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ON THE EXISTENCE OF EXCEPTIONAL LEAVES
IN FOLIATIONS OF CO-DIMENSION ONE

by Richard SACKSTEDER

1. Introduction.

Let M be a compact n-manifold \( n \geq 2 \) with a foliated structure of co-dimension one. A leaf of such a foliation is said to be \textit{exceptional} if it is nowhere dense in M, but its topology as a subset of M is not the same as its topology as an \((n-1)\)-manifold. Reeb [2] has asked if it is possible for exceptional leaves to exist in sufficiently smooth foliations, and he showed in [2] that, under certain conditions, exceptional leaves do not exist. The author proved other theorems of this type in [3] and [4]. Here we shall answer Reeb’s question by giving an example of a 3-manifold with a C\(^\infty\) foliated structure of co-dimension one in which there are exceptional leaves. Moreover, these leaves will be contained in a minimal set of the foliation.

2. Diffeomorphisms of \( S^1 \).

We shall first construct a group of C\(^\infty\) diffeomorphisms of \( S^1 \) which has a perfect, nowhere dense, minimal set C. Let \( S^1 \) be represented as the interval \([0, 2]\) with its endpoints identified. The set C is defined as follows: At the first step the intervals \((1/3, 2/3)\), \((1, 4/3)\), and \((5/3, 2)\) are removed from \([0, 2]\). At the k’th step the middle third of each closed interval which remains after the \((k-1)\)st step is removed, as in the
usual construction of a Cantor set. The set $C$ is the set which
remains after all of the steps have been completed. $C$ is perfect
and nowhere dense.

The group of diffeomorphisms of $S^1$ will be the group gene-
rated by the diffeomorphisms $f$ and $g$ defined by

\[ f(x) = x + 2/3 \quad (\text{mod. } 2) \quad \text{for } x \in [0, 2] \]
\[ g(x) = x/3 \quad \text{if } 0 \leq x \leq 1 \]
\[ g(x) = 3x - 10/3 \quad \text{if } 4/3 \leq x \leq 5/3 \]

$g(x)$ is defined elsewhere in $[0, 2]$ so that $g$ is of class $C^\infty$,
g(2) = 2, and $g^{-1}$ exists and is of class $C^\infty$ on $S^1$. Clearly this
can be done. Let $G$ denote the group generated by $f$ and $g$.

**Lemma.** — $C$ is a minimal set under the action of $G$.

**Proof.** — It is easy to verify that $C$ is closed and invariant
under $G$. Let $C_k$ denote the set which remains after the $k$'th
step in the construction of $C$ has been carried out. Then $C_k$
is the union of $3.2^{k-1}$ disjoint closed intervals,

\[ I_k^i, i = 1, \ldots, 3.2^{k-1}, \]

and $C = \cap \{C_k : k = 1, 2, \ldots\}$. To verify that $C$ is minimal it
suffices to prove that any interval $I_k^i$ is mapped onto $[0, 1/3]$
by an element of $G$. This is proved by induction on $k$. For
$k = 1$, either $f$ or $f^2$ will work. If $k > 1$, some power of $f$
will map $I_k^i$ into the interval $[0, 1/3]$, hence it can be assumed
that $I_k^i \subset [0, 1/3]$. But then $g^{-1}(I_k^i) = I_{k-1}^j$ for some $j$,
hence the induction hypothesis shows that $I_k^i$ is mapped onto
$[0, 1/3]$ by an element of $G$. This proves the lemma.

3. The Example.

In the example, $M$ is the product manifold $M = S^1 \times M_2$, where
$M_2$ is the sphere $S^2$ with two handles attached. $M_2$ is
a disjoint union of three sets $A$, $B$, and $C$, where $A$ is a « band »
diffeomorphic to $S^1 \times [0, 1]$ passing around a handle once,
and $B$ is another such band, disjoint from $A$ and passing
around the other handle. The foliated structure of $M$ will be
defined separately on the sets $T_A = S^1 \times A$, $T_B = S^1 \times B$,
and $T_C = S^1 \times C$. 
Let $\varphi$ be a function of $v$ defined for $v \in [0, 1]$ with the properties that: (a) $\varphi$ is increasing and of class $C^\infty$, (b) $\varphi(0) = 0$, $\varphi(1) = 1$, (c) all derivatives of $\varphi$ vanish for $v = 0$ and $v = 1$.

