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ON Φ-BOUNDED HARMONIC FUNCTIONS

by MITSURU NAKAI

1. Throughout this paper, we denote by $\Phi(t)$ a *non-negative* real-valued function defined on the half real line $[0, \infty) = (t\,; 0 \leqslant t < \infty)$. *A harmonic function u on a Riemann surface* R *is called Φ-bounded if the composite function $\Phi(|u|)$ admits a harmonic majorant on R,* i. e. there exists a harmonic function h such that $\Phi(|u|) \leqslant h$ on R. We denote by

$$\mathrm{H}\,\Phi = \mathrm{H}\,\Phi\,(\mathrm{R})$$

the totality of Φ-bounded harmonic functions on a Riemann surface R and by $\mathrm{O}_{\mathrm{H}\Phi}$ the class of all Riemann surfaces on which every Φ-bounded harmonic function reduces to a constant. In our study, the following two quantities will play an important role :

$$\bar{d}\,(\Phi) = \limsup_{t \to \infty} \Phi\,(t)/t$$

$$\underline{d}\,(\Phi) = \liminf_{t \to \infty} \Phi\,(t)/t$$

The properties of HΦ-functions on Riemann surfaces and the class $\mathrm{O}_{\mathrm{H}\Phi}$ are first investigated by Parreau [3] for the special $\Phi\,(t)$ which is increasing and convex (1). In the present paper we shall investigate the same problem for general $\Phi\,(t)$. Our conclusion is, roughly speaking, that Parreau's result about $\mathrm{O}_{\mathrm{H}\Phi}$ holds essentially for general $\Phi\,(t)$ and his result about properties of HΦ-functions can be derived by assuming $\underline{d}\,(\Phi) > 0$ instead of increasingness and convexity which is, in a sense, the weakest condition.

2. As for the class $\mathrm{O}_{\mathrm{H}\Phi}$, Parreau [3] showed that the class $\mathrm{O}_{\mathrm{H}\Phi}$ for

(1) For such a function, it is well-known that $\bar{d}\,(\Phi) = \underline{d}\,(\Phi) > 0$.

increasing and convex $\Phi(t)$ coincides with O_{HP} or O_{HB} (2) according to $\bar{d}(\Phi) < \infty$ or $\bar{d}(\Phi) = \infty$, respectively. Now we ask what can be said about $O_{H\Phi}$ for general $\Phi(t)$. The answer is given by

THEOREM 1. — *If* $\bar{d}(\Phi) < \infty$ *(resp.* $\bar{d}(\Phi) = \infty$*), then* $O_{H\Phi} \subset O_{HP}$ *(resp.* $O_{H\Phi} \supset O_{HB}$*).*

This was proved implicitly in our former paper [2] by using Wiener's compactification of Riemann surfaces. We shall again give an alternating elementary proof in § 1. In this theorem, we cannot replace the inclusion relation by the equality in general. But the function $\Phi(t)$, by which the equality does not hold in the above theorem, is very singular and trivial one from the view point of HΦ-functions as the following shows :

THEOREM 2. — (i) *If* $\Phi(t)$ *is bounded on* $[0, \infty)$*, then* $O_{H\Phi}$ *consists of all closed Riemann surfaces;*

(ii) *If* $\Phi(t)$ *is completely unbounded* (3) *on* $[0, \infty)$*, then* $O_{H\Phi}$ *consists of all open Riemann surfaces;*

(iii) *If* $\Phi(t)$ *is not bounded and not completely unbounded, then* $O_{H\Phi} = O_{HP}$ *or* O_{HB} *according to* $\bar{d}(\Phi) < \infty$ *or* $\bar{d}(\Phi) = \infty$*, respectively.*

This was proved in [2] and determines the class $O_{H\Phi}$ completely for any possible $\Phi(t)$. This is easily proved by using Theorem 1. We will do this also in § 1.

Observing Theorem 2, we are tempted to conclude that HΦ-property is closely related to positiveness or boundedness properties except trivial Φ's as in (i) or (ii). Next we consider this problem. To state the problem formally, let us recall three notions for harmonic functions : essentially positive, quasi-bounded and singular.

3. A harmonic function u on a Riemann surface R is called *essentially positive* if u can be represented as a difference of two HP-functions on R, or equivalently, if u admits a harmonic majorant on R. We denote the totality of essentially positive harmonic functions on R by

$$HP' = HP'(R).$$

(2) As usual, HP (R) (resp. HB (R)) denotes the totality of non-negative (resp. bounded) harmonic functions on R. The meaning of O_{HP} and O_{HB} is similar to that of $O_{H\Phi}$.

(3) We say that $\Phi(t)$ is *completely unbounded* on $[0, \infty)$ if $\Phi(t)$ is not bounded at any neighbourhood of any point in $[0, \infty)$.

