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# ON EXTENDING POTENTIAL THEORY TO ALL STRONG MARKOV PROCESSES (*)

by C. T. SHIH

## Introduction.

By a strong Markov process we mean one with stationary transition probabilities, taking values in a locally compact separable metric space, and having almost all paths right continuous ; moreover, the $\sigma$-fields relative to which stopping times and so the strong Markov property are defined are required to be right continuous and completed, and the resolvent is assumed to map bounded Borel measurable functions into such functions(1). Such a process $(X_t)$ is called a standard process (respectively a Hunt process) provided it satisfies the quasi-left-continuity on $[0, \xi)$, $\xi$ the lifetime (respectively on $[0, \infty)$) : if a sequence of stopping times $T_n$ increases to T, $X(T_n)$ converges to $X(T)$ almost surely on $\{T < \xi\}$ (respectively on $\{T < \infty\}$, with $X_t$ defined to be the point at infinity for $t \geqslant \xi$). The following facts are proved in [3] for a Hunt process $(X_t)$ : (i) for any analytic set A in the state space, the (first) hitting time

$$T_A(\omega) = \inf\{t > 0 \mid X_t(\omega) \in A\}$$

is a stopping time ; (ii) given such a set A and a fixed initial distribution for the process, there exists an increasing sequence of compact subsets $F_n$ of A such that $T_{F_n}$ decreases to $T_A$ almost surely ; (iii) if

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(1) This last condition is assumed in [1] in its definition of a standard process but not in [6] ; in view of the right continuity of the paths this condition is satisfied if the transition function $P(t, x, B)$ is Borel measurable in $x$ for each $t$ and Borel B. If the state space is only homeomorphic to a Borel set of a locally compact separable metric space while other conditions are satisfied, then as pointed out in [6] the process can be imbedded in a process as described here, consequently our results below (with the word analytic changed to Borel) hold for such a process.

the initial distribution does not charge the set of points in A but irregular for A, there exists a decreasing sequence of open supersets $U_n$ of A such that $T_{U_n}$ increases to $T_A$ almost surely. Although of a rather technical nature, these properties are instrumental in the development of Hunt's theory. For a standard process the same proof shows that (i) and (ii) are also valid ; while (iii) fails in general Blumenthal and Getoor [2] proved the following modification : (iii′) given any analytic set A and a fixed initial distribution, there exists a decreasing sequence of finely open supersets $U_n$ of A such that $T_{U_n}$ increases to $T_A$ almost surely. As a result standard processes have been taken as the basic class of processes to study potential theory, as is done in [1].

Now (i) is in fact true for any right continuous process $(X_t)$ on a locally compact separable metric space with right continuous and completed $\sigma$-fields, and in particular true for a strong Markov process. This follows from a general result, see [5 ; p. 72]. However, (ii) and (iii′) have not been known to hold for an arbitrary strong Markov process. The main result (Theorem 1) of this article is that this is indeed the case. In fact we are able to prove (ii) and (iii′) in slightly stronger versions. Also proven is a result (Theorem 2) which may perhaps be a sufficient substitute for the quasi-left-continuity itself in many important situations. Thus it seems reasonable to expect that basic potential theory can be studied for any strong Markov process.

The approach is to study the path behavior of a given strong Markov process in an enlarged state space which is again locally compact separable metric. In this enlarged space the resolvent operators map continuous functions vanishing at infinity into such functions. Thus this article is related to the papers of many authors (D. Ray, H. Kunita and T. Watanabe, F.B. Knight, P.A. Meyer, and J.L. Doob) on constructing strong Markov processes with nice path behavior from resolvents, where various ideas of enlarging the state space appear. The interested reader may like to compare the enlargement here with that in these papers, although our problem is different. Also, there have appeared in some of these papers results similar to Theorem 2 (for the processes constructed there).

