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## ON THE AXIOMATIC OF HARMONIC FUNCTIONS II

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In this paper we shall use constantly the notations and definitions from [2].

1. The fine topology of  $X$  is the least fine topology on  $X$ , which is finer than the given topology and with respect to which the superharmonic functions (non necessarily defined on the whole space  $X$ ) are continuous. A set  $V \ni x$  is a fine neighbourhood of  $x$  if and only if either  $x$  is an interior point of  $V$  for the initial topology or there exists a superharmonic function  $s$ , defined on a neighbourhood of  $x$ , such that

$$s(x) < \liminf_{X-V \ni y \rightarrow x} s(y).$$

**THEOREM 1.** — *Any point of  $X$  possesses a fundamental system of fine neighbourhoods which are compact and connected in the initial topology of  $X$ .*

Since the theorem has a local character we may suppose that there exists a harmonic positive function and a positive potential on  $X$ . Dividing the sheaf of harmonic functions by a positive harmonic function the fine topology does not change; we may suppose therefore that the constants are harmonic functions.

Let  $x \in X$  and  $V$  be a fine neighbourhood of  $x$ . It is sufficient to suppose that  $x$  is not an interior point of  $V$ . There exists then a superharmonic function  $s$  on  $X$  such that

$$s(x) < \liminf_{X-V \ni y \rightarrow x} s(y).$$

Let  $\alpha$  be a real number

$$s(x) < \alpha < \liminf_{X-V \ni y \rightarrow x} s(y)$$

and  $U$  be a regular domain containing  $x$  such that  $s$  is greater than  $\alpha$  on  $\bar{U} - V$ . We denote by  $F$  the set

$$F = \{y \in \bar{U} | s(y) \leq \alpha\}$$

and by  $K$  the component of  $F$  containing  $x$ . It is sufficient to prove that  $K$  is a fine neighbourhood of  $x$ .

Let  $C$  be a component of  $F$  contained in  $U$ . There exists then an open set  $G$ ,  $C \subset G \subset \bar{G} \subset U$ ,  $F \cap \partial G = \emptyset$ . Since  $s > \alpha$  on  $\partial G$  it follows  $s > \alpha$  on  $G$  which contradicts the inequality  $s \leq \alpha$  on  $C$ . Consequently any component of  $F$  has a non-empty intersection with  $\partial U$ .

Let  $\beta$  and  $\varepsilon$  be positive numbers such that  $s + \beta > \alpha$  on  $\partial U$  and  $s(x) + \varepsilon\beta < \alpha$ . Let  $K'$  be a compact set in  $\partial U - K$  such that

$$\omega_x^U(\partial U - (K \cup K')) < \varepsilon.$$

We denote by  $u$  the function on  $\bar{U}$  equal to

$$y \rightarrow \omega_y^U(\partial U - (K \cup K'))$$

on  $U$  equal to 1 on  $\partial U - (K \cup K')$  and equal to 0 on  $K \cup K'$ .  $u$  is lower semicontinuous. We denote by  $F_\varepsilon$  the set

$$F_\varepsilon = \{y \in \bar{U} | s(y) + \beta u(y) \leq \alpha\}.$$

Obviously  $F_\varepsilon \subset F$  and  $x \in F_\varepsilon$ . There exists an open set  $G$ ,  $K \subset G$ ,  $K' \cap F \cap G = \emptyset$ ,  $F \cap \partial G = \emptyset$ . Let  $y \in F_\varepsilon \cap G$  and  $C_y$  be the component of  $F_\varepsilon$  containing  $y$ .  $C_y$  is contained in  $G$  since  $F_\varepsilon \cap \partial G = \emptyset$ .  $C_y \cap \partial U$  is not empty, as it was shown above; let  $z \in C_y \cap \partial U$ . If  $z \notin K$  then

$$s(y) + \beta u(y) = s(y) + \beta > \alpha$$

which contradicts the relation  $z \in F_\varepsilon$ . Hence  $z \in K$  and  $C_y \subset K$ ,  $y \in K$ ,  $F_\varepsilon \cap G \subset K$ . Since

$$\begin{aligned} \liminf_{U - K \ni y \rightarrow x} (s(y) + \beta u(y)) &= \liminf_{G - K \ni y \rightarrow x} (s(y) + \beta u(y)) \\ &\geq \liminf_{G - F_\varepsilon \ni y \rightarrow x} (s(y) + \beta u(y)) \geq \alpha > s(x) + \beta u(x). \end{aligned}$$

$K$  is a fine neighbourhood of  $x$ .