Clearly HP′ (R) ⊃ HP (R). For two functions u and v in HP′ (R), there always exists the least harmonic majorant (resp. the greatest harmonic minorant) of u and v, which we denote by $u \vee v$ (resp. $u \wedge v$). Then HP′ (R) forms a vector lattice with lattice operations $\vee$ and $\wedge$. For u in HP′ (R), we denote by Mu the function $u \vee 0 + (-u) \vee 0$, which is the least harmonic majorant of $|u|$. Next first for u in HP (R), we denote by Bu the HP-function defined by sup ($v(p)$; $u \geqslant v \in$ HB (R)) on R. Clearly B is order-preserving, linear and $B^2 = B$ on HP (R) (see Ahlfors-Sario [1], p. 210). Next for u in HP′ (R), we put $Bu = Bu_1 - Bu_2$, where $u = u_1 - u_2$ and u_1 and u_2 are in HP (R). Here, by the linearity of B on HP (R), Bu does not depend on the special decomposition of u into HP-functions. Again the operator B is order-preserving, linear and $B^2 = B$ on HP′ (R) and moreover B commutes with M, $\vee$, and $\wedge$. This is clear on HP (R) by definitions of B, $\vee$, $\wedge$ and M. For the general case, we have only to show that $B(u \vee 0) = (Bu) \vee 0$. Since

$$Bu = B(u \vee 0) - B((-u) \vee 0)$$

and

$$\begin{aligned} B(u \vee 0) \wedge B((-u) \vee 0) &= B((u \vee 0) \wedge ((-u) \vee 0)) \\ &= B0 = 0, \end{aligned}$$

$B(u \vee 0)$ is the positive part of the Jordan decomposition of Bu.

An HP′-function u is called *quasi-bounded* (resp. *singular*) if $Bu = u$ (resp. $Bu = 0$). These notions were introduced by Parreau [3]. We denote the totality of quasi-bounded harmonic functions on R by

$$\text{HB}' = \text{HB}'(\text{R}).$$

Clearly HB′ ⊃ HB. Since B commutes with M, $\vee$ and $\wedge$, we see that $Bu = u$ is equivalent to $BMu = Mu$. Hence we can also define

$$\text{HB}'(\text{R}) = (u \in \text{HP}'(\text{R})\,;\, BMu = Mu).$$

4. Parreau [3] showed that, for increasing and convex function $\Phi(t)$, HΦ ⊂ HP′ and if moreover $\bar{d}(\Phi) = \infty$, then HΦ ⊂ HB′. Our next problem is to investigate whether such relations hold or not for general $\Phi(t)$. The answer is negative in general: we shall single out in § 4 an increasing continuous unbounded function $\Phi(t)$ with $\bar{d}(\Phi) < \infty$ and $\underline{d}(\Phi) = 0$ and an HΦ-function in the open unit disc which is not an HP′-function there (*Example 2*). This shows the invalidity of HΦ ⊂ HP′

in general. Only for this aim, we may take bounded $\Phi(t)$. But we are interested in unbounded $\Phi(t)$. We shall also construct in § 3 an increasing continuous function $\Phi(t)$ with $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$ and an HΦ-function in the open unit disc which is not an HP'-function there (*Example 1*). This shows the invalidity of the relation $H\Phi \subset HP'$ and so of the relation $H\Phi \subset HB'$ even if $\bar{d}(\Phi) = \infty$.

Then there arises the question when can we conclude the relation $H\Phi \subset HP'$ or HB'. Both examples above show that unboundedness, not completely unboundedness, increasingness, continuity or all of them cannot give the condition. In both examples above, $\underline{d}(\Phi) = 0$. This suggests us that the required condition may be $\underline{d}(\Phi) > 0$. This is really the case. Firstly the answer for $H\Phi \subset HP'$ is given completely by the following which includes Parreau's case :

THEOREM 3. — *In order that* $H\Phi(R) \subset HP'(R)$ *for any Riemann surface* R, *it is necessary and sufficient that* $\underline{d}(\Phi) > 0$ (*no matter whether* $\bar{d}(\Phi)$ *is finite or infinite*).

The proof of this will be given in § 5. Similarly we ask about the condition which assures the relation $H\Phi \subset HB'$. In this case, even in the Parreau's case, we must assume that $\bar{d}(\Phi) = \infty$ as the following simple example shows : $\Phi(t) = t$, $R = (z\,;\, 0 < |z| < 1)$ and $u(z) = -\log|z|$. The best possible general conclusion is as follows :

THEOREM 4. — *If* $\bar{d}(\Phi) = \infty$, *then* $H\Phi(R) \cap HP'(R) \subset HB'(R)$.