## 1. Preliminaries and Results.

Let K be a compact metric space, with its metric denoted by $\rho$ and $\sigma$-field of Borel sets by $\mathcal{B}$. Let $\mathcal{M}$ be the space of bounded real-valued Borel measurable functions on K and $\mathcal{C}$ be its subspace of continuous functions. Let $\Delta$ be a fixed point in K. We shall consider a strong Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with K as its state space, in which $\Delta$ is the death point. Our notation follows that of [1]. Thus in the above $\mathcal{F}$ is the completion of $\sigma(X_t, t \geq 0)$, the $\sigma$-field generated by all the $X_t$, with respect to the family of measures $P^\mu$, $\mu$ a probability measure in $(K, \mathcal{B})$, and $\mathcal{F}_t$ is the completion of $\sigma(X_s, s \leq t)$ with respect to $\mathcal{F}$ and the family of measures $P^\mu$. A stopping time T is of course one relative to $(\mathcal{F}_t)$, i.e., satisfying $\{T \leq t\} \in \mathcal{F}_t$ for every $t$. The definition of the hitting time $T_A$ of a set A in K is already given. The time $D_A(\omega) = \inf\{t \geq 0 \mid X_t(\omega) \in A\}$ (of course $D_A(\omega) = \infty$ if $X_t(\omega) \in A$ for no $t$) will be called the (first) *entry* time of A. We assume that X satisfies the following conditions : (1) the paths $t \longrightarrow X_t(\omega)$ are right continuous almost surely(²) ; (2) $\mathcal{F}_t = \mathcal{F}_{t+} \equiv \bigcap_{s>t} \mathcal{F}_s$ for every $t$(³) ; (3) for every $\alpha > 0$, $f \in \mathcal{M}$, $U_\alpha f(x) = \int_0^\infty e^{-\alpha t} E^x[f(X_t)]\,dt$ is in $\mathcal{M}$. It should be remarked that the existence of shift operators for X is unessential, and for the same reason the lack of shift operators for the process Y introduced in section 2 does not cause difficulty.

We shall now define a new metric $\hat{\rho}$ on K. It is proved by Knight [4 ; Lemma 1] that there exists a subalgebra $\mathcal{A}$ of $\mathcal{M}$ that contains $\mathcal{C}$, has a countable dense subset, and is such that $U_\alpha(\mathcal{A}) \subset \mathcal{A}$ for all $\alpha > 0$. In fact, $\mathcal{A}$ may be obtained as follows. Choose $\{\alpha_k\}$ dense in $(0, \infty)$ ; define $\mathcal{A}_n$ inductively by setting $\mathcal{A}_1 = \mathcal{C}$, and after choosing $\{f_{nm}\}$ dense in the unit ball of $\mathcal{A}_n$, setting $\mathcal{A}_{n+1}$ to be the minimal algebra containing both $\mathcal{A}_n$ and $\{U_{\alpha_k} f_{nm}, k \geq 1, m \geq 1\}$. Then let $\mathcal{A} = \bigcup_n \mathcal{A}_n$. To define the metric $\hat{\rho}$ choose $\{g_m\}$ dense in the unit ball of $\mathcal{A}$ and let

(²) "Almost surely" (a.s.) means a.s. $P^\mu$ for every probability measure $\mu$ in $(K, \mathcal{B})$ ; this meaning will not be changed when we discuss the process Y below.

(³) This is the case if and only if $(\Omega, \mathcal{F}, \mathcal{F}_{t+}, X_t, \theta_t, P^x)$ is strong Markov.

$$\hat{\rho}(x\,,y) = \rho(x\,,y) + \sum_{m,k} 2^{-m-k}\alpha_k\,|\,U_{\alpha_k}g_m(x) - U_{\alpha_k}g_m(y)\,|\;.$$

Let $(\hat{K}\,,\hat{\rho})$ be the completion of $(K\,,\hat{\rho})$ ; it is compact. The functions in $\mathfrak{A}$ are uniformly continuous under $\hat{\rho}$, therefore have continuous extensions on $\hat{K}$. Thus $\mathfrak{A}$ can be regarded as a subalgebra of $\hat{\mathcal{C}}$, the space of real-valued continuous functions on $\hat{K}$. Clearly $\mathfrak{A}$ separates points of $\hat{K}$, and so is dense in $\hat{\mathcal{C}}$. Thus the $U_\alpha$ are extended by continuity as operators on $\hat{\mathcal{C}}$, and then through the Riesz representation theorem extended as operators on $\hat{\mathfrak{M}}$, the space of bounded real-valued Borel measurable functions on $\hat{K}$. Let $\hat{\mathcal{B}}$ denote the $\sigma$-field of Borel sets in K. Then clearly $\mathcal{B} = \hat{\mathcal{B}} \cap K$. We also have $K \in \hat{\mathcal{B}}$. For if $\eta$ is the continuous extension on $\hat{K}$ of the identity mapping from $(K\,,\hat{\rho})$ to $(K\,,\rho)$, then $K = \{z \in \hat{K} \mid U_{\alpha_k}g_m(\eta\,(z)) = U_{\alpha_k}g_m(z)$ for all $m$ and $k\}$, $U_{\alpha_k}g_m$ being the extended functions. Now $(X_t)$ can be regarded as a process in $(\hat{K}\,,\hat{\mathcal{B}})$, and as such we have for $f \in \hat{\mathfrak{M}}$, $x \in K\,,\alpha > 0$

$$U_\alpha f(x) = \int_0^\infty e^{-\lambda t}\,E^x\,[f(X_t)]\,dt\;,$$

where $U_\alpha$ is the extended operator on $\hat{\mathfrak{M}}$. Our results can now be stated as follows.

