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ON THE ANALYTICITY
OF GENERALIZED EIGENFUNCTIONS
(CASE OF REAL VARIABLES)

by Eberhard GERLACH

The present note is a direct continuation of Chapter III in our paper [1]; its purpose is to extend the results on analyticity of the generalized eigenfunctions to the case of proper functional Hilbert spaces consisting of functions which are (real-) analytic in a domain in Euclidean space. We continue to use the notation and numbering from Chap. III in [1].

Our basic tool will be the following.

Proposition 4. — Let $G$ be a domain in Euclidean space $\mathbb{R}^n$, and $\mathcal{B}$ a class of functions defined everywhere in $G$ and analytic there, and suppose that these form a proper functional Banach space $\{\mathcal{B}, G\}$. Then there exists a common domain $\tilde{G}$ in complex space $\mathbb{C}^n$, containing $G$, to which all $f \in \mathcal{B}$ can be extended analytically.

Proof. — Since $\{\mathcal{B}, G\}$ is a p.f. Banach space, to every $x \in G$ there is an $L(x) \in \mathcal{B}'$ ($\mathcal{B}'$ is the continuous dual of $\mathcal{B}$) such that $f(x) = \langle f, L(x) \rangle$. This defines a function $L$ from $G$ into $\mathcal{B}'$ which is weakly-* real-analytic. It is well-known that Banachspace-valued functions defined on a complex domain which are weakly or weakly-* analytic are complex-analytic also in the strong topology. We shall show that $L$ is strongly (real-) analytic; then it can be extended to a strongly analytic function $\tilde{L}$ (still into $\mathcal{B}'$) in some complex domain $\tilde{G}$ containing $G$. Finally each $f \in \mathcal{B}$ will be extended to an analytic function $\tilde{f}$ on $\tilde{G}$ by setting $f(z) = \langle f, \tilde{L}(z) \rangle$ for $z \in \tilde{G}$. 
Recall (cf. for instance [2]) that for any function \( g \) which is analytic in the fixed domain \( D \subset \mathbb{C}^1 \) and for any compact \( K \subset D \), there exists a finite number \( M(g; K) \) such that for any choice of \( \zeta, \zeta + \alpha, \zeta + \beta \) in \( K \):

\[
(6) \quad \left| \frac{1}{\alpha - \beta} \left\{ \frac{1}{\alpha} [g(\zeta + \alpha) - g(\zeta)] - \frac{1}{\beta} [g(\zeta + \beta) - g(\zeta)] \right\} \right| \leq M(g; K).
\]

The same is true if instead of \( D \) one has a fixed open interval \( I \subset \mathbb{R}^1 \).

We shall establish existence of the strong derivatives \( \frac{\partial}{\partial x_i} L(x) \) in \( \mathcal{B}' \). These derivatives exist in the weak-* topology since for each \( f \in \mathcal{B} \)

\[
\frac{\partial}{\partial x_i} f(x) = \lim_{h \to 0} \frac{1}{h} \left( f(x + \varepsilon_i h) - f(x) \right)
= \lim_{h \to 0} \frac{1}{h} \langle f, L(x + \varepsilon_i h) - L(x) \rangle.
\]

Let \( N \) be a compact neighborhood of \( x \); then there are numbers \( M(f; N) \) so that for all sufficiently small \( h \) and \( k \)

\[
\left| \langle f, \left\{ \frac{1}{h - k} \left[ L(x + \varepsilon_i h) - L(x) \right] - \frac{1}{k} L(x + \varepsilon_i k) - L(x) \right\} \rangle \right| \leq M(f; N).
\]

Then by the uniform boundedness theorem, there is a constant \( M(N) \) such that \( \left\| \frac{1}{h - k} \{ \ldots \} \right\| \leq M(N) \). Letting \( h \) and \( k \) tend to zero, one now obtains existence of the strong derivative \( \frac{\partial}{\partial x_i} L(x) \). Since all derivatives of the \( f \in \mathcal{B} \) are analytic, the preceding procedure can be repeated; thus \( L \) possesses strong derivatives of all orders. It is easy to check that \( L \) and all its derivatives are strongly continuous.

