 1 - r(z', \rho)^2 = \frac{|1 - \rho\rho'e^{i\varphi}|^2 - |\rho'e^{i\varphi} - \rho|^2}{|1 - \rho\rho'e^{i\varphi}|^2} \\ &= \frac{(1 - \rho^2)(1 - \rho'^2)}{(1 - \rho\rho')^2 + 2\rho\rho'(1 - \cos \varphi)} \geq \frac{A(1 - \rho)(1 - \rho')}{\varphi^2}, \end{aligned}$$

while

$$\left| \frac{1 + z'}{1 - z'} \right| = \frac{1 + 2\rho' \cos \varphi + \rho'^2}{(1 - \rho')^2 + 2\rho'(1 - \cos \varphi)} \leq \frac{A}{\varphi^2}.$$

Since (4.1) is false, we deduce

$$\frac{A}{\varphi^2} \geq \frac{A}{(1 - t)^2} = \frac{A(1 - r)^{4\epsilon}}{(1 - \rho)^2} \geq \frac{A}{(1 - \rho)^2} \left[\frac{(1 - \rho)(1 - \rho')}{\varphi^2} \right]^{4\epsilon},$$

$$|\varphi|^{2(1 - 4\epsilon)} \leq A(1 - \rho)^{2 - 4\epsilon} (1 - \rho')^{-4\epsilon} \leq A(1 - \rho')^{2(1 - 4\epsilon)},$$

and so, since $\lambda = 1 - 4\epsilon > 0$,

$$|\varphi| \leq A(\epsilon)(1 - \rho').$$

This inequality thus holds in any case if (4.1) is false. Thus in this case

$$1 - r^2 = \frac{(1 - \rho^2)(1 - \rho'^2)}{(1 - \rho\rho')^2 + 2\rho\rho'(1 - \cos \varphi)} > \frac{A(\epsilon)(1 - \rho)}{(1 - \rho')}. \quad (4.4)$$

We now put

$$\nu(z) = c\Re \left(\frac{1 + z}{1 - z} \right)^{1 - \epsilon} \geq c \sin \left(\frac{\pi\epsilon}{2} \right) \left| \frac{1 + z}{1 - z} \right|^{1 - \epsilon},$$

where c is so chosen that

$$\nu(z') = (1 - r)^\lambda.$$

Then we have for $|z| < 1$

$$\begin{aligned} u(z) + \nu(z) &\geq \sin \left(\frac{\pi}{2} \epsilon \right) \left\{ \left(\frac{1 + \rho}{1 - \rho} \right)^{1 - \epsilon} \left| \frac{1 - z}{1 + z} \right|^{1 - \epsilon} + c \left| \frac{1 + z}{1 - z} \right|^{1 - \epsilon} \right\} \\ &\geq \sin \left(\frac{\pi}{2} \epsilon \right) \left[c \left(\frac{1 + \rho}{1 - \rho} \right)^{1 - \epsilon} \right]^{\frac{1}{2}}. \end{aligned}$$

We have

$$(1-r)^\lambda = \nu(z') \leq c \left| \frac{1+z'}{1-z'} \right|^{1-\epsilon} \leq c \left(\frac{1+|z'|}{1-|z'|} \right)^{1-\epsilon},$$

so that

$$c \geq \frac{1}{2} (1-\rho')^{1-\epsilon} (1-r)^\lambda \geq A(1-\rho')^{1-\epsilon} \left(\frac{1-\rho}{1-\rho'} \right)^{1-4\epsilon},$$

by (4. 4). Thus

$$\begin{aligned} u(z) + \nu(z) &\geq A(\epsilon) [(1-\rho')^{3\epsilon} (1-\rho)^{1-4\epsilon} (1-\rho)^{\epsilon-1}]^{\frac{1}{2}}, \\ &\geq A(\epsilon) \left(\frac{1-\rho'}{1-\rho} \right)^{\frac{3}{2}\epsilon} \geq A(\epsilon) (1-r)^{-\frac{3}{2}\epsilon}, \end{aligned}$$

again by (4. 4), so that Lemma 4 follows also in this case.

