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## A SURVEY OF MINIMAL SETS

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Let  $(X, T)$  be a transformation group, that is,  $X$  is a topological space,  $T$  is a topological group, to each  $t \in T$  there is assigned a homeomorphism  $\pi^t$  of  $X$  onto  $X$  such that the map  $t \rightarrow \pi^t$  is a group homomorphism, and finally the action  $\pi: X \times T \rightarrow X$  defined by  $xt = (x, t)\pi = x\pi^t$  is continuous. The transformation group  $(X, T)$  is said to be a *discrete flow* in case  $T$  is the discrete additive group  $\alpha$  of integers, and the transformation group  $(X, T)$  is said to be a *continuous flow* in case  $T$  is the additive group  $\mathfrak{R}$  of real numbers with its usual topology. A discrete flow is determined simply by a homeomorphism  $\pi^1$  of a topological space  $X$  onto  $X$ . A continuous flow on a manifold  $X$  is determined in the familiar fashion under general conditions by a given autonomous system  $dx/dt = f(x)$  of ordinary differential equations where  $x$  and  $f$  are vectors or, what is the same, by a contravariant vector field over the manifold  $X$ . Consequently, the notion of transformation group subsumes the two classical cases occurring in the qualitative theory of ordinary differential equations as founded by Poincaré. We may define « topological dynamics » as the study of transformation groups with respect to those properties, wholly or largely topological in nature, whose prototypes occurred in classical dynamics [12, 13]. We may here identify dynamics with the theory of ordinary differential equations.

For simplicity and convenience, we suppose that all topological spaces mentioned are Hausdorff and that every transformation group  $(X, T)$  considered has compact *phase space*  $X$ .

Since the flows, discrete or continuous, are crucial special cases, it is essential that we do not assume that the *phase group*  $T$  is compact.

The property of periodicity is certainly centrally located in the qualitative theory of ordinary differential equations. It is possible to define a periodic point of a transformation group  $(X, T)$ . A subset  $A$  of  $T$  is said to be a *syndetic* in case  $T = AK$  for some compact subset  $K$  of  $T$ . A point  $x$  of  $X$  is said to be *periodic* provided there exists a syndetic subset  $A$  of  $T$  such that  $xA = x$ . This notion generalizes the notion of periodic point under a flow.

Now periodic points do not always exist for a given transformation group. It is then natural to consider a less stringent notion: almost periodic point. A point  $x$  of  $X$  is said to be *almost periodic* provided that for each neighborhood  $U$  of  $x$  there exists a syndetic subset  $A$  of  $T$  such that  $xA \subset U$ . That almost periodic points are much more plentiful than periodic points is borne out by several theorems.

A subset  $M$  of  $X$  is said to be *minimal* in case  $M$  is nonempty and the orbit  $xT$  of every point  $x$  of  $M$  is dense in  $M$ . The basic connection between almost periodic point and minimal set is this: If  $x \in X$  then  $x$  is almost periodic if and only if the orbit closure  $\overline{xT}$  of  $x$  is minimal [12]. Furthermore, there always exists a minimal set and therefore almost periodic points for every transformation group  $(X, T)$  whose phase space  $X$  is compact.

It is to be observed a subset  $M$  of  $X$  is minimal if and only if  $M$  is non empty closed invariant and  $M$  is minimal with respect to these properties. This characterization accounts for the designation « minimal set ». Perhaps the simplest example of a nonperiodic minimal set is a spinning circle whose angle of rotation is an irrational multiple of  $\pi$ ; this example, of course, is a discrete flow. The notion of minimal set under a continuous flow determined by an autonomous system was introduced by G. D. Birkhoff in 1912 [5]. The phrases « minimal set » and « minimal transformation group » are used virtually interchangeably.

The notion which permits the comparison of transformation groups  $(X, T)$  and  $(Y, T)$  with the same phase group  $T$  is the notion of homomorphism. A *homomorphism of  $(X, T)$  into*

or onto  $(Y, T)$  is defined to be a continuous map  $h$  of  $X$  into or onto  $Y$  such that  $xth = xht$  for all  $x \in X$  and all  $t \in T$ .

