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## GLOBAL PHASE-PORTRAIT OF A PLANE AUTONOMOUS SYSTEM

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### Introduction.

Consider an autonomous system on the plane  $\mathbb{R}^2$

$$(S) \quad x' = f(x) \quad (' = d/dt),$$

where  $x = (x_1, x_2)$ ,  $f(x) = (f_1(x), f_2(x))$ .

The purpose of the present note is to study the global structure of (S) under the main assumption that the divergence of the right-hand side of (S) is nonpositive everywhere on  $\mathbb{R}^2$ ; that is we assume the following condition  $H_1$ .  $f(x)$ , in (S), is of class  $C^1$  and

$$\operatorname{div} f(x) = \partial f_1 / \partial x_1 + \partial f_2 / \partial x_2 \leq 0 \quad \text{on} \quad \mathbb{R}^2.$$

Some other restrictions of  $f(x)$  will be also introduced below, one concerning the behaviour of  $f(x)$  in a neighbourhood of infinity and the other concerning the number of singular points of (S).

This note is a continuation of the author's paper [1]. In particular Theorems 5 and 6 of [1] are special cases of the situation we are going to discuss here.

### 1. Further introductory remarks and propositions.

Let  $x(t, P)$  denote the unique solution of (S) satisfying the initial condition  $x(0, P) = P$ . Let  $D$  be a region of  $\mathbb{R}^2$  with regular boundary  $C$ . An important and well known

consequence of  $H_1$  is that the area of  $D(t) = x(t, D)$  is non-increasing function of  $t$ . This may suggest that solutions of (S), if  $H_1$  is assumed, should exhibit some stability property as  $t$  increases as well as instability for negative  $t$ . However for establishing such properties of solutions of (S), the fact that the area of  $D(t)$  is decreasing seems to be rather useless since the shape of a particular  $D(t)$  may change considerably with  $t$ . Thus we shall make use of  $H_1$  rather through the Geren's formula

$$(1.1) \quad \int_{\int_D} \operatorname{div} f(x) dx_1 dx_2 = \int_C f_1 dx_2 - f_2 dx_1.$$

Notice that the right-hand side of (1.1) is zero along those parts of  $C$  which are segments of solution curves of (S). Consider now the system

$$(S^*) \quad x' = f^*(x),$$

where  $f^*(x) = (f_2(x), -f_1(x))$ .  $(S^*)$  is a system for orthogonal trajectories to solution curves of (S). Let  $x^*(t)$  be a solution of  $(S^*)$  and suppose  $P = x^*(t_P)$  and  $Q = x^*(t_Q)$ ,  $t_P < t_Q$ . By  $I^*(P, Q)$  we denote the set  $\{x : x = x^*(t), t_P < t < t_Q\}$  and will call it a segment of solution curve of  $(S^*)$ . Suppose now that  $I^*(P, Q) \subset C$ , where  $C$  is as in (1.1). Then it is easy to see that

$$(1.2) \quad \int_{I^*(P, Q)} f_1 dx_2 - f_2 dx_1 = \pm \int_{t_P}^{t_Q} |f(x^*(t))|^2 dt,$$

where the sign «  $-$  » should be chosen if solution curves of (S) crossing  $I^*(P, Q)$  enter  $D$  as  $t$  increases and «  $+$  » in the opposite case. (Notice that the way of crossing  $I^*(P, Q)$  by solutions of (S) is the same at each point of the segment). The absolute value of the integral in the right-hand side of (1.2) we denote shortly by  $L(P, Q)$ . Therefore  $L(P, Q)$  is a function of two points of  $R^2$  defined for such pairs  $P, Q$  which can be joined by a solution curve of  $(S^*)$  and unless the solution curve of  $(S^*)$  passing through  $P$  and  $Q$  is a closed orbit,  $L(P, Q)$  is well defined. However we shall use  $L(P, Q)$  also in the case  $I^*(P, Q)$  is a segment of a closed orbit and the context will exclude any misunderstanding.

The following two propositions can be obtained from (1.1) and (1.2) if  $H_1$  holds.