Here we cannot drop HP' (R) in the above relation as Example 1 shows. The above theorem will be proved in § 6. Now assume that $\underline{d}(\Phi) > 0$, then by Theorems 3 and 4, $H\Phi(R) \subset HB'(R)$. Conversely if $H\Phi(R) \subset HB'(R)$ for any R, then $H\Phi(R) \subset HP'(R)$ for any R and by Theorem 3, $\underline{d}(\Phi) > 0$. Thus we get the following which includes Parreau's case :

THEOREM 5. — *Assume that* $\underline{d}(\Phi) = \infty$. *In order* $H\Phi(R) \subset HB'(R)$ *for any Riemann surface* R, *it is necessary and sufficient that* $\underline{d}(\Phi) > 0$.

1. Proofs of Theorems 1 and 2.

1. *Proof of Theorem* 1. — **I.** The case $\bar{d}(\Phi) = \infty$: Assume that there exists a non-constant HΦ-function u on R. By the definition of

Φ-boundedness, there exists an HP-function h such that $\Phi(|u|) \leqslant h$ on R. We have to show that $R \notin O_{HB}$. Contrary to the assertion, assume that $R \in O_{HB}$. Since $\bar{d}(\Phi) = \infty$, we can find a strictly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers r_n such that $\lim_{n\to\infty} r_n = \infty$, $\Phi(r_n) > 0$, $G_n = (p \in R\,;\, |u(p)| < r_n) \neq O$ and $\lim_{n\to\infty} a_n = 0$, where $a_n = r_n/\Phi(r_n)$. Then clearly

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots,\ R = \bigcup_{n=1}^{\infty} G_n.$$

First we show that $G_n \notin SO_{HB}$ for some n on (4). If this is not the case, then $G_n \in SO_{HB}$ for all n's. Then since $a_n h - |u|$ is superharmonic and bounded from below on G_n and

$$a_n h - |u| \geqslant a_n \Phi(|u|) - |u| = a_n \Phi(r_n) - r_n = 0$$

on ∂G_n, we can conclude that $a_n h - |u| \geqslant 0$ on G_n. Since $a_n \searrow 0$, we must have $u \equiv 0$ on R, which is clearly a contradiction. Hence we may assume that $G_n \notin SO_{HB}$ $(n = 1, 2, 3, \ldots)$ by choosing a suitable subsequence of (r_n), if necessary.

Next we assert that $G_n - \bar{G}_1 \in SO_{HB}$ $(n = 1, 2, 3, \ldots)$. For, if there exists a G_n with $G_n - \bar{G}_1 \notin SO_{HB}$, then there would exist two disjoint non-empty open sets G_1 and $G_n - \bar{G}_1$ not belonging to SO_{HB}. By the so called "two domains criterion", we must have that $R \notin O_{HB}$ (see Ahlfors-Sario [1], p. 213). But this contradicts our assumption $R \in O_{HB}$.

Now consider the function $w_n = a_n h + r_1 - |u|$ on G_n, which is superharmonic and bounded from below on G_n and so on $G_n - \bar{G}_1$. By the similar manner as before, we see that $w_n \geqslant a_n h - |u| = 0$ on ∂G_n. Clearly $w_n \geqslant r_1 - |u| = 0$ on ∂G_1. Hence $w_n \geqslant 0$ on $\partial(G_n - \bar{G}_1)$. Since $G_n - \bar{G}_1 \in SO_{HB}$, we can conclude that $w_n \geqslant 0$ on G_n or $|u| \leqslant a_n h + r_1$ on G_n. Hence by the fact that $a_n \searrow 0$, we get that $|u| \leqslant r_1$ on R. This contradicts our assumption that $R \in O_{HB}$. Thus we must have $R \notin O_{HB}$.

II. The case $\bar{d}(\Phi) \leqslant \infty$: Assume that there exists a non-constant HP-function u on R. By $\bar{d}(\Phi) < \infty$, we can find a point s in $[0, \infty)$ such that there exists a finite positive constant C with $\Phi(t) \leqslant Ct$ $(s \leqslant t < \infty)$. Let $v = s + u$. Clearly v is a non-constant HP-function on R with

(4) An open subset G of a Riemann surface R with smooth relative boundary ∂G is said to belong to SO_{HB} if every HB-function on G with continuous boundary value zero at ∂G reduces to a constant zero.

$v \geqslant s$ on R. Hence $\Phi(|v|) = \Phi(v) \leqslant Cv$ on R. Thus v is a non-constant HΦ-function on R and so $R \notin O_{H\Phi}$.

2. *Proof of Theorem 2.* — *Ad* (*i*) : If $\Phi(t)$ is bounded, then every non-constant harmonic function is an HΦ-function. Thus $O_{H\Phi}$ consists of all Riemann surfaces carrying no non-constant harmonic function, which are closed Riemann surfaces.