Another general fact that insures the existence of many minimal sets is the following. If  $(X, T)$  and  $(Y, T)$  are minimal transformation groups, then there exists a minimal subset  $M$  of the product transformation group  $(X \times Y, T)$  defined by  $(x, y)t = (xt, yt)$  and it follows that the projections of  $X \times Y$  onto  $X$  and  $Y$  actually map  $(M, T)$  homomorphically onto  $(X, T)$  and  $(Y, T)$ . The argument can be generalized. If  $(X_i, T)/i \in I$  is a family of minimal transformation groups, then there exists a minimal subset  $M$  of the product transformation group  $(\prod_{i \in I} X_i, T)$  and  $(M, T)$  is homomorphic to  $(X_i, T)$  for all  $i \in I$ . The process may be used to produce a *universal* minimal set generated by a given topological group  $T$ . Since the cardinal of  $X$  is bounded by  $\exp \exp \text{crd } T$  for any given minimal transformation group  $(X, T)$  we may consider a family  $((X_i, T)/i \in I)$  of minimal transformation groups representing all isomorphism types of minimal transformation groups  $(X, T)$  and hence assure the existence of a minimal set  $(M, T)$  which is homomorphic to every minimal set  $(X, T)$ . An argument due to Ellis [8] shows that the universal minimal set is isomorphically unique.

As for any kind of mathematical structure, there are two very basic problems regarding minimal sets: (1) The classification problem; (2) The construction problem. The construction problem (namely, to construct all minimal sets systematically) is presumably a weaker question than the classification problem. The classification and construction problems seem to be nowhere near solution even for the flows, that is,  $T = \alpha$  or  $T = \mathbb{R}$ . All that can be done at the moment is to offer some isomorphism invariants and the construction of some minimal sets. It may be proved that there are exactly  $\aleph_1$  (= the cardinal of the real number continuum) isomorphism types of minimal continuous flows with compact metrizable phase spaces. A similar statement with « continuous » replaced by « discrete » also holds.

Given a transformation group  $(X, T)$  whether minimal or not, there is associated with  $(X, T)$  an isomorphism invariant called the *structure group*  $\Gamma(X, T)$  of  $(X, T)$  which is a compact

group [9]. Let  $H$  be the group of all homeomorphisms of  $X$  onto  $X$ , and let  $H$  be provided with its usual topology so that  $H$  becomes a topological group. Consider the *transition group*  $G = \{\pi^t/t \in T\}$  of  $(X, T)$ . Now  $G$  is a subgroup of  $H$ . If  $(X, T)$  is equicontinuous, that is, if  $G$  is equicontinuous, then the closure  $\bar{G}$  of  $G$  in  $H$  is a compact group and we define  $\Gamma(X, T) = \bar{G}$ . We extend the definition to arbitrary  $(X, T)$ . It may be observed that there exists a least closed invariant equivalence relation  $R$  in  $X$  such that  $(X/R, T)$  is equicontinuous. Now define  $\Gamma(X, T)$  to be  $\Gamma(X/R, T)$ . For example, the structure group of the Floyd minimal set is the triadic group and the structure group of the Jones minimal set is the circle group [1, 12].

However, the structure group is of limited use in classifying minimal sets. First of all, nonisomorphic minimal sets may have the same structure group. But something more dramatic may occur. A transformation group  $(X, T)$  is said to be *totally minimal* in case  $(X, S)$  is minimal for every closed syndetic invariant subgroup  $S$  of  $T$ . It may be proved that if  $T$  is locally compact connected abelian (for example, if  $T = \mathbb{R}$ ), then  $(X, T)$  is totally minimal if and only if  $(X, T)$  is minimal and  $\Gamma(X, T)$  is a singleton. The horocycle minimal sets are totally minimal so that the notion of structure group does not help to analyze the horocycle minimal sets [17, 18]. Of course, any property possessed by some minimal sets but not all is a means of partial classification. Many such properties are known [12].

Apart from a relatively few particular constructions of certain minimal sets which do not seem to generalize readily, there appears to be presently two rather large classes of minimal sets: (1) coset transformation groups; (2) symbolic flows.