PROPOSITION 1.1. — *Suppose the boundary C of D in (1.1) can be decomposed into disjoint parts*

$$C = I^*(P, Q) \cup C_1 \cup C_2$$

where  $C_1$  is composed of segments of solution curves of (S) and

$$\left| \int_{C_2} f_1 dx_2 - f_2 dx_1 \right| < L(P, Q).$$

Then, if  $H_1$  holds on D, solution curves of (S) crossing  $I^*(P, Q)$  enter D as  $t$  increases.

The other proposition involves the following construction. Suppose we have given a segment  $I^*(P, Q)$ ; that is there is a solution  $x^*(t)$  of (S\*) such that, say,  $x^*(0) = P$  and  $x^*(t_0) = Q$  for some  $t_0 > 0$ . Consider solutions  $x(t, P)$  and  $x(t, Q)$  of (S) defined by initial conditions  $x(0, P) = P$  and  $x(0, Q) = Q$ , respectively. Put  $P(t) = x(t, P)$  for  $t \geq 0$ , hence  $P(0) = P$ . There is  $\epsilon > 0$  such that for any  $0 \leq t < \epsilon$ , the solution  $x^*(\tau, P(t))$  ( $x^*(0, P(t)) = P(t)$ ) of (S\*) crosses  $x(t, Q)$ ; more precisely, there is  $\tau(t)$  and  $s(t)$  for any  $0 \leq t < \epsilon$  such that  $x^*(\tau(t), P(t)) = x(s(t), Q)$  and if we require  $s(t)$  and  $\tau(t)$  to be continuous functions of  $t$  satisfying conditions  $s(0) = 0$  and  $\tau(0) = t_0$ , then  $s(t)$  and  $\tau(t)$  are uniquely defined and  $s(t)$  is increasing function of  $t$ . Since  $s(t)$  can be defined locally it can be continued to the maximal interval  $[0, \omega)$  open to the right. Put  $Q(t) = x(s(t), Q)$  for  $0 \leq t < \omega$ , of course  $Q(0) = Q$ , and we can consider segments

$$I^*(P(t), Q(t)) = \{x : x = x^*(\tau, P(t)), \quad 0 < \tau < \tau(t)\}.$$

We have now the following.

PROPOSITION 1.2. — *If  $H_1$  is assumed then the function*

$$(1.3) \quad \varphi(t) = L(P(t), Q(t)) = \int_0^{\tau(t)} |f(x^*(\tau, P(t)))|^2 d\tau$$

is continuous and non-increasing for  $t \in [0, \omega)$ .

## 2. Negative limiting sets and singular points.

In the sequel we shall use the following notations.  $x(t, P)$ , as above, will stand for the solution of (S) satisfying the initial condition  $x(0, P) = P$ . The maximal interval on which

$x(t, P)$  exists we denote by  $(a(P), b(P))$ . By  $I(P)$  we denote the set of  $R^2$  defined by  $\{x: x = x(t, P), a(P) < t < b(P)\}$ .  $I(P)$  will be called solution curve or trajectory of (S) passing through  $P$ . Correspondingly,  $I^+(P)$  and  $I^-(P)$  will stand for the positive and negative half trajectory of (S) issuing from  $P$ . We shall write  $I, I^+$  and  $I^-$  for trajectory, positive half trajectory and negative half trajectory respectively, if we do not wish to indicate any particular point they initiate from. By  $\alpha(I)$  and  $\omega(I)$  we denote, as usual, the negative and the positive limiting set of  $I$  respectively.

We wish to discuss in this section some results concerning the negative limiting sets and isolated singular points of (S). We start with the following theorem which can be obtained using Proposition 1.1.

**THEOREM 2.1.** — *Assume  $H_1$  holds. Let  $\alpha(I(P)) \neq \emptyset$  ( $\emptyset$  stands for the empty set) for some non-singular  $P(f(P) \neq 0)$ . Then for all  $Q, Q \in I^*(P), Q \neq P$ , with one possible exception we have*

$$\alpha(I(P)) \cap \alpha(I(Q)) = \emptyset.$$

In other words, it follows from  $H_1$  that non-singular trajectories of (S) with non-empty negative limiting set are isolated in the sense that its neighbouring trajectories have negative limiting set either empty or disjoint with that of  $I(P)$ . In particular, such trajectories are orbitally unstable for  $t < 0$ .