THEOREM 1. – *Let* A *be an analytic set in* $(K\,,\rho)$. *Then given any initial distribution* $\mu$, *there exist an increasing sequence of* $\hat{\rho}$-*compact*(4) *subsets* $F_n$ *of* A *and a decreasing sequence of* $\hat{\rho}$-*open supersets* $U_n$ *of* A *such that* $D_{F_n} \downarrow D_A$ a.s. $P^\mu$ *and* $D_{U_n} \uparrow D_A$ a.s. $P^\mu$.

THEOREM 2. – *There exists a Markov kernel (transition probability)* $\nu(z\,,B)$ *on* $(\hat{K}\,,\hat{\mathcal{B}})$ *such that with* $G = \{z \mid \nu(z\,,K) = 1\}$ *we have : for any sequence of stopping times* $T_n$ *increasing to* T, i) $\hat{\rho}\text{-}\lim_n X(T_n) \in G$ a.s. *on* $\{T < \infty\}$ ; ii) *given any initial distribution* $\mu, f \in \mathfrak{M}$,

$$E^\mu\{f(X(T)) \mid \bigvee_n \mathcal{F}(T_n)\} = \int_K \nu(\hat{\rho}\text{-}\lim_n X(T_n)\,,dx)\,f(x)\,(^5)$$

(4) I.e. compact with respect to $\hat{\rho}$ ; $\hat{\rho}$-open, $\hat{\rho}$-right-continuous, etc. have similar meanings.

(5) We use the convention $X_\infty \equiv \Delta$. $\mathcal{F}(S)$ for a stopping time S is the $\sigma$-field of sets $\Lambda$ satisfying $\Lambda \cap \{S \leqslant t\} \in \mathcal{F}_t$ for every $t$, and $\bigvee_n \mathcal{F}(T_n)$ is the minimal $\sigma$-field containing all the $\mathcal{F}(T_n)$.

on $\{T < \infty\}$, where $\hat{\rho}$-$\underset{n}{\text{limit}}$ $X(T_n)$ stands for any limit point of $X(T_n)$ in $\hat{K}$, (necessarily, the measure $\nu(\hat{\rho}\text{-limit } X(T_n), \cdot)$ is independent of the choice of the limit point, a.s. on $\{T < \infty\}$).

Theorem 1 implies properties (ii) and (iii′) about hitting times stated in the introduction (see [1] or [6]) ; in fact for (ii) the compact sets may be chosen among $\hat{\rho}$-open sets. That $\hat{\rho}$-open sets are finely open follows from the fact that almost all paths are $\hat{\rho}$-right-continuous, which will be established on the way of proving Theorem 1. Theorem 2 and the strong Markov property imply that if stopping times $T_n$ increase to T, then for any $\mu$ and bounded $\mathcal{F}$-measurable $\varphi$

$$E^{\mu}\{\varphi(\theta_T) \mid \bigvee_n \mathcal{F}(T_n)\} = \int_K \nu(\hat{\rho}\text{-}\underset{n}{\text{limit}}\ X(T_n), dx)\, E^x(\varphi)$$

on $\{T < \infty\}$.

In the proof of the above theorems we shall not need the full force of $U_\alpha(\hat{\mathcal{C}}) \subset \hat{\mathcal{C}}$ for all $\alpha > 0$ but only $U_\alpha(\hat{\mathcal{C}}) \subset \hat{\mathcal{C}}$ for a sequence of $\alpha$ increasing to infinity. Thus if we choose $\alpha_k \longrightarrow \infty$, find a subalgebra $\mathcal{A}$ of $\mathcal{M}$ containing $\mathcal{C}$ and a countable dense subset and such that $U_{\alpha_k}(\mathcal{A}) \subset \mathcal{A}$ for all $k$, and define $\hat{\rho}$ with the same formula, the same results will hold.

