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EVERY COMPACT SET
IN $\mathbb{C}^n$
IS A GOOD COMPACT SET

by Jan-Erik BJORK

Introduction.

The aim of this note is to establish a conjecture of A. Martineau in [1]. Before we state the Main Theorem which settles this conjecture we introduce some notations.

We denote the $n$-dimensional complex vector space by $\mathbb{C}^n$ and we assume that $n$ coordinates $z_1 \ldots z_n$ are given. By a polynomial we always mean a polynomial in the coordinate functions $z_1 \ldots z_n$. Next we let $W$ be an open set in $\mathbb{C}^n$. We denote by $O(W)$ the algebra of all holomorphic functions on $W$. If $K$ is a compact subset of $W$ we shall put $O^s(W) = \{ f \in O(W) : \text{there exists a sequence } (P_n) \text{ of polynomials and some open neighborhood } U_f \text{ of } K \text{ such that } \lim |P_n - f|_{U_f} = 0, \text{i.e. } f \text{ can be uniformly approximated by polynomials in some neighborhood of } K \}$.

It is obvious that $O_K(W)$ is a subalgebra of $O(W)$. But it is not clear that $O_K(W)$ is a closed subalgebra of the Frechet algebra $O(W)$, where $O(W)$ is equipped with the topology of uniform convergence on compact subsets of $W$. The Main Theorem shows that $O_K(W)$ is a closed subalgebra of $O(W)$, which means that $K$ is a good compact set in the sense of ([1], p. 18, Definition 1.10).

Main Theorem. — Let $K$ be a compact set in $\mathbb{C}^n$ and let $W$ be an open set containing $K$. Then there exists an open neighborhood $U_0$ of $K$ such that if $f \in O_K(W)$, then $\lim |P_n - f|_{U_0} = 0$ for some sequence of polynomials.

Before we begin the proof we remark that we shall use some basic ideas which are developed in ([2], Chapter 1, Section G), and
except for a direct application of the Oka-Weil Theorem concerning polynomial approximation in $\mathbb{C}^n$ no deep methods of several complex variables are used. The discussion which follows contains the preliminary steps towards the proof of the Main Theorem, and it is entirely based on the content from [2].

Preliminaries.

Let $W$ be an open subset of $\mathbb{C}^n$ and suppose that $A$ is a closed subalgebra of the Frechet algebra $O(W)$. We also assume that $A$ contains the polynomials and that $A$ is closed under derivation, i.e. if $f \in A$ then $\delta f/\delta z_i \in A$ for each $i = 1 \ldots n$. We denote by $\text{Spec}(A)$ the collection of all non zero continuous complex-valued homomorphisms on the Frechet algebra $A$. The continuity means that to each point $x$ in $\text{Spec}(A)$ there exists a compact set $S$ in $W$ such that $|x(f)| \leq |f|_S$ for all $f$ in $A$. Each element $f$ of $A$ may be considered as a complex-valued function $\hat{f}$ on $\text{Spec}(A)$ if we define $\hat{f}(x) = x(f)$. Because $A$ contains the polynomials we can determine a map $\pi$ from $\text{Spec}(A)$ into $\mathbb{C}^n$ as follows. If $x \in \text{Spec}(A)$ then the map which sends each polynomial $P$ into $P(x)$ determines a unique point $\pi(x)$ in $\mathbb{C}^n$ for which $P(\pi(x)) = P(x)$ holds for all polynomials.

Next we use the fact that $A$ is closed under derivations to obtain some properties of the map $\pi$. Take a point $x_0 \in \text{Spec}(A)$ and suppose that $|\hat{f}(x_0)| \leq |f|_K$ holds for all $f$ in $A$ and for some compact set $K$ in $W$. We choose $\varepsilon > 0$ so small that if $K_{\varepsilon} = \{z \in \mathbb{C}^n : d(z, K) \leq \varepsilon\}$, then $K_{\varepsilon} \subset W$. Here we are using the metric

$$d(z, w) = \sup \{|z_i - w_i| : i = 1 \ldots n\}$$

in $\mathbb{C}^n$. If we now take a point $z$ in $\mathbb{C}^n$ for which $d(z, 0) < \varepsilon$ and if we define the map $L_z : f \rightarrow \Sigma T_k(f)z^k : k = (k_1 \ldots k_n)$ and where $T_k(f) = (D^k f)^*(x_0)/k!$, then $L_z$ is a homomorphism on $A$ for which $|L_z(f)| \leq |f|_{K_{\varepsilon}}$ holds. (See p. 47 in [2]). It follows that $L_z$ determines a point $x(z)$ in $\text{Spec}(A)$, and here $\pi(x(z)) = \pi(x_0) + z$ holds.