Ad (*ii*) : For any non-constant harmonic function u on R, since u is open map of R into $[0, \infty)$ by the maximum principle, $\Phi(|u|)$ is completely unbounded on R along with $\Phi(t)$ and so u is not HΦ-function. Thus there exists no non-constant HΦ-function on any Riemann surface and $O_{H\Phi}$ consists of all Riemann surfaces.

Ad (*iii*) : Assume that $\bar{d}(\Phi) = \infty$ and that there exists a non-constant HB-function u on R. As $\Phi(t)$ is not completely unbounded, so there exists an interval (a, b) in which $\Phi(t) < c =$ const. Let

$$v = (a+b)/2 + ((b-a)/2)\,(\sup_{R} |u|)^{-1} u.$$

Then v is a non-constant HB-function and $\Phi(|v|) = \Phi(v) < c$ on R. Thus $O_{H\Phi} \subset O_{HB}$. This with Theorem 1 gives $O_{H\Phi} = O_{HB}$.

Next assume that $\bar{d}(\Phi) < \infty$. By Theorem 1, $O_{HP} \supset O_{H\Phi}$. Contrary to the assertion, assume that there exists an R in $O_{HP} - O_{H\Phi}$. Let u be a non-constant HΦ-function on R. Then $\Phi(|u|) \leqslant c =$ constant on R. Since $\Phi(t)$ is unbounded and $|u|(R)$ is connected in $[0, \infty)$ and contains 0, u must be bounded on R. Then $\sup_{R} |u| + u$ is a non-constant HP-function on R, which contradicts the assumption that $R \in O_{HP}$. Hence $O_{H\Phi} = O_{HP}$.

2. Preparations for Examples.

Let $U = (z; |z| < 1)$ and A be an arc in $\partial U = (z; |z| = 1)$. We denote by $w(z; A)$ *the harmonic measure* of A calculated at z in U with respect to U. It is well known that

$$w(z; A) = (2\beta - \alpha)/2\pi, \tag{1}$$

where α is the length of A and β is the angle seeing the arc A from z.

We denote by L_A the line segment connecting both end points of A. Then from (1), we easily see that

(2) $$w(0;A) = \alpha;$$

(3) $$w(z;A) = 1 - \alpha/2\pi \quad \text{on} \quad L_A.$$

Next let A_j be the arc in $\partial U = (z; |z| = 1)$ with end points 1 and $e^{i\alpha_j} (j = 1, 2)$ such that $0 < \alpha_1 < \alpha_2$, $\alpha_1 < \pi/2$, $\alpha_2 < \pi/2$. We denote by $\tilde{A}_j$ (resp. A_j') the arc with end points 1 and $e^{-i\alpha_j}$ (resp. $A_j' = A_j \cup \tilde{A}_j$). For simplicity, we set $L_2 = L_{A_2'}$, i.e. L_2 is the line segment connecting two end points of A_2'. Then we get the following inequality which plays an important role in our forth-coming examples : there exists a universal constant s_0 ($\leqslant 4^{-1}\pi^4$) such that

(4) $$|w(z;A_1) - w(z;\tilde{A}_1)| \leqslant s_0 \alpha_1^2/(\alpha_2^2 - \alpha_1^2)^2 \quad \text{on} \quad L_2.$$

Proof. — We denote the points $e^{i\alpha_1}$, $e^{-i\alpha_1}$, $(e^{i\alpha_1} + e^{-i\alpha_1})/2$, 1, $(e^{i\alpha_2} + e^{-i\alpha_2})/2$ and z on L_2 with $\operatorname{Im}(z) \geqq 0$ by D, E, F, G, H and P respectively. We set $DF = FE = d$, $FH = k$, $DP = a$, $PF = b$ and $PE = c$. By (1), $w(z;A_1) - w(z;\tilde{A}_1) = (\measuredangle DPG - \measuredangle GPE)/\pi$. Let $\measuredangle DPF = \theta_1$ and $\measuredangle FPE = \theta_2$. Then clearly $\measuredangle DPG \leqslant \theta_1$ and $\measuredangle GPE \geqslant \theta_2$. Hence we have $0 \leqslant w(z;A_1) - w(z;\tilde{A}_1) \leqslant (\theta_1 - \theta_2)/\pi$. Applying the cosine theorem to triangles ΔDPF and ΔFPE and then Pappos' identity to the triangle ΔDPE, we have

$$\sin 2^{-1}(\theta_1 - \theta_2) = (c - a)(8abc \sin 2^{-1}(\theta_1 + \theta_2))^{-1}(4d^2 - (a-c)^2).$$

Here we have

$$\begin{aligned} ac \sin 2^{-1}(\theta_1 + \theta_2) &\geqslant ac \sin 2^{-1}(\theta_1 + \theta_2) \cos 2^{-1}(\theta_1 + \theta_2) \\ &= 2^{-1} ac \sin \measuredangle DPE = \Delta DPE \\ &= \Delta DHE = dk. \end{aligned}$$