## 2. An Auxiliary Process(6).

We shall now regard X as a process in $(\hat{K}, \hat{\mathcal{B}})$ (any initial distribution has to concentrate on K). The right continuity of almost all paths is then lost ; but as we shall see eventually it is only apparently so. Define a new process $(Y_t)$ as follows. First let

(6) Professor P.A. Meyer has pointed out that it is unnecessary to introduce the auxiliary process Y. Indeed, from his theorem [6 ; XIV, T11] it follows that almost all paths of X are $\hat{\rho}$-right-continuous. Using this fact our proofs would be somewhat simplified (Theorem 1 would then be proved at the end of section 2). However, the proof of the important [6 ; XIV, T11] is based on the deep theorem [5 ; VIII, T21], also due to Meyer. The present paper establishes the approximation property (ii) of hitting times directly and using this property one can also prove [6 ; XIV, T11] (see [1 ; p. 75]). Hence we have kept the present proof of Theorem 1 for the interest of exhibiting an alternative proof of [6 ; XIV, T11].

$$Y_t = X_t \quad , \quad t \geqslant 0 \text{ rational} .$$

Now given $\alpha > 0$, $f \in \mathfrak{M}$, the function $t \longrightarrow U_\alpha f(X_t(\omega))$ on the non-negative rationals has right hand limits at all $s \in [0 , \infty)$ (and left hand limits at all $s \in (0 , \infty)$) almost surely. Letting $\alpha$ run through $\{\alpha_k\}$ and $f$ run through $\{g_m\}$ and using the right continuity of X in $(K , \rho)$, we obtain the following : almost surely $t \longrightarrow Y_t(\omega)$ has right hand limits in $(\hat{K} , \hat{\rho})$ at all $s \in [0 , \infty)$ (and has left hand limits in $(\hat{K} , \hat{\rho})$ at all $s \in (0 , \infty)$ for which it has a left hand limit in $(K , \rho)$). Now for any irrational $t$ define $Y_t(\omega)$ to be the limit of $Y_s(\omega)$ in $(\hat{K} , \hat{\rho})$ as $s \downarrow t$ through the rationals, if this limit exists, and any point in $\hat{K}$ otherwise.

If stopping times $T_n \downarrow T_\infty$, then for $\alpha > 0$, $f \in \mathfrak{M}$,

$$\lim_n U_\alpha f(X(T_n)) = U_\alpha f(X(T_\infty)) \text{ a.s.}$$

This is because for $f \geqslant 0$, $\{e^{-\alpha T_n} U_\alpha f(X(T_n)), \mathcal{F}(T_n), 1 \leqslant n \leqslant \infty\}$ is a reversed submartingale with respect to any $P^\mu$,

$$E^\mu \{e^{-\alpha T_n} U_\alpha f(X(T_n)) \mid \mathcal{F}(T_\infty)\} \uparrow E^\mu \{e^{-\alpha T_\infty} U_\alpha f(X(T_\infty)) \mid \mathcal{F}(T_\infty)\}$$

and

$$\mathcal{F}(T_\infty) = \bigcap_{1 \leqslant n < \infty} \mathcal{F}(T_n),$$

the last fact following from the right continuity of $(\mathcal{F}_t)$. It follows that $X(T_n)$ converges to $X(T_\infty)$ under $\hat{\rho}$ almost surely. The same argument shows that if $t \downarrow s$ through the rationals, then $X(t)$ converges to $X(s)$ under $\hat{\rho}$ almost surely ; this implies immediately i) of the following.

PROPOSITION 2.1. – i) $t \longrightarrow Y_t(\omega)$ *is $\hat{\rho}$-right-continuous a.s.* ii) *For any stopping time* T, $Y(T) = X(T)$ a.s. (*by convention* $X(\infty) \equiv Y(\infty) \equiv \Delta$). iii) $(Y_t)$ *and* $(X_t)$ *are equivalent under and* $P^\mu$. iv) $Y = (\Omega , \mathcal{F}, \mathcal{F}_t , Y_t , P^x)$ *is a strong Markov process, and* $\{U_\alpha , \alpha > 0\}$ *is the resolvent of* Y *on* $\mathfrak{M}$.

*Proof.* – To show ii) choose stopping times $T_n$ taking rational values and decreasing to T. Then $Y(T_n) = X(T_n)$ converges to $X(T)$ under $\hat{\rho}$ a.s. Now ii) follows from i). iii) of course needs no proof. iv) follows from ii), iii), and the strong Markov property of X.

Note that both X and Y cannot start at points in $\hat{K} - K$. Also Y does not have shift operators ; but as remarked earlier it does not concern us. We now study the entry times $\tau_A = \inf\{t \geqslant 0 \mid Y_t \in A\}$, $A \subset \hat{K}$, for the process Y. Because of a result stated in the introduction $\tau_A$ is a stopping time for any analytic A in $(\hat{K}, \hat{\rho})$. In the rest of this section only the space $(\hat{K}, \hat{\rho})$ will be involved, and we shall avoid mentioning it every time. Let $\{f_n\}$ be dense in $\hat{\mathcal{C}}$ and let

$$E_1 = \{z \in \hat{K} \mid \lim_{\alpha \to \infty} \alpha U_\alpha f_n(z) = f_n(z) \text{ for all } n\},$$