The Taylor series for \( L \) will converge strongly to the values of \( L \) if \( \| (x!)^{-1} D_x L(x) \| \frac{1}{x} \) is uniformly bounded on compacts \( K \subset G \), with a bound independent of \( x \). (Here the \( x_i \) are non-negative integers, \( x = (x_1, x_2, \ldots, x_n) \),...
\[ |\alpha| = \sum \alpha_i, \quad D_\alpha = \partial^{\alpha}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}, \quad \alpha! = \alpha_1! \ldots \alpha_n! \). Since all \( f \in \mathcal{B} \) are analytic,

\[
| (\alpha!)^{-1} D_\alpha f(x) | \frac{1}{|\alpha|} = | \langle f, (\alpha!)^{-1} D_\alpha L(x) \rangle | \frac{1}{|\alpha|}
\]

(for fixed \( f \)) is uniformly bounded on compacts \( K \subset G \), independent of \( \alpha \). But for variable \( f \), this expression is a sub-additive continuous functional on \( \mathcal{B} \). By the uniform boundedness theorem then

\[
\sup_{|\alpha| \leq 1} | \langle f, (\alpha!)^{-1} D_\alpha L(x) \rangle \frac{1}{|\alpha|} = \left( \sup_{|\alpha| = 1} | \langle \ldots \rangle | \right) \frac{1}{|\alpha|}
\]

\[
= | (\alpha!)^{-1} D_\alpha L(x) | \frac{1}{|\alpha|}
\]

is uniformly bounded on compacts \( K \subset G \), independent of \( \alpha \). Thus \( L \) has a strongly convergent power series expansion in some neighborhood of any point \( x \in G \).

For each \( x \in G \), let \( S(x) \) be the largest open ball in \( C^n \), centered at \( x \), in which the Taylor series for \( L \) about \( x \) converges and set

\[
\hat{G} = \bigcup_{x \in G} S(x).
\]

Then the series expansions yield an analytic continuation \( \tilde{L} \) of \( L \) from \( G \) to \( \hat{G} \). Finally, for \( f \in \mathcal{B} \), define

\[
\tilde{f}(\zeta) = \langle f, \tilde{L}(\zeta) \rangle \quad \text{for} \quad \zeta \in \hat{G} \quad \text{and} \quad |\tilde{f}| = |f|;
\]

this gives us a p.f. Banach space \( \{\mathcal{B}, \hat{G}\} \) which is isometrically isomorphic to \( \{\mathcal{B}, G\} \). The proof of Proposition 4 is complete.

From now on, \( \{\mathcal{F}, G\} \) will denote a p.f. Hilbert space consisting of analytic functions on a domain \( G \subset \mathbb{R}^n \). Our aim is to extend the results of Corollary 2. III and Theorem 3. III in [1] to such spaces.

The anti-space \( \mathcal{F}' \) of the Hilbert space \( \mathcal{F} \) is identified with the dual \( \mathcal{F}' \), and \( \mathcal{F} \) itself with its continuous anti-dual \( \mathcal{F}^* \)

\[ (= \mathcal{F}' = \mathcal{F}' \) \] by means of the canonical mappings \( J \) and \( \theta : \mathcal{F}' = J\mathcal{F} \) where \( J \) is the anti-isomorphism \( f \to Jf = (., f) \)

\[ (1) \] For these notations, cf. L. Schwartz [3].
and
\[ T^* = \varphi T \] where \( \varphi \) is the isomorphism \( f \rightarrow \varphi f = (f, \cdot) \).
If \( K \) is the reproducing kernel of \( T \) then for \( f \in F \)
\[ f(x) = (f, K(\cdot, x)) = (f, L(x)) \] for every \( x \in G \)
where \( (\cdot, \cdot) \) denotes the pairing of \( T \) and \( T' \). Thus
\[ L(x) = JK_x \] and
\[ K(x, y) = (K_y, K_x) = (K_y, JK_x) = (J^{-1}L(y), L(x)). \]
By Proposition 4, \( L \) and \( F \) extend analytically to a complex domain \( \tilde{G} \); we obtain the p.f. Hilbert space \( \{ \tilde{T}, \tilde{G} \} \) with r.k. \( \tilde{K} \):
\[ \tilde{f}(z) = (f, \tilde{L}(z)) = (f, J^{-1}\tilde{L}(z)) \]
and
\[ \tilde{K}(z, \omega) = (J^{-1}\tilde{L}(\omega), \tilde{L}(z)) = (\tilde{K}_z, \tilde{K}_\omega). \]
Since the function \( \tilde{L} \) is strongly analytic from \( \tilde{G} \) into \( \tilde{T} \) and \( \tilde{K}_z = J^{-1}\tilde{L}(z) \), we note that \( \tilde{K}(\cdot, z) \) is strongly anti-analytic for \( z \in \tilde{G} \) (i.e. the function \( \bar{z} \rightarrow \tilde{K}(\cdot, z) \) is strongly analytic from \( \tilde{G} = \{z|z \in \tilde{G}\} \) into \( \tilde{T}'\)). Let \( U \) denote the extension isomorphism \( U : T \rightarrow \tilde{T} \) constructed by Proposition 4. If \( \{g_k\} \) is a complete orthonormal system in \( \tilde{G} \), then so is \( \{g_k = Ug_k\} \) in \( \tilde{T} \) and \( \tilde{K}(z, \omega) = \sum_{k=1}^{\infty} \tilde{g}_k(z)\bar{\tilde{g}}_k(\omega) \) for \( z, \omega \in \tilde{G} \), i.e., \( \tilde{K} \) is also a « direct » continuation of \( K \).

