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ON REGULAR FOLIATIONS

by Enrique VIDAL

Given a foliation in a differentiable manifold, we establish several properties equivalent to the fact that the foliation is regular and we deduce two consequences.

1. Regular foliations and applications of decomposition.

Let us consider an n-dimensional differentiable manifold (M, ψ), determined by a set M and an atlas ψ, real, C^∞, in which there exists a foliation F, of codimension q, given by a completely decomposable differentiable form of degree q

$$\omega = \omega^1 \wedge \dots \wedge \omega^q \qquad \text{local decomposition} \tag{1}$$

The necessary and sufficient condition for determining a foliation is that $d\omega = \lambda \wedge \omega$ (1), which is equivalent to the following (Chevalley [1]) ; there exists a chart at each point such that ω is locally equivalent to

$$dy^1 \wedge \dots \wedge dy^q.$$

The coordinate neighbourhood is of the form $U_x \times U_y$ (U_x, U_y being cubical coordinate neighbourhoods) which we call flat coordinate neighbourhoods. Every subspace $U_x \times s$ will be a slice, $s \in U_y$.

A foliation is said to be regular on a point p, if there exists a flat neighbourhood of p intersected by each leaf in at most one slice.

Let B be another differentiable manifold. An application $\Phi : M \to B$ is called of *decomposition* for the foliation (Hermann [2]) if :

1) Φ is a submersion
2) $\dim B + \dim F = \dim M$

(1) λ is determined *modulo* $\mathcal{J}_1$, $\mathcal{J}_1$ being the $\mathcal{F}$-module of the 1-forms that are null on the leaves.

3) $F_x = \Phi_*^{-1}(0)$, (F_x is the tangent space to the leaf on the point $x \in M$).

A decomposition application may not exist globally; we say that it exists locally if M can be covered by open sets (U) in such a way that there exists a decomposition application $\Phi_U : U \to B_U$, B_U being an open neighbourhood of B.

2. Equivalences between regularity and other properties.

THEOREM.

Let M *be an n-dimensional differentiable manifold in which there exists a foliation* F *of codimension q. The following conditions are equivalent :*

1) *There exists an application of* M *in another differentiable manifold of decomposition for the foliation.*

2) *There exists a differentiable application of maximum rank, which is constant on the leaves of* F, B *being q-dimensional.*

3) *There exists a differentiable application* Φ *of* M *in a differentiable manifold* B, *q-dimensional, such that the sequence*

$$0 \to F_x \xrightarrow{i_*} M_x \xrightarrow{\Phi_*} B_{\Phi(x)} \to 0$$

is exact (*i, application inclusion*).

4) *The foliation* F *is regular* (*The manifold* B *is in this case* M/F *and the application* $\pi_F : M \to M/F$ *will be of decomposition*).

5) M/F *is a differentiable manifold.* (Palais)

6) *There exists a differentiable application* Φ *of* M *in another differentiable manifold* N *where there exists a regular foliation* E. Φ *is a submersion and* $\Phi_* (F_x) = E_{\Phi(x)} \cdot (E_{\Phi(x)}$ *is the tangent space at the point* $\Phi(x)$ *to the leaf* E *of the foliation given in* N).

7) a) $M \times_{M/F} M$ *is a submanifold of* $M \times M$.

b) $pr_2 : M \times_{M/F} M \to M$ *is a submersion* (Godement).

Demonstration.

1) is equivalent to 2). In fact, if the application is of decomposition we will prove that Φ is constant on the leaves. Let $(x_1, ..., x_n ; U)$ be a

cubical system of local coordinates, centered on x, flat with respect to the foliation, and let $(y_1, ..., y_q ; V)$ be another cubical system of local coordinates centered on $\Phi(x)$. By the condition 3) of the decomposition application :

$$\Phi_* \left(\frac{\partial}{\partial x_a}\right)_x = 0, \tag{3}$$

$$(1 \leqslant a \leqslant n-q\,; \quad 1 \leqslant u, \quad v \leqslant q)$$

whence

$$\frac{\partial}{\partial x_a}(y_u \circ \Phi) = \Phi_* \left(\frac{\partial}{\partial x_a}\right) y_u = 0 \tag{4}$$

$$\frac{\partial}{\partial x_u}(y_u \circ \Phi) \neq 0 \tag{5}$$

(4) proves that $y_u \circ \Phi$ is independent from the coordinates $(x_1, ..., x_p)$. That is to say : to every slice there corresponds only one point, and to two different slices, two different points on account of (5). Since the leaf can be decomposed into slices that have common points we have proved that to each leaf there corresponds only one point, and that it cannot cut the neighbourhood in more than one slice ; therefore it is a regular foliation, from which we deduce that condition 1) implies condition 4).