$$E_2 = \{z \in \hat{K} \mid \lim_{\alpha \to \infty} \alpha U_\alpha f_n(z) \text{ exists for all } n\} - E_1,$$

$$E_3 = \hat{K} - E_1 - E_2 .$$

These are Borel sets in $\hat{K}$ since $\overline{\lim_{\alpha \to \infty}} \alpha U_\alpha f_n$ and $\underline{\lim_{\alpha \to \infty}} \alpha U_\alpha f_n$ are Borel measurable, $\alpha U_\alpha f_n$ being continuous. Note that

$$\lim_{\alpha \to \infty} \alpha U_\alpha f_n(z) = f_n(z)$$

for all $n$ implies $\lim_{\alpha \to \infty} \alpha U_\alpha f(z) = f(z)$ for all $f \in \hat{\mathcal{C}}$. We shall show that both $\tau_{E_2}$ and $\tau_{E_3}$ are infinite almost surely.

PROPOSITION 2.2. – $K \subset E_1$.

*Proof.* – Let $f \in \hat{\mathcal{C}}$, $x \in K$. Then

$$\alpha U_\alpha f(x) = \alpha \int_0^\infty e^{-\alpha t} E^x [f(Y_t)]\, dt = E^x \int_0^\infty \alpha e^{-\alpha t} f(Y_t) dt \longrightarrow f(x)$$

as $\alpha \longrightarrow \infty$, by Proposition 2.1.

Let $E_4 = \bigcup_m \bigcup_n \bigcup_k \bigcap_{\alpha > k} \{z \in \hat{K} \,\|\, \alpha U_\alpha f_n(z) - f_n(z) | \geqslant 1/m\}$. Then $E_2 \subset E_4 \subset \hat{K} - E_1$ and $E_4$ is $\sigma$-compact. Now if F is a compact subset of $\hat{K} - E_1$, then $Y(\tau_F) \in \hat{K} - E_1$ a.s. on $\{\tau_F < \infty\}$ ; but $\tau_F$ being a stopping time, we have $Y(\tau_F) = X(\tau_F)$ a.s. from Proposition 2.1. Hence Proposition 2.2 implies $\tau_F = \infty$ a.s. If follows that $\tau_{E_4} = \infty$ a.s. and so we have

PROPOSITION 2.3. – $\tau_{E_2} = \infty$ a.s.

In order to prove $\tau_{E_3} = \infty$ a.s. we introduce the (first) *contact* times for the process Y. The contact time $\sigma_A$ of a subset A of $\hat{K}$ is defined by :

$$\sigma_A(\omega) = \inf\{t \geq 0 \mid \overline{Y_{[0,t]}(\omega)} \cap A \neq \emptyset\}$$

where $\overline{Y_{[0,t]}(\omega)}$ is the closure of the set $\{z \in \hat{K} \mid z = Y_s(\omega)$ for some $s$ in $[0, t]\}$. Since almost all paths of Y are right continuous, it is clear that for an open set U, $\sigma_U$ is a stopping time and $\sigma_U = \tau_U$ a.s. If F is compact and $U_n$ are decreasing open sets with $\bar{U}_n \downarrow F$, then clearly $\sigma_{U_n} \downarrow \sigma_F$. Based on this fact one can apply Choquet's capacity theorem to establish the following result (see [6] or [1]).

PROPOSITION 2.4. – *Let* A *be an analytic set in* $\hat{K}$. *Then* i) $\sigma_A$ *is a stopping time ;* ii) *given any* $\mu$, *there exist an increasing sequence of compact subsets* $F_n$ *and a decreasing sequence of open supersets* $U_n$ *of* A *such that* $\sigma_{F_n} \downarrow \sigma_A$ a.s. $P^\mu$ *and* $\sigma_{U_n} \uparrow \sigma_A$ a.s. $P^\mu$.

Since $\sigma_A \leq \tau_A$ for any A, we shall prove $\tau_{E_3} = \infty$ a.s. by showing $\sigma_{E_3} = \infty$ a.s. In view of ii) of the above proposition it suffices to prove the following

PROPOSITION 2.5. – $\sigma_F = \infty$ a.s. *for any compact subset* F *of* $E_3$.