**Corollary 2'.** — Let \( G \) be an arbitrary domain in \( \mathbb{R}^n \) and \( \{T, G\} \) any p.f. Hilbert space of functions (real-) analytic in \( G \). Then \( \{T, G\} \) is Hilbert-Schmidt expansible.

**Proof.** — By Corollary 2, there is an H.S. operator \( T \) in \( T \) such that \( K_\zeta \in T \tilde{T} \) for all \( \zeta \in \tilde{G} \). Now \( S = U^{-1}TU \) is H.S. in \( T \), and \( K_\zeta \in S \tilde{T} \) for all \( \zeta \in G \).

Now let \( \tilde{A} \) be a selfadjoint operator in \( \tilde{T} \) with resolution of identity \( E(.) \) and spectral measure \( \mu \). Then \( (f, g) = (Uf, Ug) \) for all \( f, g \in T \). The operator \( \tilde{A} = UAU^{-1} \) is selfadjoint in \( \tilde{T} \) and unitarily equivalent to \( A \); its resolution of identity is \( \tilde{E}(.) = UE(.)U^{-1} \), and \( \mu \) is also a spectral measure for \( \tilde{A} \).
Both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are H.S.-expansible. Let $\tilde{\Lambda}_0$ denote the complement in $\mathbb{R}^1$ of the set of all $\lambda$ for which
\[
\frac{d}{d\mu(\lambda)} \left( \frac{1}{d\mu(\lambda)} \right) = K(z, \omega; \lambda) \text{ exists and is finite for all } z, \omega \in \tilde{G}
\]
(similar definition for $\Lambda_0$, without tildas). Then $\tilde{\Lambda}_0 \supset \Lambda_0$ and $\mu(\tilde{\Lambda}_0) = 0$. Let $\mathcal{F}_0^{(0)} (\tilde{\mathcal{F}}_0^{(0)})$ be the p.f. Hilbert space on $\tilde{G}(G)$ defined by the r.k. $K(., .; \lambda)$ ($K(., .; \lambda)$). For $f \in \mathcal{F}$, let $\tilde{\Lambda}_{f, (\tilde{G})}$ be the smallest set containing $\tilde{\Lambda}_0$ such that for all $\lambda \in \tilde{\Lambda}_{f, (\tilde{G})}$:
\[
\begin{align*}
\frac{d}{d\mu(\lambda)} \left( \frac{1}{d\mu(\lambda)} \right) &= f(z; \lambda) \text{ exists, is finite} \\
\text{and } \mu(\tilde{\Lambda}_0) &= 0 \text{ whenever } K(z, z; \lambda) = 0, \text{ for all } z \in \tilde{G}
\end{align*}
\]
and
\[
\tilde{f}(., \lambda) \in \mathcal{F}_0^{(0)}, \quad \frac{d\|E(\lambda)f\|^2}{d\mu(\lambda)} \text{ exists and equals } \|f(., \lambda)\|_{\mathcal{F}_0^{(0)}}^2
\]
(similar definition for $\Lambda_{f, (G)}$, without tildas). The correspondence $f \mapsto \tilde{f}(., \lambda)$ defines $\tilde{\mathcal{F}}_0^{(0)}$ with domain $\tilde{\mathcal{F}}_0^{(0)} = \{f \lambda \in \tilde{\Lambda}_{f, (\tilde{G})}\}$. For $f \in \mathcal{F}_0^{(0)}$, $f(., \lambda)$ is just the restriction of $\tilde{f}(., \lambda)$ to the domain $\tilde{G}$.

**Theorem 3'.** — Let $A$ be an arbitrary selfadjoint operator in $\{\mathcal{F}, G\}$ with spectral measure $\mu$. Then there is a set $\Lambda$ on the real line, $\mu(\Lambda) = 0$, which is determined by Theorem 3 (and also Corollary 2, Theorem 11. I, and the above considerations) such that the generalized eigenfunctions
\[
\frac{dE(\lambda)f(x)}{d\mu(\lambda)} = f(x; \lambda) \in \mathcal{F}_0^{(0)} \text{ for } \lambda \in \Lambda \text{ and } f \in \mathcal{F}_0^{(0)}
\]
are real-analytic in the whole domain $G$.

**Proof.** — According to the preceding preparations, set $Uf = \tilde{f}$. If $\lambda \in \Lambda$ and $\tilde{f} \in \mathcal{F}_0^{(0)}$ then $\tilde{f}(., \lambda)$ is analytic in $\tilde{G}$ by Theorem 3, and consequently its restriction $f(., \lambda)$ is (real-) analytic in $G$. 

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