By the triangle inequality applied for ΔDPE, $c - a \leqslant 2d$. Thus by noticing $b \geqslant k$, we have $\sin 2^{-1}(\theta_1 - \theta_2) \leqslant d^2 k^{-2}$. As

$$\sin\theta \geqslant (2/\pi)\theta \qquad (0 \leqslant \theta \leqslant 2^{-1}\pi),$$

so $\theta_1 - \theta_2 \leqslant \pi d^2 k^{-2}$. Now we have $d = \sin\alpha_1 \leqslant \alpha_1$ and

$$\begin{aligned} k &= \cos\alpha_1 - \cos\alpha_2 \\ &= 2\sin^{-1}(\alpha_1 + \alpha_2)\sin 2^{-1}(\alpha_2 - \alpha_1) \geqslant 2\pi^{-2}(\alpha_2^2 - \alpha_1^2). \end{aligned}$$

Hence

$$0 \leqslant w(z;A_1) - w(z;\tilde{A}_1) \leqslant 4^{-1}\pi^4 \alpha_1^2/(\alpha_2^2 - \alpha_1^2)^2. \quad Q.E.D.$$

We shall use (4) in the particular case where $0 < \alpha_1 < \alpha_2/\sqrt{2}$. In this case, by using universal constant s ($\leqslant \pi^4$), we get

$$(5) \qquad |w(z;A_1) - w(z;\tilde{A}_1)| \leqslant s(\alpha_1^2/\alpha_2^4) \quad \text{on} \quad L_2.$$

3. Example 1.

We are now able to construct an example of a function Φ which is continuous, increasing, $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$; and an $H\Phi$-function u in then open unit disc $U = (z; |z| < 1)$ which is not an HP′-function.

EXAMPLE 1. *Let p be a constant such that $0 < p < \min(1/4, 1/4s)$, where s is the constant in (5) in § 2, and $(p_n)_{n=1}^{\infty}$ be a sequence defined by $p_n = (p^{4^\nu})^{2^\nu + \mu}$ for $n = 2^\nu + \mu$ ($\nu = 0, 1, 2, \ldots$; $\mu = 1, 2, 3, \ldots, 2^\nu$). Let A_n and $\tilde{A}_n$ be arcs on the unit circumference such that*

$$A_n = (e^{i\theta}; 0 \leqslant \theta \leqslant 2\, p_n\, \pi/n)$$

and

$$\tilde{A}_n = (e^{i\theta}; -2\, p_n\, \pi/n \leqslant \theta \leqslant 0).$$

Let $(r_\nu)_{\nu=1}^{\infty}$ and $(b_\nu)_{\nu=1}^{\infty}$ be two sequences of positive numbers defined by $r_\nu = 2/(p^{4^{\nu-1}})^{2^\nu}$ and $b_\nu = 2^{\nu/2} \cdot r_\nu$. Define the function $\Phi(t)$ on $[0, \infty]$ by

$$\Phi(t) = \begin{cases} 0, & t \in [0, r_1]; \\ b_1(t - r_1), & t \in [r_1, r_1 + 1]; \\ b_\nu, & t \in [r_\nu + 1, r_{\nu+1}] \ (\nu = 1, 2, \ldots); \\ b_\nu + (b_{\nu+1} - b_\nu)(t - r_{\nu+1}), & t \in [r_{\nu+1}, r_{\nu+1} + 1] \ (\nu = 1, 2, \ldots) \end{cases}$$

and the function $u(z)$ in U by

$$u(z) = \sum_{n=1}^{\infty} (w(z; A_n) - w(z; \tilde{A}_n))/p_n.$$

Then the following hold:

(a) *$\Phi(t)$ is continuous, increasing, $\bar{d}(\Phi) = \infty$ and $\underline{d}(\Phi) = 0$;*

(b) *$u(z)$ is well defined in U and harmonic there;*

(c) *$u(z) \in H\,\Phi(U)$;*

(d) *$u(z) \notin HP'(U)$.*

Proof of (a). — Is immediate by the definition of $\Phi(t)$.