*Proof.* – Let $U_n$ be open with $\bar{U}_n \downarrow F$. Set $T_n = \sigma_{U_n}$ and $T = \lim_n T_n = \sigma_F$. For a fixed $P^\mu$ let $Q(\omega, B)$ be a regular conditional probability distribution of $Y(T)$ given $\bigvee_n \mathcal{F}(T_n)$. (Recall $Y_\infty \equiv \Delta$). Since $Y(T) \in K$ a.s., $Q(\omega, \cdot)$ is concentrated on K a.s. $P^\mu(d\omega)$. We show that for any $\alpha > 0$, $f \in \hat{\mathfrak{M}}$

$$\lim_n U_\alpha f(Y_{T_n}(\omega)) = \int Q(\omega, dz)\, U_\alpha f(z) \tag{2.1}$$

a.s. $P^\mu(d\omega)$ on $\{T < \infty\}$. That the left hand side exists a.s. on $\{T < \infty\}$ follows from Proposition 2.1. An easy application of the strong Markov property of Y shows

$$E^\mu\{U_\alpha f(Y(T)) \mid \bigvee_n \mathcal{F}(T_n)\} = E^\mu\{\lim_n U_\alpha f(Y(T_n)) \mid \bigvee_n \mathcal{F}(T_n)\}$$

on $\{T < \infty\}$ ; see e.g. [6 ; p. 52]. (Regard the integrand on the right as 0 on $\{T = \infty\}$, here and below). Thus (2.1) follows from the definition of $Q(\omega, B)$. Now there exists a $P^\mu$-null set $\Gamma$ such that if $\omega \in \{T < \infty\} - \Gamma$ then the equality (2.1) holds symmultaneously for all $f \in \{f_m\}$, and for all rational $\alpha > 0$. We may assume furthermore that for $\omega \notin \Gamma$, $Q(\omega, \cdot)$ is concentrated on K and $t \longrightarrow Y_t(\omega)$ is right continuous. Let $\omega \in \{T < \infty\} - \Gamma$. Then as $\alpha \longrightarrow \infty$,

$$\int Q(\omega, dz)\, \alpha U_\alpha f_m(z) \longrightarrow \int Q(\omega, dz)\, f_m(z)$$

in view of Proposition 2.2. Hence $\lim_n \alpha U_\alpha f_m(Y_{T_n}(\omega))$ converges as $\alpha \longrightarrow \infty$, for every $m$. But let $z$ be a limit point of $Y_{T_n}(\omega)$, say $z = \lim_k Y_{T_{n_k}}(\omega)$ ; then since $Y_{T_n}(\omega) \in U_n$, we have $z \in F$. Now the continuity of $U_\alpha f_m$ implies $\lim_k U_\alpha f_m(Y_{T_{n_k}}(\omega)) = U_\alpha f_m(z)$. The above then implies that $\alpha U_\alpha f_m(z)$ converges as $\alpha \longrightarrow \infty$, for every $m$. Since this contradicts the fact $z \in F \subset E_3$, we must have $\{T < \infty\} - \Gamma = \emptyset$, so that $T = \infty$ a.s. $P^\mu$.

As remarked before, we now have $\sigma_{E_3} = \infty$ a.s. and so

COROLLARY 2.6. – $\tau_{E_3} = \infty$ a.s.

PROPOSITION 2.7. – $\tau_F = \sigma_F$ a.s. *for any compact subset* F *of* $E_1$.

*Proof.* – Again let $U_n$ be open with $\bar{U}_n \downarrow F$, and let $T_n = \sigma_{U_n}$ and $T = \lim_n T_n = \sigma_F$. Since $\lim_{\alpha\to\infty} \alpha U_\alpha f_m = f_m$ on $E_1$, the functions $\alpha U_\alpha f_m$, $m \geq 1$, $\alpha$ rational, distinguish points in F. For a fixed $P^\mu$, there exists a $P^\mu$-null set $\Gamma$ such that if $\omega \in \{T<\infty\} - \Gamma$, $\alpha U_\alpha f(Y_{T_n}(\omega))$ converges as $n \longrightarrow \infty$ for all $f \in \{f_m\}$ and all rational $\alpha > 0$, and $t \longrightarrow Y_t(\omega)$ is right continuous. For such an $\omega$, $\{Y_{T_n}(\omega)\}$ cannot have two distinct limit points since they must be in F. Hence $Y(T_n)$ converges a.s. $P^\mu$ on $\{T < \infty\}$. Now for $f \in \hat{\mathcal{C}}$ we have

$$\begin{aligned} E^\mu\{\alpha U_\alpha f(Y(T)) ; T < \infty\} &= E^\mu\{\lim_n \alpha U_\alpha f(Y(T_n)) ; T < \infty\} \\ &= E^\mu\{\alpha U_\alpha f(\lim_n Y(T_n)) ; T < \infty\}. \end{aligned}$$

Since $Y(T)$ and $\lim_n Y(T_n)$ are in $E_1$ a.s. on $\{T < \infty\}$, the above yields as $\alpha \longrightarrow \infty$

$$E^{\mu}\{f(Y(T)) \; ; \; T < \infty\} = E^{\mu}\{f(\lim_n Y(T_n)) \; ; \; T < \infty\} .$$

This being true for all $f \in \hat{\mathcal{C}}$, it holds for all $f \in \hat{\mathfrak{M}}$. With $f = 1_F$ we obtain $P^{\mu}(Y(T) \in F \; ; T < \infty) = P^{\mu}(\lim_n Y(T_n) \in F \; ; T < \infty) = P^{\mu}(T < \infty)$. Thus $\tau_F \leqslant T$ a.s. $P^{\mu}$ and so $\tau_F = \sigma_F$ a.s. $P^{\mu}$.