5. COMPLETION OF PROOF OF THEOREM 1. — We can now construct our harmonic functions $U_n(z)$ to satisfy (3. 7) and (3. 8). Let $z_n = \rho_n e^{i\theta_n}$ be the members of our sequence and suppose that

$$\rho_n \leq \rho_{n+1}, \quad n = 1, 2, \dots$$

Set

$$V_n(z) = \Re \left\{ \frac{1 + \rho_n}{1 - \rho_n} \cdot \frac{1 - ze^{-i\theta_n}}{1 + ze^{-i\theta_n}} \right\}^{1-\epsilon}.$$

Then after a rotation of the unit circle we can deduce from Lemma 4 that we can, for $m < n$, construct a function $u_{m,n}(z)$, positive and harmonic in $|z| < 1$ and such that

$$u_{m,n}(z_m) = (1 - r_{m,n})^\lambda,$$

and

$$u_{m,n}(z) + V_n(z) \geq A(\epsilon) (1 - r_{m,n})^{-\epsilon}, \quad |z| < 1.$$

Set now

$$U_m(z) = V_m(z) + \sum_{n=m+1}^{\infty} u_{m,n}(z).$$

Then

$$\begin{aligned} U_m(z_m) &= 1 + \sum_{n=m+1}^{\infty} (1 - r_{m,n})^\lambda \\ &\leq 1 - \sum_{n=m+1}^{\infty} \log [1 - (1 - r_{m,n})^\lambda] \leq C \quad (5. 1) \end{aligned}$$

by (1. 3). On the other hand if $m < n$ and $|z| < 1$

$$\begin{aligned} \max \{U_m(z), U_n(z)\} &\geq \frac{1}{2} [U_m(z) + U_n(z)] \\ &\geq \frac{1}{2} [u_{m,n}(z) + V_n(z)] \geq A(\epsilon)(1 - r_{m,n})^{-\epsilon}. \end{aligned} \tag{5. 2}$$

If we write $A(\epsilon)U_n(z)$ instead of $U_n(z)$ in (5. 1), (5. 2) we obtain (3. 7), (3. 8) as required. This completes the proof of Theorem 1.

6. PROOF OF THEOREM 2. — We proceed to deduce Theorem 2 from Theorem 1. We prove first the sufficiency part of Theorem 2. Suppose that $z = \rho e^{i\theta}, z' = \rho' e^{i\theta'}$, where $\rho \leq \rho'$. Then

$$\begin{aligned} 1 - r(z, z')^2 &= \frac{(1 - \rho^2)(1 - \rho'^2)}{(1 - \rho\rho')^2 + 2\rho\rho'[1 - \cos(\theta - \theta')]} \\ &\leq \frac{(1 - \rho^2)(1 - \rho'^2)}{1 - \rho\rho'} = 1 - r(\rho, \rho')^2. \end{aligned}$$

Thus also

$$1 - r(z, z') \leq 1 - r(\rho, \rho') = 1 - \frac{\rho' - \rho}{1 - \rho\rho'} = \frac{(1 - \rho')(1 + \rho)}{1 - \rho\rho'} \leq \frac{2(1 - \rho')}{1 - \rho}.$$

Suppose now that $z_n = \rho_n e^{i\theta_n}$ is the sequence of Theorem 2 and that we have for $n \geq n_0$,

$$1 - |z_{n+1}| < K(1 - |z_n|)$$

where $K < 1$. Then for $n > m \geq n_0$ we have

$$1 - |z_n| \leq K^{n-m}(1 - |z_m|)$$

and hence for $n > n_0, m > n_0$

$$1 - r_{m,n} < 2K^{|n-m|}. \tag{6. 1}$$

Similarly if $n > n_0, m \leq n_0$

$$1 - r_{m,n} \leq 2K^{n-n_0}. \tag{6. 2}$$

Finally since $r_{m,n} \neq 0$, for $m < n \leq n_0$, we have for $m < n \leq n_0$