As a consequence of Theorem 2.1 we have

**THEOREM 2.2.** — *If  $H_1$  holds and if for a given non-singular trajectory  $I$  of (S) the negative limiting set  $\alpha(I)$  is not empty then either  $\alpha(I) = I$ , hence  $I$  is a closed orbit, or  $\alpha(I)$  is composed of singular point of (S) only.*

To see this theorem from Theorem 2.1, notice that if  $\alpha(I)$  contains a regular point  $P$  which does not belong to  $I$  then  $I^*(P)$  has to be crossed by  $I$  at least in three different points, but this contradicts Theorems 2.1.

In the case singular points of (S) are isolated we have

**THEOREM 2.3.** — *Assume  $H_1$  and suppose the singular points of (S) are isolated. Then for any trajectory  $I$  of (S) we*

have: either  $\alpha(I)$  is empty, or  $\alpha(I) = I =$  closed orbit or  $\alpha(I) = \{P\}$ , where  $P$  is a singular point of  $(S)$ .

Suppose now  $x_0$  is an isolated singular point of  $(S)$ .  $x_0$  is said to be stable if for any  $\varepsilon > 0$  there is  $\delta > 0$  such that for any solution  $x(t)$  of  $(S)$ ,  $|x(0) - x_0| < \delta$  implies that

$$|x(t) - x_0| < \varepsilon \quad \text{for} \quad t > 0.$$

Let us notice that  $x_0$  is not stable if and only if there exists a trajectory  $I$  of  $(S)$  such that  $\alpha(I) = \{x_0\}$ . Suppose the last is the case for  $x_0$ , then by Theorem 2.1 there is a finite number, at the most, of trajectories of  $(S)$  having  $x_0$  as negative limiting set, since any of them must be isolated. Thus also elliptic sectors in a neighbourhood of  $x_0$  are excluded (cf. for terminology [2]). Between two trajectories of  $(S)$  approaching  $x_0$  as  $t \rightarrow -\infty$  there must exist a trajectory  $I$  for which

$$\omega(I) = \{x_0\}$$

or an attractive fan (cf. [2], p. 208). Therefore in a neighbourhood of an isolated and unstable singular point of  $(S)$  we have the following picture. There is a neighbourhood  $N$  of  $x_0$  which can be decomposed into disjointed parts

$$N_1 \cup N_2 \cup N_3 \cup \{x_0\} = N$$

in such a way that 1) if  $P \in N_1$  then  $I^-(P) \subset N_1$  and  $\alpha(I(P)) = x_0$ , and  $N_1$  is contained in a union of finite number of trajectories of  $(S)$ , 2)  $N_2$  is composed of a finite number of half trajectories or attractive fans and if  $P \in N_2$  then  $I^+(P) \subset N_2$  and

$$\omega(I(P)) = x_0,$$

3)  $N_3$  is composed of finite number of hyperbolic sectors (cf. [2]) and if  $P \in N_3$  then neither  $I^+(P) \subset N$  nor  $I^-(P) \subset N$ , 4)  $N_1$  and  $N_2$  are closed in  $N - \{x_0\}$  while  $N_3$  is open, but  $x_0 \in \bar{N}_3$  — the closure of  $N_3$ . Such a singular point we shall call in the sequel a generalised saddle point. Manifestly a saddle point in the usual sense is a generalized saddle point, too.

Consider now the case when  $x_0$  is stable isolated singular point of  $(S)$ . Then either  $x_0$  is asymptotically stable or any neighbourhood of it contains a closed orbit of  $(S)$ . Suppose the

second possibility is the case and let  $I$  be a closed orbit of  $(S)$  containing inside  $x_0$  as only one singular point of  $(S)$ . Then by  $H_1$  it follows that  $\operatorname{div} f(s) \equiv 0$  inside  $I$ . On the other hand it follows from Theorem 2.2 that any solution curve inside  $I$  has to be a closed orbit. Therefore  $x_0$  is a center in that case. We have the following result.

**THEOREM 2.4.** — *Assume  $H_1$ . Then the singular points of  $(S)$  if they are isolated, fall under the following categories: generalized saddle points, points of attraction and centers.*

A stable singular point of  $(S)$  is a center if and only if there is a neighbourhood of it on which  $\operatorname{div} f(x) \equiv 0$ .