THEOREM 2.8. – *Let* A *be any analytic set in* $\hat{K}$. *Then given any* $\mu$, *there exist an increasing sequence of compact subsets* $F_n$ *of* $A \cap E_1$ *and a decreasing sequence of open supersets* $U_n$ *of* $A \cap E_1$ *such that* $\tau_{F_n} \downarrow \tau_A$ a.s. $P^{\mu}$ *and* $\tau_{U_n} \uparrow \tau_A$ a.s. $P^{\mu}$.

*Proof.* – In view of Proposition 2.3 and Corollary 2.6, we may assume $A \subset E_1$. For a fixed $P^{\mu}$, there exist increasing compact subsets $F_n$ of A such that $\sigma_{F_n} \downarrow \sigma_A$ a.s. $P^{\mu}$ (Proposition 2.4). Now $\sigma_A \geqslant \tau_A$ and by Proposition 2.7 $\tau_{F_n} = \sigma_{F_n}$ a.s. $P^{\mu}$ for all $n$. Hence $\tau_{F_n} \downarrow \tau_A$ a.s. $P^{\mu}$. Since $\tau_U = \sigma_U$ a.s. for any open U, from Proposition 2.4 ii) there exist decreasing open supersets $U_n$ of A such that $\tau_{U_n} \uparrow \tau_A$ a.s. $P^{\mu}$.

## 3. Proof of Theorems 1 and 2.

Recall the continuous extension $\eta$ on $\hat{K}$ of the mapping $x \longrightarrow x$ from $(K, \hat{\rho})$ to $(K, \rho)$. For $A \subset K$, let $\hat{A} = \eta^{-1}(A)$. If A is an analytic set in $(\hat{K}, \hat{\rho})$, $\hat{A}$ is an analytic set in $(\hat{K}, \hat{\rho})$. Since $Y_t = X_t$ for rational $t$, and almost surely $t \longrightarrow X_t(\omega)$ is $\rho$-right-continuous and $t \longrightarrow Y_t(\omega)$ is $\hat{\rho}$-right-continuous, we have for almost all $\omega$, $\eta(Y_t(\omega)) = X_t(\omega)$ for all $t$. Thus for any A, $\{D_A \neq \tau_{\hat{A}}\}$ is a $P^{\mu}$-null set for every $\mu$.

THEOREM 3.1. – *Let* A *be an analytic set in* $(K, \rho)$. *Then given any* $\mu$, *there exists an increasing sequence of* $\rho$*-compact subsets* $F_n$ *of* A *such that* $D_{F_n} \downarrow D_A$ a.s. $P^{\mu}$.

*Proof.* – Since $\hat{A}$ is analytic in $(\hat{K}, \hat{\rho})$, from Theorem 2.8 there exist increasing $\hat{\rho}$-compact subsets $C_n$ of $\hat{A}$ such that $\tau_{C_n} \downarrow \tau_A$ a.s. $P^\mu$. Let $F_n = \eta(C_n)$. Then the $F_n$ are $\rho$-compact subsets of A. Since $C_n \subset \hat{F}_n$ we have $\tau_{\hat{F}_n} \leqslant \tau_{C_n}$. Hence $D_{F_n} = \tau_{\hat{F}_n} \downarrow \tau_{\hat{A}} = D_A$ a.s. $P^\mu$.

The above theorem implies that the same result holds for the hitting times $T_A$, $T_{F_n}$, in place of the entry times $D_A$, $D_{F_n}$. It then follows that for $\alpha > 0, f \in \mathfrak{M}$, $t \longrightarrow U_\alpha f(X_t(\omega))$ is right continuous on $[0, \infty)$ and has left hand limits on $(0, \infty)$ almost surely ; see [1 ; p. 75] and footnote 6). Hence we have the following.

COROLLARY 3.2. – *Almost surely,* $t \longrightarrow X_t(\omega)$ *is* $\hat{\rho}$*-right-continuous (and has a* $\hat{\rho}$*-left-hand-limit at any* $t > 0$ *for which a* $\rho$*-left-hand-limit exists).*

In view of the corollary we have almost surely, $X_t(\omega) = Y_t(\omega)$ for all $t$. Thus the two processes X and Y not merely are equivalent, but have identical paths almost surely. In particular $D_A = \tau_A$ a.s. for any $A \subset \hat{K}$.