Condition 3) is equivalent to the fact that the application is of maximum rank, so the equivalence of 1) and 3) is immediate. The equivalence 4) and 5) has been proved by Palais [4]. Let us see that 4) implies 1) : if the foliation is regular there exists a decomposition application of M on another differentiable manifold. This manifold is M/F; $\Pi_F : M \to M/F$ will be the decomposition application. In fact, let $(x_1, ..., x_n, U)$ be a coordinate system, flat with respect to F, centered at x. The fact that M/F has a manifold structure (Palais [4]) implies that there exists a coordinate system $(y_1, ..., y_q ; V)$ such that

$$y_u \cdot \Pi_F = x_{p+u}$$

which proves the differentiability of Π_F in $x \in U$, and also that Π_F is a submersion

$$(\Pi_F)_* \left(\frac{\partial}{\partial x_{p+u}}\right) = \frac{\partial}{\partial y_u}\,.$$

Because

$$(\Pi_F)_* \left(\frac{\partial}{\partial x_a} \right) = 0 \Rightarrow (\Pi_F)_* (F_x) = 0;$$

Π_F is therefore a decomposition application.

Let us see that 6) implies 4). Indeed, if the submersion of M exists in another differentiable manifold with regular foliation E, we get a similarly regular foliation, determined by the field of tangent subspaces obtained by considering on each point the tangent subspace applied by Φ_* on the space $E_{\Phi(x)}$ tangent to the leaf of the foliation E. The condition of coincidence with the leaf F given in M is secured by establishing $\Phi_* (F_x) = E_{\Phi(x)}$. If F is regular, Π_F is a particular case of submersion. That is to say : 4) implies 6). The points of M/F constitute the leaves of a regular zero-dimensional system.

The equivalence of 4) and 7) has been proved by Godement (Vid. : Serre [6]).

COROLLARY 1. — *If the foliation is determined by the q-form* (1) *such that* $d\omega = \lambda \wedge \omega$, *a necessary condition for the foliation to be regular is that* $\lambda \mid F$, *restriction of* λ *to the leaf, be in the null cohomology class.*

Demonstration.

If the foliation is regular by the condition 1) of the theorem, there exists an application of global decomposition, which implies that the group of holonomy of each leaf with respect to the foliation is the identity.

There exists a homomorphism between the fundamental group $\pi_1 (L)$ and the holonomy group of the leaf. In a coordinate system, an element of the holonomy group is represented by a $q \times q$ matrix. Let $m(\sigma)$ be the matrix corresponding to the element $\sigma \in \pi_1 (L)$. The application $\sigma \rightarrow \log \mid \det m(\sigma) \mid$ defines a homomorphism between $\pi_1 (L)$ and the additive group of the real numbers; that is to say it defines a real cocycle $\lambda_L : H_1 (L, R) \rightarrow R$. In the case we are considering, because

$$\lambda_L : H_1 (L, R) \rightarrow 0,$$

we deduce that the class of cohomology of λ_L, and therefore that of $\lambda \mid F$, is null, since the class of cohomology of λ_L and $\lambda \mid F$ are equal (Reeb [5]).

COROLLARY 2. — *Supposing that* M/F *is a differentiable connected, paracompact variety, a necessary and sufficient condition for a function f to exist, so that* $f\omega$ *be a measure of leaves in* M, *invariant for the foliation, is that the cohomology class of* λ *be zero mod* $\mathfrak{I}_1$.

Demonstration.

On account of the regularity of F, Π_F is an application of global decomposition. If $\lambda = dg$, g being a function, then if we put $\Omega = e^{-g}\,\omega$ we deduce $d\Omega = 0$, and therefore Ω is a measure of leaves (Vid. : Lichnerowicz [3]). Reciprocally, if $f\omega \neq 0$ in M is a measure of the leaves, invariant for the foliation, it will be $f\omega = \Pi_F^{*}(\theta_{M/F})$; $\theta_{M/F}$ is a measure in M/F,

$$d\,(f\omega) = df \wedge \omega + f\lambda \wedge \omega = \left(\frac{df}{f} + \lambda\right) \wedge f\omega = 0$$

which implies $\lambda = -\,d\,(\log f) \bmod \mathfrak{I}_1$, as we wanted to demonstrate.

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