### 3. Positive limiting sets and behaviour at infinity.

In order to avoid some undesired behaviour of trajectories near infinity we adopt the following hypothesis in which follows.  $H_2$ . There is a finite number of singular points of  $(S)$ , if any.  $H_3$ . For any fixed  $P$  and any  $Q_n \in I^*(P)$  ( $n = 1, 2, \dots$ ) the boundness of  $L(P, Q_n)$  implies boundness of  $|P - Q_n|$ , where  $| \cdot |$  indicates any norm on  $\mathbb{R}^2$ .

Let us remark that  $H_3$  will be satisfied if

$$(3.1) \quad \int_{|x|=r}^{\infty} \min |f(x)| dr = \infty.$$

The following result is a crucial point of our argument.

**THEOREM 3.1.** — *Suppose  $H_1$ ,  $H_2$  and  $H_3$  hold true. Let  $I(P)$  be a non-singular trajectory of  $(S)$  and assume  $\omega(I(P))$  does not reduce to a singular point, hence is empty or contains regular points.*

Then

i) there is an open segment  $I^*(Q_1, Q_2)$  of the orthogonal trajectory  $I^*(P)$ , containing  $P$  and such that for any

$$Q \in I^*(Q_1, Q_2), \quad \omega(I(Q))$$

does not reduce to a singular point,

ii) for each  $Q \in I^*(Q_1, Q_2)$  the function  $s(t)$  of section 1 (cf. p. 3) can be defined for  $t \in [0, b(P))$  and  $s(t) \rightarrow b(Q)$  as  $t \rightarrow b(P)$ , and the function  $\varphi(t)$  in (1.3) is non-increasing.

Theorem 3.1. can be derived easily from Proposition 1.2. Let us remark that assumption  $H_3$  is needed only in the case  $I(P)$  is unbounded for  $t > 0$ . For such trajectories the above result has been proved in slightly less general form in [1] (cf. Lemma 2). For a generalization of it to  $n$ -dimensional case cf. [3].

In the case  $I(P)$  is a closed orbit, hence  $x(t, P)$  is periodic, it follows from Theorem 3.1 that for any neighbouring solution  $x(t)$  there is a function  $s(t)$  such that either

$$|x(s(t)) - x(t, P)|$$

is periodic in  $t$  or  $|x(s(t)) - x(t, P)|$  tends to zero as  $t \rightarrow +\infty$ . The first case is the only possible for  $x(t)$  from inside  $I(P)$ .

The second case takes place if and only if  $\operatorname{div} f(x)$  is not identically zero in any neighbourhood of  $I(P)$ . If this is the case then  $I(P)$  is a one side limit cycle.

Thus we have for any closed orbit  $I$  of (S): either  $I$  is a one side limit cycle for positive  $t$  or there is a neighbourhood of  $I$  filled up by closed orbits.

Notice that inside of each limit cycle there must be at least one singular point of (S) and there cannot be any other limit cycle. Hence if  $H_2$  is assumed, then there is a finite number of limit cycles if any.

As for a neighbourhood of infinity we have the following consequence of Theorem 3.1. Following V. V. Niemytskii (cf. [4], p. 434) we say that (S) has a saddle point at infinity or an improper saddle point if there is a sequence of points  $P_n$  and two sequences  $t_n$  and  $\tau_n$  of reals,  $0 < \tau_n < t_n$  such that  $P_n \rightarrow P$ ,  $x(t_n, P_n) \rightarrow Q$  while  $x(\tau_n, P_n)$  does not contain any convergent subsequence, hence  $|x(\tau_n, P_n)| \rightarrow \infty$ .

We have the following result.

**THEOREM 3.2.** — *If  $H_1 - H_3$  hold then there is no saddle point at infinity.*

Theorem 3.1. shows that the generalized saddle points and the trajectories of (S) which tend to a saddle point as  $t \rightarrow +\infty$  are exceptional in the sense that the neighbouring trajectories may behave differently as  $t \rightarrow +\infty$ . This is the very property of system (S) which leads to the global phase-portrait described in the next section.