*Proof of Theorem 1.* – We have $A = \hat{A} \cap K$. Since $\hat{A}$ is analytic in $(\hat{K}, \hat{\rho})$ and so is K, being a Borel set in $\hat{K}$, A is analytic in $(\hat{K}, \hat{\rho})$. The theorem then follows from Theorem 2.8 and the remark preceding the proof.

For $z \in E_1 \cup E_2$, $\alpha U_\alpha f(z)$ converges as $\alpha \longrightarrow \infty$ for every $f \in \hat{\mathcal{C}}$. Hence there exists a probability measure $\nu(z, \cdot)$ on $\hat{K}$ such that for $f \in \hat{\mathcal{C}}$, $\lim\limits_{\alpha \to \infty} \alpha U_\alpha f(z) = \int \nu(z, dz') f(z')$. Of course $\nu(z, \cdot)$ is the unit mass at $z$ for $z \in E_1$. Define $\nu(z, \cdot)$ to be the unit mass at $z$ for $z \in E_3$. Then $\nu(z, B)$ is a Markov kernel (transition probability) on $(\hat{K}, \hat{\mathcal{B}})$. Let $G = \{z \in \hat{K} \mid \nu(z, \cdot)$ is concentrated on $K\}$. We have $K \subset G \subset E_1 \cup E_2$.

*Proof of Theorem 2.* – Since $\sigma_{E_3} = \infty$ a.s., $\{X_{T_n}(\omega)\}$ has no limit point (under $\hat{\rho}$) in $E_3$ for almost every $\omega$ with $T(\omega) < \infty$. Now for any $\mu$, $f \in \hat{\mathfrak{M}}$, $\alpha > 0$, and $\Lambda \in \bigvee_n \mathcal{F}(T_n)$

$$E^\mu\{\alpha U_\alpha f(X(T)) ; \Lambda, T < \infty\} = E^\mu\{\lim_n \alpha U_\alpha f(X(T_n)) ; \Lambda, T < \infty\}. \tag{3.1}$$

If $f \in \hat{\mathcal{C}}$, the integrand on the right can be written as

$$\alpha U_\alpha f(\hat{\rho}\text{-}\lim_n \mathrm{it}\ X_{T_n}(\omega)) \tag{3.2}$$

on $\{T < \infty\}$, where, again, $\hat{\rho}\text{-}\lim_n \mathrm{it}\ X_{T_n}(\omega)$ denotes any limit point of $\{X_{T_n}(\omega)\}$ in $(\hat{K}, \hat{\rho})$. Since $\hat{\rho}\text{-}\lim_n \mathrm{it}\ X(T_n) \notin E_3$ a.s. on $\{T < \infty\}$, (3.2) converges as $\alpha \longrightarrow \infty$ to

$$\int_{\hat{K}} \nu(\hat{\rho}\text{-}\lim_n \mathrm{it}\ X_{T_n}(\omega), dz) f(z) \tag{3.3}$$

a.s. on $\{T < \infty\}$. As $f$ runs through a countable dense set in $\hat{\mathcal{C}}$, we see that for almost every $\omega$ in $\{T < \infty\}$ the measure $\nu(\hat{\rho}\text{-}\lim_n \mathrm{it}\ X_{T_n}(\omega), \cdot)$ is independent of the choice of the limit point of $\{X_{T_n}(\omega)\}$. In particular (3.3) stands unambiguously as an $\omega$-function on $\{T < \infty\}$ for any $f \in \hat{\mathfrak{M}}$, which is of course measurable with respect to

$$\bigvee_n \mathcal{F}(T_n) \cap \{T < \infty\} .$$

Letting $\alpha \longrightarrow \infty$ in (3.1) we obtain

$$E^\mu\{f(X(T)); \Lambda, T<\infty\} = E^\mu \left\{ \int_{\hat{K}} \nu(\hat{\rho}\text{-}\lim_n \mathrm{it}\ X(T_n), dz) f(z); \Lambda, T<\infty \right\}$$

for all $f \in \hat{\mathfrak{M}}$. With $f = 1_K$, the left hand side equals $P^\mu(\Lambda \cap \{T < \infty\})$; it follows that $\nu(\hat{\rho}\text{-}\lim_n \mathrm{it}\ X(T_n), K) = 1$ a.s. $P^\mu$ on $\Lambda \cap \{T < \infty\}$. i) of the theorem follows by taking $\Lambda = \Omega$ and ii) follows from i) and the above equality.

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