GIOVANNI VIDOSSICH

Characterization of separability for $LF$-spaces

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CHARACTERIZATION
OF SEPARABILITY FOR LF-SPACES
by Giovanni VIDOSSICH

This note characterizes the separability of LF-spaces by five equivalent conditions, one being that all members of a given defining sequence (=suite de définition according to [2]) are separable. These conditions imply that the dual space must be hereditarily separable and Lindelöf for the weak* topology (= topology σ(X', X) of [2]).

Concerning uniform spaces, we shall employ the terminology (and results) of the first two chapters of [4]. We shall denote by

\[ F \]
the scalar field, which is \( \mathbb{R} \) or \( \mathbb{C} \); and we shall say

weak topology induced by \( H \subseteq F(X, Y) \)

the less fine topology on \( X \) making continuous all members of \( H \) (caution that this is a purely topological definition). Finally, an \( \aleph_0 \)-space is — according to [5] — a regular space \( X \) where there exists a countable pseudobase \( \mathcal{P} \), i.e. a countable \( \mathcal{P} \subseteq \mathcal{P}(X) \) such that for every compact \( K \subseteq X \) and every open \( U \subseteq X \) which contains \( K \) it follows \( K \subseteq P \subseteq U \) for a suitable \( P \in \mathcal{P} \).

Theorem. — Let \( X \) be an LF-space and \( (E_n)_{n=1}^\infty \) a defining sequence of \( X \). The following statements are pairwise equivalent:

1. \( X \) is separable.
2. \( X \) is weakly separable.
3. Every weakly* compact subset of \( X' \) is weakly* metrizable.
Every equicontinuous subset of $X'$ has a countable base for the weak* topology.

(5) Every $E_n$ is separable.

(6) $X$ is an $\aleph_0$-space.

Proof. — (1) $\rightarrow$ (2) : Clear.

(2) $\rightarrow$ (3) : Let $e : X \rightarrow X''$ be the canonical map $x \mapsto (f(x))_{f \in X'}$ and $K$ a weakly* compact subset of $X'$. Then $e' : x \mapsto e(x)|_K$ is a continuous map from $X$ equipped with the weak topology into the topological subspace $e'(X)$ of $F_{p}(K, F)$ ($=$ product space of $\text{Card}(K)$ copies of $F$).

By (2), $e'(X)$ is separable: let $H$ be a countable dense subset of it. The weak* topology of $K$ is exactly the weak topology induced by $e'(X) \subseteq F(K, F)$: by [3, p. 175, Footnote], this topology equals the weak topology induced by $H \subseteq F(K, F)$ and therefore it is metrizable.

(3) $\rightarrow$ (4) : Because the weak* closure of an equicontinuous set is weakly* compact by [2, Th. 3].

(4) $\rightarrow$ (5) : By [2, Cor. to Th. 3], there is a linear homeomorphism $e$ from $X$ onto a subspace $e(X)$ of $L_{\mathfrak{g}}(X', F)$, $\mathfrak{C}$ being a suitable cover of $X'$ consisting of weakly* compact subsets of $X'$ and $L_{\mathfrak{g}}(X', F)$ the space of weakly* continuous linear functionals on $X'$ with the topology of uniform convergence on members of $\mathfrak{C}$. By [2, Th. 3], the members of $\mathfrak{C}$ are equicontinuous and hence weakly* metrizable by (4). By a well known theorem contained in [5, (J) and (D)], $C_{u}(K, F)$ ($=$ uniform space made of all weakly* continuous maps $K \rightarrow F$ and the uniformity of uniform convergence) is separable for all $K \in \mathfrak{C}$ and therefore the uniformity of $C_{u}(K, F)$ has a basis of countable uniform coverings (as follows easily, if you want, from [4, ii. 33 and ii. 9]). Consequently the uniformity $\pi$ of $\prod_{K \in \mathfrak{C}} C_{u}(K, F)$ — and hence the trace of $\pi$ on every subset — has a basis of countable uniform covers as it follows directly from the definition of product uniformity [4, Exercise ii. 2] (alternatively, this result may be deduced from [1, Prop. 3] and [4, ii. 33 and ii. 9]). It is well known that there is a uniform embedding $e^* : L_{\mathfrak{g}}(X', F) \rightarrow \prod_{K \in \mathfrak{C}} C_{u}(K, F)$, the last space being equipped with the product uniformity $\pi$. 

GIOVANNI VIDOSSICH
By what has been proved, the uniformity induced by \( \pi \) on \( e^\ast(e(X)) \) has a basis consisting of countable uniform coverings: consequently — because a linear homeomorphism is a uniform isomorphism for the (canonical) uniformities of linear topological spaces — the (canonical) uniformity of the linear topological space \( X \) has a basis of countable uniform covers, as well as its trace on every \( E_n \). But this trace coincides with the (canonical) uniformity of the linear topological space \( E_n(n \in \mathbb{Z}^+) \), hence it is metrizable and consequently separable (if \( \{ (U_{m,n})_{n=1}^\infty | n \in \mathbb{Z}^+ \} \) is a countable base of countable uniform covers for the uniformity of \( E_n \) and if \( x_{m,n} \) is an element of \( U_{m,n} \) whenever this set is not empty, then \( \{ x_{m,n} | m, n \in \mathbb{Z}^+ \} \) is dense in \( E_n \)).

(5) \( \rightarrow \) (6): By [2, Prop. 4], every compact subset of \( X \) is contained in some \( E_n \). This, together with the fact that \( X \) induces the original topology on each \( E_n \), implies that

\[
\bigcup_{n=1}^\infty \mathfrak{B}_n \text{ is a countable pseudobase for } X \text{ whenever } \mathfrak{B}_n \text{ is for } E_n(n \in \mathbb{Z}^+).
\]

(6) \( \rightarrow \) (1): By [5, (D) and (E)].

We remark that the idea of countable uniform covers may be used to show directly that every metrizable subgroup of a separable topological group must be separable.

The above theorem points out some important examples of non-metrizable \( \aleph_0 \)-spaces. [5, (J) and 10,4] imply some results on spaces of mappings between separable LF-spaces, of which we note only the following one.

**Corollary.**— *If an LF-space \( X \) is separable, then \( X' \) is weakly* hereditarily Lindelöf and separable.*

**Proof.**— By « (1) \( \longleftrightarrow \) (6) » of the above theorem and [5, (J) and (D), (E)].

BIBLIOGRAPHIE


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Giovanni Vidossich,
Vle XX Sett. 225,
54031 Avenza (Italie).