Proof of (b). — For each $n = 1, 2, \ldots$, set

$$v_n(z) = w(z; A_n) - w(z; \tilde{A}_n), \qquad u_n(z) = \sum_{k=1}^{n} v_k(z)/p_k.$$

Then v_n and u_n are harmonic in U, positive in the upper half of U and $v_n(-z) = -v_n(z)$ and $u_n(-z) = -u_n(z)$ in U. Hence to show that the series defining $u(z)$ is convergent in U and defines a harmonic function there, we have only to prove that $(u_n(i/2))_{n=1}^{\infty}$ is convergent. By (5) in § 2, we have that

$$0 < v_n(i/2) \leqslant s(2p_n \pi/n)^2/(\pi/2)^4 \leqslant s' p_n^2,$$

where s' is a constant independent of $n \geqslant 1$. Thus

$$0 < u_{n+m}(i/2) - u_n(i/2) = \sum_{k=n+1}^{n+m} v_k(i/2)/p_k \leqslant s' \sum_{k=n+1}^{n+m} p_k < s' p^n/(1-p).$$

This shows that $(u_n(i/2))_{n=1}^{\infty}$ is convergent.

Proof of (c). — For each $\nu = 1, 2, \ldots$, we denote by L_ν the line segment $L_{A_{2^\nu}}$, i.e. the line segment connecting two end points of $A'_{2^\nu} = A_{2^\nu} \cup \tilde{A}_{2^\nu}$. Since $|v_k(z)| < 1$ in U, we have

$$|v_k(z)/p_k| \leqslant 1/p_k \leqslant 1/(p^{4^{\nu-1}})^k \qquad (1 \leqslant k \leqslant 2^\nu)$$

on U and so on L_ν. Next for $k = 2^\nu + \mu$ ($\mu = 1, 2, \ldots$) and $z \in L_\nu$, by (5) in § 2, we have that

$$\begin{aligned} v_k(z)/p_k &\leqslant s(2p_k \pi/k)^2/(2p_{2^\nu}\pi/2^\nu)^4\, p_k \\ &= s(2^{4^\nu}/4\pi^2 k^2)\,[p_k/p_2^4\,] \\ &\leqslant s(2^{4^\nu}/4\pi^2 k^2)\,[(p^{4^\nu})^k/((p^{4^{\nu-1}})^{2^\nu})^4] \\ &= s(2^{4^\nu}/4\pi^2 k^2)\, p^{4^\nu \mu} \leqslant p^{4^\nu(\mu-1)}. \end{aligned}$$

Hence for z in L_ν, we get that

$$\begin{aligned} |u(z)| &\leqslant \sum_{k=1}^{2^\nu} |v_k(z)/p_k| + \sum_{k=2^\nu+1}^{\infty} |v_k(z)/p_k| \\ &\leqslant \sum_{k=1}^{2^\nu} 1/(p^{4^{\nu-1}})^k + \sum_{\mu=1}^{\infty} p^{4^\nu(\mu-1)} \\ &\leqslant 2/(p^{4^{\nu-1}})^{2^\nu} = r_\nu. \end{aligned}$$

Since $u(z)$ is quasi-bounded in the upper half of U and in the lower half of U respectively, we have, for $e^{i\theta}$ in $U - A'_{2^\nu}$, that

$$|u(e^{i\theta})| = \sum_{k=1}^{2^\nu} |v_n(e^{i\theta})| \leqslant \sum_{k=1}^{2^\nu} 1/p_k$$

$$= \sum_{k=1}^{2^\nu} 1/(p^{4^{\nu-1}})^k \leqslant r_\nu.$$

Hence by the maximum principle, $0 \leqslant u(z) \leqslant r_\nu$ in the intersection of the upper half of U and the left side of L_ν in U. Hence $|u(z)| \leqslant r_\nu$ in the left side of L_ν in U. By (3) in § 2, we see that $w(z; A'_{2^\nu}) \geqslant 1 - p_{2^\nu}/2^\nu$ on L_ν and so on the right side of L_ν in U. Hence if z lies between L_ν and $L_{\nu+1}$ in U, $b_\nu w(z; A'_{2^\nu}) \geqslant b_\nu - 2^{-\nu/2+2} \geqslant \Phi(|u(z)|) - 2^{-\nu/2+2}$, or $\Phi(|u(z)|) \leqslant b_\nu w(z; A'_{2^\nu}) + 2^{-\nu/2+2}$, since $\Phi(t) \leqslant b_\nu$ if $t \leqslant r_{\nu+1}$. On the other hand,

$$2\pi\, b_\nu\, w(0; A'_{2^\nu}) = b_\nu (4 p_{2^\nu} \pi/2^\nu) = 8\pi\, 2^{-\nu/2}.$$

Hence if we set $w(z) = \sum_{\nu=1}^{\infty} (b_\nu w(z; A_{2^\nu}) + 2^{-\nu/2+2})$, then $w(0) = 8 \cdot \sum_{\nu=1}^{\infty} 2^{-\nu/2} < \infty$ and so $w(z) \in \mathrm{HP}(U)$. Thus

$$\Phi(|u(z)|) \leqslant b_\nu w(z; A'_{2^\nu}) + 2^{-\nu/2+2} \leqslant w(z)$$

between L_ν and $L_{\nu+1}$ in U. As ν is arbitrary, so $\Phi(|u(z)|) \leqslant w(z)$ in U [5]. This shows that $u \in \mathrm{H}\,\Phi(U)$.