The most general notion of coset transformation group that I know of is the following. Let  $G$  be a topological group, let  $H$  be a closed subgroup of  $G$ , let  $K$  be a compact subgroup of  $G$ , let the double coset space  $X = H \backslash G / K = \{HgK/g \in G\}$  be provided with its partition topology so that  $X$  becomes a Hausdorff space, let  $N$  be the normalizer of  $K$  in  $G$ , let  $T$  be a topological group, let  $\varphi$  be a continuous group homomorphism of  $T$  into  $N$ , and let  $\pi_\varphi: X \times T \rightarrow X$  be such that  $g \in G$  and  $t \in T$  implies  $(HgK, t) \pi_\varphi = HgK(t\varphi) = Hg(t\varphi)K$ .

Then  $(X, T, \pi_\varphi)$  is a *coset transformation group*. To insure that  $X = H \backslash G / K$  is compact, let  $H$  be syndetic in  $G$ . In applications,  $G$  could be a Lie group,  $T$  could be  $\mathbb{R}$ , and  $\varphi$  could be a one-parameter subgroup of  $G$ . Horocycle flows and certain geodesic flows are coset transformation groups [10, 11, 17, 18].

Questions about equicontinuous minimal sets are reduced to the theory of topological groups by the following theorem: For a given topological group  $T$ , to construct all isomorphism types of all equicontinuous minimal transformation groups  $(X, T)$  where  $X$  is a compact Hausdorff, let  $(G, \varphi)$  be a group compactification of  $T$ , let  $H$  be a closed subgroup of  $G$ , and form the coset transformation group  $(H \backslash G, T, \pi_\varphi)$ .

In constructing nonequicontinuous minimal sets from coset transformation groups, the following theorem shows that it appears necessary to start from relatively complicated groups  $G$ : Let  $G$  be a locally connected group, let  $H$  be a discrete syndetic subgroup of  $G$ , and let  $(H \backslash G, G)$  be a transformation group with action  $H \backslash G \times G \rightarrow H \backslash G$  defined by  $(H g_0, g) \rightarrow H g_0 g$ . Then  $(H \backslash G, G)$  is equicontinuous if and only if the left and right uniformities of  $G$  coincide.

Recently, L. Auslander, L. Green, F. Hahn and L. Markus have demonstrated various theorems including the result that certain nilmanifolds carry nonequicontinuous distal minimal continuous flows [3, 4]. Their flows are certain kinds of coset transformation groups. A transformation group  $(X, T)$  is said to be *distal* provided that if  $x, y \in X$  with  $x \neq y$ , then there exists a neighborhood  $V$  of the diagonal of  $X \times X$  such that  $(xt, yt) \notin V$  for all  $t \in T$ . A useful theorem of Ellis [7] says that  $(X, T)$  is distal if and only if  $(X \times X, T)$  is pointwise almost periodic.

Symbolic dynamics is a major source of minimal sets. Let  $P$  be a finite nondegenerate set with its discrete topology, let  $X = P^\alpha$  be the set of all bisequences in  $P$ , that is, let  $X = P^\alpha$  be the set of all functions on  $\alpha$  to  $P$ , let  $X$  be provided with its product topology whence  $X$  is homeomorphic to the Cantor discontinuum, and let  $\sigma$  be the homeomorphism of  $X$  onto  $X$  defined by  $x\sigma = (x_{i+1}/i \in \alpha)$  and called the *shift transformation of  $X$* . Symbolic dynamics is by definition the study of the discrete flow  $(X, \sigma)$ .

If  $x = (x_i/i \in \alpha) \in X$ , then we may visualize  $x$  as

$$\dots x_{-1}, \dot{x}_0, x_1 \dots$$

where the dot indicates the value of the bisequence at 0 and  $x\sigma$  is then

$$\dots x_{-1}, x_0, \dot{x}_1, x_2 \dots$$

The name « symbolic dynamics » comes from two facts: (I) the discrete flows  $(X, \sigma)$  can be used to help describe the geodesic flows on certain surfaces of constant negative curvature [19, 20].

2) The elements of  $P$  are conventionally called « symbols ». The discrete flow  $(X, \sigma)$  may be called a *symbolic* flow; clearly,  $(X, \sigma)$  depends isomorphically only on the cardinal of  $P$ . A *symbolic* minimal set is then a minimal subset of  $(X, \sigma)$  [12].