#### 4. The global structure of (S).

Let us assume  $H_1 - H_3$ . Denote by  $Z$  the set composed by the union of all generalized saddle points of (S), of all trajectories of (S) tending to a saddle point as  $t \rightarrow +\infty$  and of all limit cycles of (S). Since there is a finite number of saddle points and limit cycles, if any, therefore  $Z$  is closed in  $\mathbb{R}^2$ . Let  $W$  be the complement of  $Z$ . Therefore  $W$  is open. One can prove that  $W$  is a union of a finite number of regions, connected components, say,  $W_1, \dots, W_m$ . Of course each

$$W_i (i = 1, \dots, m)$$

is invariant for (S), that is if  $P \in W_i$  then  $I(P) \subset W_i$ .

Let us fix  $W_i$ , for simplicity we shall write simply  $W$  without any subscript. Let us distinguish three cases

A.  $W$  is bounded.

B.  $W$  is unbounded and contains a singular point of (S).

C.  $W$  is unbounded and does not contain any singular point of (S).

In the case A we have two subcases.

$A_1$ .  $W$  contains a singular point. Then it contains exactly one singular point and it is a center. In this case  $W$  can be mapped topologically onto a two-cell  $|x| < r_0$  in such a way that the singular point goes into  $x = 0$  and any other trajectory contained in  $W$  is mapped onto a circumference  $|x| = r < r_0$ .

$A_2$ .  $W$  does not contain any singular point of (S), then  $W$  is filled up with closed orbits and topologically it is an annulus  $0 < r_1 < |x| < r_2$  with trajectories being the circumferences  $|x| = r, r_1 < r < r_2$ .

In the case B we have also two subcases.

$B_1$ . The singular point contained in  $W$  is a point of attraction, then  $W$  is the set of attraction to this point, that is the union of all trajectories tending to the point of attraction as  $t \rightarrow +\infty$ .

$B_2$ .  $W$  contains a center, then  $\operatorname{div} f(x) = 0$  on  $W$  and any trajectory of (S) within  $W$  is a closed orbit. Actually,  $W$  must

be identical with  $\mathbb{R}^2$  in this case and there is a homeomorphism of  $\mathbb{R}^2$  onto  $\mathbb{R}^2$  mapping the center into 0 and any trajectory of (S) onto a circumference  $|x| = r$ .

At last, let us consider the case C. The picture now may be as follows.

$C_1$ . W contains a solution curve I of (S) such that  $\omega(I) = \emptyset$ , then the same holds for any trajectory contained in W and W can be mapped topologically onto  $\mathbb{R}^2$  such that each trajectory of (S) contained in W has as the image a straight line parallel to a fixed direction. Hence the family of trajectories of (S) composing W is, as we say, parallelizable.

$C_2$ . W contains a trajectory I of (S) for which  $\omega(I) \neq \emptyset$ . Then  $\omega(I)$  is bounded and we have two subcases:

$C_{21}$ .  $\omega(I) \neq I$ . Hence  $\omega(I)$  is either a limit cycle or a closed trajectory-polygon and W is identical with the union of all trajectories of (S) which spirals towards  $\omega(I)$  as  $t \rightarrow +\infty$  (W is the set of attraction of  $\omega(I)$ , as one may say).

$C_{22}$ .  $\omega(I) = I$ , then each trajectory of (S) within W is a closed orbit and the boundary of W reduces to a bounded trajectory-polygon. Hence the complement of W is bounded. Again as in  $A_2$ , topologically W is an annulus  $0 < r_0 < |x|$  with trajectories being circumferences  $|x| = r$ ,  $r > r_0$ .

Among seven possibilities we have listed for the phase-portrait of a connected component  $W_i$  of the set W,  $B_2$  excludes any other and  $C_{22}$  excludes all but  $A_1$  and  $A_2$ . Notice also that  $A_1$ ,  $A_2$ ,  $B_2$ ,  $C_{21}$  and  $C_{22}$  (either one of them) implies the existence of a center for (S). Therefore if there is no center then only  $B_1$  and  $C_1$  are possible cases for the picture within any  $W_i$ .