Proof of (d). — Contrary to the assertion, assume that $u \in \mathrm{HP}'(U)$. Then $|u(z)|$ has a harmonic majorant $h(z)$ on U. As $u(z)$, $u_n(z)$ and $v_n(z)$ are positive in the upper half of U and antisymmetric with respect to the real line (i.e. $u(z) = -u(-z)$ etc.), so $h(z) \geqslant |u(z)| \geqslant |u_n(z)|$ in U. Clearly $|u_n(z)| = \sum_{k=1}^{n} |w(z; A_k) - w(z; \tilde{A}_k)|/p_k$ and the least harmonic majorant of the subharmonic function $|u_n(z)|$ is $\sum_{k=1}^{n} w(z; A'_k)/p_k$, where $A'_k = A_k \cup \tilde{A}_k$ as before. Hence

$$\sum_{k=1}^{n} w(z; A'_k)/p_k \leqslant h(z)$$

on U for any $n = 1, 2, \dots$. Thus in particular, $\sum_{k=1}^{\infty} w(0; A'_k)/p_k \leqslant h(0)$, which gives the following contradiction:

$$\infty = 2 \sum_{k=1}^{\infty} 1/k = \frac{1}{2\pi} \sum_{k=1}^{\infty} (4 p_k \pi/k)/p_k \leqslant h(0).$$

[5] Notice that if z lies in the left of L_1 in U, then $|u(z)| \leqslant r_1$ and so $0 = \Phi(|u(z)|) \leqslant w(z)$ there.

4. Example 2.

Consider functions

$$\begin{cases} \Phi(t) = \log^+ t = \max(\log t, 0) & \text{on} \quad [0, \infty); \\ u(z) = r^{-1} \cos\theta \ (z = r e^{i\theta}) & \text{on} \quad U_0 = (z;\ 0 < |z| < 1). \end{cases}$$

Then $\Phi(t)$ is unbounded, increasing, continuous and $\bar{d}(\Phi) = \underline{d}(\Phi) = 0$. We can also easily see that $u(z)$ is an H Φ-function in U_0 but not an HP′-function in U_0. But this example deeply depends on the weakness of the special boundary point 0 of U_0. However, without using such a special boundary property, we can construct such an example in the open unit disc $U = (z; |z| < 1)$ by the aid of Example 1.

EXAMPLE 2. *Let $\Phi(t)$ and $u(z)$ be as in Example* 1. *Let*

$$\Phi_a(t) = \min(\Phi(t), at) \quad (0 < a < \infty).$$

Then the followings hold :

(a) $\Phi_a(t)$ *is increasing, continuous,* $\bar{d}(\Phi_a) = a$ and $\underline{d}(\Phi_a) = 0$;

(b) $u(z) \in \mathrm{H}\Phi_a(\mathrm{U})$;

(c) $u(z) \notin \mathrm{HP}'(\mathrm{U})$.

5. Proof of Theorem 3.

First we prove that $\mathrm{H}\Phi(\mathrm{R}) \subset \mathrm{HP}'(\mathrm{R})$ for any R if $\underline{d}(\Phi) > 0$. Let $u \in \mathrm{H}\Phi(\mathrm{R})$ and $\underline{d}(\Phi) = 2c > 0$. Then there exists a point t_0 in $[0, \infty)$ such that $\Phi(t) > ct \ (t \geqslant t_0)$. Then for any t in $[0, \infty)$, $\Phi(t) + ct_0 \geqslant ct$. As $\Phi(|u|)$ possesses a harmonic majorant h on R, so

$$h + ct_0 \geqslant \Phi(|u|) + ct_0 > c\,|u|$$

on R. Thus u possesses a harmonic majorant $(h + ct_0)/c$, i.e. $u \in \mathrm{HP}'(\mathrm{R})$.

Conversely, if $\mathrm{H}\Phi(\mathrm{R}) \subset \mathrm{HP}'(\mathrm{R})$ for any R, then Examples 1 and 2 show that $\underline{d}(\Phi) > 0$ no matter whether $\bar{d}(\Phi)$ is finite or infinite.

6. Proof of Theorem 4.

Let $u \in \mathrm{H\Phi}(\mathrm{R}) \cap \mathrm{HP}'(\mathrm{R})$. We have to show that $u \in \mathrm{HB}'(\mathrm{R})$. As $u \in \mathrm{H\Phi}(\mathrm{R})$, so there exists an HP-function h such that $\Phi(|u|) \leqslant h$ on R. Since $u \in \mathrm{HP}'(\mathrm{R})$, we can consider $\mathrm{M}u = u \vee 0 + (-u) \wedge 0 \geqslant |u|$ and $\mathrm{B}u$. To show that $u \in \mathrm{HB}'(\mathrm{R})$, we have to prove that $\mathrm{B}u = u$ or equivalently, $\mathrm{BM}u = \mathrm{M}u$ (see 3 in the introductory part of this paper).