2. We shall suppose in this paragraph that there exists a positive potential on  $X$ .

**THEOREM 2.** — *For any non-negative superharmonic function  $s$  on  $X$  and any set  $E \subset X$ ,  $\hat{R}_s^E$  is equal to  $s$  on the fine interior of  $E$ .*

Let  $x$  be a fine interior point of  $E$ . Then [3] (pag. 435)

$$\lim_{\mathcal{U}, \mathcal{U}} \int_{(X-E) \cap \partial U} \bar{d}\omega_x^U = 0,$$

where  $\mathcal{U}$  denotes the filter of sections of regular neighbourhoods of  $x$ . If  $s$  is bounded in a neighbourhood of  $x$  then

$$\begin{aligned} s(x) &\geq \hat{R}_s^E(x) = \lim_{\mathcal{U}, \mathcal{U}} \int \bar{R}_s^E d\omega_x^U \\ &\geq \lim_{\mathcal{U}, \mathcal{U}} \sup \int_{E \cap \partial U} \bar{R}_s^E d\omega_x^U = \lim_{\mathcal{U}, \mathcal{U}} \sup \int_{E \cap \partial U} s d\omega_x^U \\ &\geq \lim_{\mathcal{U}, \mathcal{U}} \int s d\omega_x^U - \lim_{\mathcal{U}, \mathcal{U}} \sup \int_{(X-E) \cap \partial U} s d\omega_x^U = s(x). \end{aligned}$$

In the general case let  $\mathcal{G}$  be the set of continuous finite positive superharmonic functions dominated by  $s$ . We have

$$s(x) \geq \hat{R}_s^E(x) \geq \sup_{s' \in \mathcal{G}} \hat{R}_{s'}^E(x) = \sup_{s' \in \mathcal{G}} s'(x) = s(x).$$

**COROLLARY 1.** — *A polar set has no fine interior points.*

**THEOREM 3.** — *Let  $G$  be a fine open set and  $s$  be a non-negative superharmonic function. Then*

- a)  $\hat{R}_s^G = R_s^G$ ;
- b)  $R_{R_s^G}^E = R_s^E$  for any  $E \subset G$ ;
- c)  $R_s^G = \sup_{s' \in \mathcal{G}} R_{s'}^G$ , where  $\mathcal{G}$  is an increasingly directed set of superharmonic functions with  $s = \sup_{s' \in \mathcal{G}} s'$  on  $G$ ;
- d)  $\hat{R}_s^G = \sup_{K \subset G} \hat{R}_s^K$  where  $K$  is compact.

**COROLLARY 2.** — *For any fine open set  $G$  and any measure  $\mu$  with compact carrier we have*

$$\int s d\mu^G = \int \hat{R}_s^G d\mu \quad (1)$$

where  $s$  is an arbitrary non-negative superharmonic function.

(1)  $\mu^G$  is the balayaged measure of  $\mu$  on  $G$  [3] (p. 447).

This relation follows from Theorem 3 c) taking  $\mathcal{S}$  as the set of all continuous finite positive superharmonic functions smaller than  $s$ .

LEMMA 1. — *Let  $s$  be a positive superharmonic function on  $X$  and  $F \subsetneq X$  be a closed non empty set, non polar if  $X \notin \mathfrak{P}$ .  $s$  is resolutive for the normed Dirichlet problem on  $X - F$  and we have*

$$R_s^F = H_s^{X-F} \quad (2)$$

on  $X - F$ .