If either one of  $A_1$ ,  $B_1$ ,  $B_2$  or  $C_1$  is the case for a particular  $W_i$ , then  $W_i$  must be simply connected. In the remaining cases  $W_i$  is always homeomorphic to an annulus. It follows from the above discussion also that each  $W_i$  can contain one singular point of (S), at the most and it does contain in the cases  $A_1$ ,  $B_1$  and  $B_2$ . Therefore the number of stable singular points of (S) can be evaluated if we are able to estimate the number  $m$  of connected components of W.

We may state now the following theorem.

**THEOREM 4.1.** — Assume  $H_1$ ,  $H_2$  and  $H_3$  and consider three cases: i)  $\operatorname{div} f(x) \equiv 0$  on  $\mathbb{R}^2$ , ii)  $\operatorname{div} f(x) \not\equiv 0$  on  $\mathbb{R}^2$ , iii)

$$\operatorname{div} f(x) \not\equiv 0$$

on any neighbourhood of each singular point of (S). Let  $Z$  and  $W$  be as defined above. Then  $Z$  is closed.  $W$  is open and composed of a finite number of components  $W_i$  each of them is homeomorphic either with  $\{x: |x| < 1\}$  or with

$$\{x: 0 < |x| < 1\}.$$

Moreover in the case i) we have: there is no limit cycle of (S), if there is a closed orbit inside which are all singular points of (S), then  $Z$  is bounded and if  $Z$  is empty then  $B_2$  holds for  $W = \mathbb{R}^2$ , if  $Z$  is not empty, then there is exactly one component of  $W$ , say  $W_1$ , which is unbounded and for  $W_1$ ,  $C_{22}$  is the case while for any  $W_i$  ( $i = 2, \dots, m$ ) either  $A_1$  or  $A_2$  holds. If there is a solution curve  $I$  of (S) such that  $\omega(I) = \emptyset$ , then on any unbounded  $W_i$  the picture is as in  $C_1$  and of course for each bounded  $W_i$   $A_1$  or  $A_2$  holds.

In the case (ii) the phase portrait within any  $W_i$  is one of  $A_1$ ,  $A_2$ ,  $B_1$ ,  $C_1$  and  $C_{21}$ .

Finally, in the case (iii) any stable singular point has to be asymptotically stable, thus there is no center of (S), each  $W_i$  is unbounded and the phase-portrait within  $W_i$  is either that of  $B_1$  or that of  $C_1$ .

We close up the discussion with a special case in which we are able to evaluate the number  $m$  of components of  $W$  provided we know the number of saddle points. We have the following.

**THEOREM 4.2.** — Suppose  $H_1 - H_3$  hold and let  $x^{(1)}, \dots, x^{(N)}$  are the singular points of (S). Assume additionally that

$$(4.1) \quad \operatorname{div} f(x) < 0, \quad \det J(x) \neq 0 \text{ for } x = x^{(i)}, i = 1, \dots, N,$$

where  $J(x)$  is the Jacobian matrix for  $f(x)$ .

If the number of saddle points of (S) is  $n$  ( $n \leq N$ ), then the number  $m$  of components  $W_i$  of  $W$  is equal  $m = n + 1$ , and each  $W_i$  is a set of attraction to a singular point of attrac-

tion or the family of trajectories contained in  $W_i$  is parallelizable. We have also the inequality

$$N \leq 2n + 1,$$

therefore if  $N = 2n + 1$  then each  $W_i$  contains a singular point and each  $W_i$  must be a set of attraction. In the last case the system (S) has the following property : for any solution curve  $I$  of (S) the positive limiting set  $\omega(I) = \{x_0\}$ , where  $x_0$  is a singular point of (S).

#### BIBLIOGRAPHY

- [1] C. OLECH, On the global stability of an autonomous system on the plane, *Contributions to Differential Equations*, vol. I (1963), 389-400.
- [2] S. LEFSCHETZ, *Differential equations: geometric Theory*, Interscience Publishers, New York, London, (1957).
- [3] P. HARTMAN and C. OLECH, On global asymptotic stability of solutions of differential equations, *Trans. Amer. Math. Soc.*, 104 (1962), 154-178.
- [4] V. V. NIEMYTSKII and V. V. STIEPANOV, *Qualitative theory of Differential equations* (Second edit., Russian), Moskow, (1949).