A *block* is a finite sequence in  $P$ . A *substitution* (over  $P$ ) is a function on  $P$  which assigns to each element of  $P$  a block of length at least 2. Let  $\theta$  be a given substitution. We show how  $\theta$  determines effectively at least one symbolic minimal set. Let  $\lambda: P^2 \rightarrow P^2 = P \times P$ , be the map such that if  $pq \in P^2$ , then  $(pq)_\lambda$  is the 2-block whose first element is the last element of  $p\theta$  and whose second element is the first element of  $q\theta$ . Let  $L$  be the set of all periodic points of  $P^2$ , that is the set of all  $pq \in P^2$  such that  $(pq)\lambda^\mu = pq$  for some positive integer  $\mu$ . If  $pq \in L$  with  $(pq)\lambda^\mu = pq$ , then  $p\dot{q} \subset p\dot{q}\theta^\mu \subset p\dot{q}\theta^{2\mu} \subset \dots$  and we may define  $\omega_{pq}$  to be the union for  $i \geq 0$  of  $\cup i \geq 0 p\dot{q}\theta^{i\mu} \in X$

Let  $L_0$  be the set of all  $pq \in L$  such that for each  $r$  belonging to the range of  $\omega_{pq}$  there exists a nonnegative integer  $i$  such that the 2-block  $pq$  occurs in  $r\theta^i$ . It may be proved that  $L_0 \neq \emptyset$  and that if  $pq \in L_0$ , then  $\omega_{pq}$  is almost periodic under the shift  $\sigma$  and thus the orbit closure  $\bar{0}(\omega_{pq, \sigma})$  is a symbolic minimal set and is called a *substitution* minimal set [16].

Here are a few examples of substitution minimal sets.

1)  $P = \{0, 1\}$

$$\theta: 0 \rightarrow 01, 1 \rightarrow 10$$

$$\lambda: 00 \rightarrow 10, 01 \rightarrow 11, 10 \rightarrow 00, 11 \rightarrow 01$$

$$L = L_0 = P^2, \mu = 2, \text{ (here } \mu \text{ is the period of } \lambda \text{ on } L_0)$$

$$\bar{v}(\omega_{pq, \sigma}) \text{ is one and the same for all } pq \in P^2 \text{ and is the}$$

Morse minimal set. In particular,

$$\begin{aligned} \theta^2: O &\rightarrow OIIO, I \rightarrow IOOI \\ (OO)\theta^0 &= \quad \quad \quad OO \\ (OO)\theta^2 &= \quad \quad \quad OIIOOIIO \\ (OO)\theta^4 &= OIIIOIOOIIOOIOIIOOIHOIOOOIHOIOHO \\ &\text{etc.} \end{aligned}$$

2) Let  $n$  be an integer such that  $n \geq 2$ .

$$P = \{O, I, \dots, n - I\}$$

$$\theta: O \rightarrow OI \dots (n - I), \dots, n - I \rightarrow (n - I)OI \dots (n - 2)$$

(cyclic interchange)

The substitution minimal set determined by  $\theta$  is the  $n$ -cyclic minimal set. This generalizes example I above. Its structure group is the  $n$ -adic group.

3) The substitution minimal set determined by the substitution  $\theta: O \rightarrow OIO, I \rightarrow IOI$  is irreversible [15]. A discrete flow  $(X, \varphi)$  is said to be *reversible* in case there exists a homeomorphism  $\rho$  of  $X$  onto  $X$  such that  $\varphi^{-1} = \rho\varphi\rho^{-1}$

Under rather general hypotheses, the structure group of a substitution minimal is an  $n$ -adic group. Not all symbolic minimal sets are substitution minimal sets, however, since there are only denumerably many substitution minimal sets and there are uncountably many pairwise nonisomorphic minimal sets (the Sturmian minimal sets, for example).

The preceding discussion of minimal sets is necessarily selective and incomplete. A relatively complete bibliography of topological dynamics up to October 1962 is available in mimeographed form [14]. Recent work concerning minimal sets includes [2, 6].

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