Since  $F$  is non-polar if  $X \notin \mathfrak{P}$ , there exists a locally bounded positive potential on any component of  $X - F$ . Let  $s_0$  be a positive continuous superharmonic function on  $X$ . We want to prove that  $X - F$  is an  $MP_0$ -set [1]. Let  $s' \in \overline{\mathcal{G}}_0^{X-F, X}$ . Then  $s' + \varepsilon s_0$  is non-negative outside a compact set contained in  $U$  for any  $\varepsilon > 0$ . From [2] (Theorem 2) it follows  $s' + \varepsilon s_0 \geq 0$ .  $\varepsilon$  being arbitrary we get  $s' \geq 0$  and  $X - F$  is an  $MP_0$ -set. By [1] (Corollary 3) the restrictions of the functions  $\min(s, ns_0)$  on  $\partial U$  are resolutive. Since  $\min(s, ns_0) \uparrow s$  for  $n \uparrow \infty$  it follows that the restriction of  $s$  on  $\partial U$  is resolutive.

Let  $\bar{s} \in \overline{\mathcal{G}}_s^{X-F, X}$  and  $s'$  be the function on  $X$  equal to  $s$  on  $F$  and equal to  $\min(s, \bar{s})$  on  $X - F$ .  $s'$  is superharmonic and dominates  $s$  on  $F$ . Hence  $\bar{s} \geq R_s^F$ ,  $H_s^{X-F} \geq R_s^F$  on  $X - F$ . The converse inequality is trivial.

THEOREM 4. — *Let  $s$  be a non-negative superharmonic function. For any  $E \subset X$  such that  $s$  is finite on  $E$*

$$R_s^E = \inf_{G \supset E} R_s^G,$$

where  $G$  is fine open.

Obviously

$$R_s^E \leq \inf_{G \supset E} R_s^G.$$

Let  $s'$  be a non-negative superharmonic function on  $X$ ,  $s' \geq s$  on  $E$  and  $\theta$  be a real number,  $0 < \theta < 1$ . The set

$$G = \{x \in X \mid s'(x) > \theta s(x)\}$$

(\*) The normed Dirichlet problem and the associated notions were introduced in [1].

is fine open and contains E. We have  $\frac{s'}{\theta} \geq s$  on G and therefore

$$\frac{s'}{\theta} \geq R_s^G \geq \inf_{G \supseteq E} R_s^G.$$

$s', \theta$  being arbitrary we get

$$R_s^E \geq \inf_{G \supseteq E} R_s^G.$$

**THEOREM 5** <sup>(3)</sup>. — *Let  $s_1, s_2$  be non-negative superharmonic functions and E be an arbitrary set. Then*

$$R_{s_1+s_2}^E = R_{s_1}^E + R_{s_2}^E, \quad \hat{R}_{s_1+s_2}^E = \hat{R}_{s_1}^E + \hat{R}_{s_2}^E.$$

If E is compact the relation follows from lemma 1. By theorem 3 d) it can be extended to fine open sets and by theorem 4 to arbitrary E subjected to the condition that  $s_1 + s_2$  is finite on E. In the general case we have

$$R_{s_1+s_2}^E \leq R_{s_1}^E + R_{s_2}^E$$

and

$$R_{s_1+s_2}^E = R_{s_1}^E + R_{s_2}^E$$

on E. We denote by E' the set

$$E' = \{y \in E | s_1(y) + s_2(y) < \infty\}.$$

Let  $x \in X - E$ . If  $R_{s_1+s_2}^E(x) = \infty$  the required equality holds at x. On the contrary case there exists a non-negative superharmonic function  $s_0$  on X finite at x and  $s_0 \geq s_1 + s_2$  on E. For any non-negative superharmonic function s on X and any  $\epsilon > 0$  we have

$$R_s^{E'} \leq R_s^E \leq R_s^{E'} + \epsilon s_0.$$

Hence

$$R_s^{E'}(x) = R_s^E(x).$$

We have therefore

$$R_{s_1+s_2}^E(x) = R_{s_1+s_2}^{E'}(x) = R_{s_1}^{E'}(x) + R_{s_2}^{E'}(x) = R_{s_1}^E(x) + R_{s_2}^E(x).$$

The second equality follows immediately from the first one.

<sup>(3)</sup> This theorem was proved by R.-M. Hervé under the supplementary hypothesis that X has a countable basis, and either  $s_1, s_2$  are continuous or E is closed or E is open or the axiom D is fulfilled [3].

## BIBLIOGRAPHY

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