Again regard $S^1$ as the interval $[0, 2]$ with its endpoints identified. Define the $C^\infty$ functions $h, k$ from $S^1 \times [0, 1]$ to $S^1$ by:

\[
     h(x, v) = x + 2/3 \varphi(v) \mod 2 \quad \text{and} \\
     k(x, v) = x + (g(x) - x)\varphi(v) \mod 2.
\]

Note that $h(x, 0) = k(x, 0) = x$ and $h(x, 1) = f(x)$,

\[
     k(x, 1) = g(x).
\]

Let $(u, v), \ u \in S^1, \ v \in [0, 1]$ represent a point of $A$, hence $(x, u, v)$ represents a point in $T_A$ if $x \in S^1$. We define the foliation on $T_A$ by agreeing that the leaf passing through $(x, u, 0)$ will contain all points $(h(x, v), u', v)$. The foliation on $T_B$ is defined similarly except that $k$ replaces $h$. The foliation on $T_C$ is defined by the condition that $x = \text{const.}$ on each leaf.

It is easy to check that the foliations defined on $T_A, T_B, T_C$ fit together to define a $C^\infty$ foliation of $M = T_A \cup T_B \cup T_C$. It is also clear that the leaves of the foliation are transversal to $S^1$ in product $M = S^1 \times M_2$. This transversality property implies that an arc in $M_2$ beginning at $b \in M_2 = A \cup B$ be «lifted» to the leaf through any point $(x, b) \in M, \ x \in S^1$. The lifted arc is uniquely determined by the initial point $(x, b)$. If $\gamma$ is a closed curve parameterized by $t(0 \leq t \leq 1)$ such that $\gamma(0) = \gamma(1) = b$, the lifted curve will end at a point $(T(x, \gamma), b) \in M$. It is easy to verify that the map $x \rightarrow T(x, \gamma)$ is of class $C^\infty$ and depends only on the homotopy class of $\gamma$.

Suppose that the closed curve $\gamma_A$ has the property that $\gamma_A$ does not intersect $A \cup B$, except for one sub-arc of $\gamma_A$ which is mapped homeomorphically on to the arc in $A$ which corresponds to $u = \text{const.}$ in terms of the $(u, v)$ coordinates established above. Then if $\gamma_A$ begins at $b \in M_2 = A \cup B$ and the mapping on the sub-arc is such that increasing $t$ corresponds to increasing $v$, $T(x, \gamma_A) = f(x)$. Similar considerations lead to a closed curve $\gamma_B$ beginning at $b$ and such that

\[
     T(x, \gamma_B) = g(x).
\]

Finally, if $\gamma_1$ and $\gamma_2$ are closed curves which begin at $b$ and
do not meet $A \cup B$ at all, $T(x, \gamma_i) = x$. Arcs $\gamma_A, \gamma_B, \gamma_1, \gamma_2$ with properties described can be chosen in such a way that their homotopy classes generate the fundamental group, $\pi_1(M_2)$. The map from $\gamma$ to $T(\gamma)$ induces a homomorphism from $\pi_1(M_2)$ to a group of $C^\infty$ diffeomorphisms of $S^1$, and it is now clear that this group is just the group $G$ defined above.

These considerations show that if $y \in Gx \subset S^1$, $(Gx$ is the orbit of $x$ under $G)$, then $(x, b)$ and $(y, b)$ are on the same leaf of the foliation. The converse is also easy to check, that is, if $(x, b)$ and $(y, b)$ are on the same leaf, $y \in Gx$.

Now if one takes $x$ to be a point of $C$, the lemma implies that the closure of the points $(y, b)$ on the leaf containing $(x, b)$ is just $C \times \{b\}$. It follows easily that the leaf through $(x, b)$ is exceptional and its closure is a minimal set.

4. The fundamental group of an exceptional leaf.

It was remarked in [4] that Lemma 12.1 of [4] suggests that the fundamental group of a nowhere dense leaf might be finitely generated. However, this is not the case, as will now be shown. In fact, the exceptional leaf just constructed has a fundamental group which is not finitely generated. To see this, let $\gamma_1$ be, as above, a generator of the fundamental group of $M_2$ which does not intersect the set $A \cup B$. Let $F$ be an exceptional leaf. There are infinitely many points of $F$ which project onto the initial point of $\gamma_1$. One can show that the lifts of $\gamma_1$ through these points are closed curves, which when connected to a base point, represent elements of the fundamental group of $F$. They cannot be represented in terms of any finite number of generators. We omit the details.
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