$$1 - (1 - r_{m,n})^{\frac{1}{2}} \geq C. \tag{6. 3}$$

This inequality remains true for general distinct m, n . In fact if $m \leq n_0 < n$ we have

$$r_{m,n} = \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| \geq \frac{\rho_n - \rho_m}{1 - \rho_n \rho_m} \geq \frac{\rho_{n_0+1} - \rho_{n_0}}{1 - \rho_{n_0} \rho_{n_0+1}} = C,$$

and if $n > m \geq n_0$, we have

$$\begin{aligned} r_{m,n} &\geq \frac{\rho_{m+1} - \rho_m}{1 - \rho_m \rho_{m+1}} = \frac{(1 - \rho_m) - (1 - \rho_{m+1})}{(1 - \rho_m) + \rho_m(1 - \rho_{m+1})} \\ &\geq \frac{(1 - K)(1 - \rho_m)}{2(1 - \rho_m)} = \frac{1 - K}{2}. \end{aligned}$$

Thus (6.3) holds in all cases.

Let now t_0 be the smallest positive integer, such that $2K^{t_0} < \frac{1}{2}$. Suppose first $n \leq n_0 + t_0$. Then

$$\Pi_n\left(\frac{1}{2}\right) = \prod_{\substack{m=1 \\ m \neq n}}^{\infty} \left[1 - (1 - r_{m,n})^{\frac{1}{2}}\right] = \prod_{\substack{m \leq n_0 + 2t_0 \\ m \neq n}} \prod_{m > n_0 + 2t_0} = \Pi' \Pi'',$$

say. Here $\Pi' \geq C$ by (6.3) and by (6.1), (6.2)

$$\Pi'' \geq \prod_{t=t_0+1}^{\infty} \left[1 - (2K^t)^{\frac{1}{2}}\right] = C > \frac{1}{2}.$$

Thus in this case $\Pi_n\left(\frac{1}{2}\right) \geq C$ in (1.3).

Similarly if $n > n_0 + t_0$

$$\begin{aligned} \Pi_n\left(\frac{1}{2}\right) &\geq \prod_{m \leq n_1} \prod_{\substack{1 \leq |m-n| \leq t_0 \\ |n-m| > t_0 \\ m > n_0}} \prod_{\substack{|n-m| > t_0 \\ m > n_0}} \left[1 - (1 - r_{m,n})^{\frac{1}{2}}\right] \\ &\geq C^{n_0} C^{2t_0} \left\{ \prod_{t=t_0+1}^{\infty} \left[1 - (2K^t)^{\frac{1}{2}}\right] \right\}^2 \geq C, \end{aligned}$$

and so (1.3) holds again with $\lambda = \frac{1}{2}$. This completes the sufficiency part of Theorem 2.

To prove necessity if the z_n are all positive, suppose that they are arranged in order of magnitude. Then (1.2) must be satisfied and it follows that

$$\begin{aligned} r_{m,m+1} &= \frac{z_{m+1} - z_m}{1 - z_m z_{m+1}} \geq C > 0, \quad m = 1 \text{ to } \infty, \\ z_{m+1} &\geq \frac{C + z_m}{1 + Cz_m}, \\ (1 - z_{m+1}) &\leq \frac{(1 - C)(1 - z_m)}{1 + Cz_m} \leq (1 - C)(1 - z_m). \end{aligned}$$

Since this holds for all m , we have (1.4). This completes the proof of Theorem 2.

Since receiving the proofs of this paper, Prof. L. Carleson has kindly shown me the proofs of a very elegant paper of his, to be published in the American Journal of Mathematics, in which he proves that the condition (1.2) is sufficient as well as necessary for z_n to be a u.i.s. However his proof is nonconstructive, so that the present paper, in which an interpolations series is actually constructed, may still have some interest.