By the assumption that $\bar{d}(\Phi) = \infty$, we can find an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers converging to ∞ such that $\Phi(r_n) > 0$ and $\lim_{n\to\infty} a_n = 0$, $a_n = r_n/\Phi(r_n)$. Let $\mathrm{G}_n = (p \in \mathrm{R}\,;\, |u(p)| < r_n)$ $(n = 1, 2, \dots)$. Clearly

$$\mathrm{G}_1 \subset \mathrm{G}_2 \subset \dots \subset \mathrm{G}_n \subset \dots, \quad \mathrm{R} = \bigcup_{n=1}^{\infty} \mathrm{G}_n.$$

Let $(\mathrm{R}_m)_{m=1}^{\infty}$ be an exhaustion of R and w_m be a harmonic function on $\mathrm{R}_m \cap \mathrm{G}_n$ with the boundary value

$$w_m = \begin{cases} \min(\mathrm{M}u - \mathrm{BM}u, r_n) & \text{on } (\partial\mathrm{R}_m) \cap \mathrm{G}_n; \\ 0 & \text{on } \partial\mathrm{G}_n. \end{cases}$$

Since $\min(\mathrm{M}u - \mathrm{BM}u, r_n)$ is superharmonic on R, w_m is subharmonic on R_m if we define $w_m = 0$ in $\mathrm{R}_m - \mathrm{G}_n$, and $w_m \geqslant w_{m+1}$ on R_m. Let w'_m be harmonic in R_m with the boundary value

$$w'_m = \begin{cases} \min(\mathrm{M}u - \mathrm{BM}u, r_n) & \text{on } (\partial\mathrm{R}_m) \cap \mathrm{G}_n; \\ 0 & \text{on } \partial\mathrm{R}_m - \mathrm{G}_n. \end{cases}$$

Then clearly $(w'_m)_{m=1}^{\infty}$ is a bounded sequence and $0 \leqslant w'_m \leqslant \mathrm{M}u - \mathrm{BM}u$, $n = 1, 2, \dots$. If w' is any limiting harmonic function of a convergent subsequence of $(w'_m)_{m=1}^{\infty}$, then $0 \leqslant w' \leqslant \mathrm{M}u - \mathrm{BM}u$. By applying the operator B, we get

$$0 \leqslant \mathrm{B}w' \leqslant \mathrm{B}(\mathrm{M}u - \mathrm{BM}u) = \mathrm{BM}u - \mathrm{B}^2\mathrm{M}u = \mathrm{BM}u - \mathrm{BM}u = 0.$$

Since w' is bounded and positive, $\mathrm{B}w' = w'$. Hence $w' \equiv 0$ on R. Thus $\lim_{m\to\infty} w'_m = 0$ on R. As we have $w'_m \geqslant w_m \geqslant 0$ on R_m, so we conclude that $\lim_{m\to\infty} w_m = 0$ on R.

On $(\partial\mathrm{R}_m) \cap \mathrm{G}_n$, $|u| \leqslant r_n$ and $|u| \leqslant \mathrm{M}u = \mathrm{BM}u + (\mathrm{M}u - \mathrm{BM}u)$ or $|u| - \mathrm{BM}u \leqslant \mathrm{M}u - \mathrm{BM}u$. Hence on $(\partial\mathrm{R}_m) \cap \mathrm{G}_n$, $|u| - \mathrm{BM}u \leqslant \min(\mathrm{M}u - \mathrm{BM}u, r_n)$ or $|u| \leqslant \mathrm{BM}u + w_m$. On $\partial\mathrm{G}_n$, we have $|u| = r_n = a_n \Phi(|u|) \leqslant a_n h$. Thus we conclude that $|u| \leqslant a_n h +$

$+ \mathrm{BM}u + w_m$ on $\partial(\mathrm{R}_m \cap \mathrm{G}_n)$. Since $|u|$ is subharmonic and $a_n h + \mathrm{BM}u + w_m$ is harmonic on $\mathrm{R}_m \cap \mathrm{G}_n$, we can conclude that

$$|u| \leqslant a_n h + \mathrm{BM}u + \mathrm{w}_m \quad \text{on } \mathrm{R}_m \cap \mathrm{G}_n.$$

By letting $m \nearrow \infty$ and then $n \nearrow \infty$, we conclude that $|u| \leqslant \mathrm{BM}u$ on R. By the definition of Mu, we must have $\mathrm{M}u \leqslant \mathrm{BM}u$ and hence $\mathrm{BM}u = \mathrm{M}u$.

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