We take the sets (defined for large values of $n$)

$$W_n(x_0) = \{x(z) : d(z, 0) < 1/n\}$$
as a basis for neighborhoods around the point \( x_0 \) in \( \text{Spec}(A) \). Since \( x_0 \) is an arbitrary point in \( \text{Spec}(A) \) this enables us to define a topology in \( \text{Spec}(A) \). In this way \( \text{Spec}(A) \) becomes a locally compact space and the map \( \pi \) defines a local homeomorphism from \( \text{Spec}(A) \) into \( \mathbb{C}^n \). It can also be proved that \( \pi \) defines an analytic structure in \( \text{Spec}(A) \) in which the functions from \( A \) become analytic. Using this fact a classical argument by Weierstrass shows that \( \text{Spec}(A) \) is a metric space too.

The open set \( W \) can be identified with an open subset of \( \text{Spec}(A) \), for each point evaluation in \( W \) determines a homomorphism on \( A \). We denote this open subset of \( \text{Spec}(A) \) with \( j(W) \) and we notice that the restriction of \( \pi \) to the set \( j(W) \) maps \( j(W) \) homeomorphically onto \( W \). If \( z \in W \) we denote by \( j(z) \) the unique point in \( j(W) \) for which \( \pi(j(z)) = z \) holds. With these notations we see that \( f(z) = \tilde{f}(j(z)) \) holds for all \( f \) in \( A \).

We remark here that the topology introduced on \( \text{Spec}(A) \) above actually coincides with the weak \( A \)-topology on \( \text{Spec}(A) \). This implies for example that \( \text{Spec}(A) \) is a Stein manifold (See p. 55, Theorem 18 in [2]). The result is deep and is originally due to K. Oka when \( A = \mathcal{O}(W) \) and proved for a general \( A \) by E. Bishop. However we shall not need this result in the proof of the Main Theorem.

**Proof of the Main Theorem.**

Firstly we show that \( \mathcal{O}_K(W) \) is a subalgebra of \( \mathcal{O}(W) \) which is closed under derivation. For if \( f \in \mathcal{O}_K(W) \) and if \( \lim |P_n - f|_{U_f} = 0 \) for some open neighborhood \( U_f \) of \( K \), then it follows that

\[
\lim |\delta P_n / \delta z_i - \delta f / \delta z_i|_V = 0
\]

if \( V \) is a subset of \( U_f \) such that its closure \( cl(V) \) is contained in \( U_f \). So if we let \( V \) be an open neighborhood of \( K \) such that \( cl(V) \subset U_f \), then it follows that \( \delta f / \delta z_i \in \mathcal{O}_K(W) \).

We denote by \( A \) the closure of \( \mathcal{O}_K(W) \) in the Frechet algebra \( \mathcal{O}(W) \). Hence \( A \) is a subalgebra of \( \mathcal{O}(W) \) satisfying the conditions of the preceding discussion. So now we can introduce the set \( \text{Spec}(A) \) and the associated map \( \pi \) from \( \text{Spec}(A) \) into \( \mathbb{C}^n \). The idea of the
proof which follows is to derive some properties of the map \( \pi \). Firstly we shall need the following property of \( A \).

Let \( f \in A \). Since \( K \) is a compact subset of \( W \) it follows that \( \lim |g_n - f|_K = 0 \) for some sequence \( (g_n) \) in \( \mathcal{O}_K(W) \). To each \( g_n \) we can determine a polynomial \( P_n \) such that \( |P_n - g_n|_K < 1/n \). It follows that \( \lim |P_n - f|_K = 0 \), i.e. we have proved that each function in \( A \) can be uniformly approximated by polynomials on \( K \).

Now we introduce the set

\[
\text{Hull}(K) = \{ x \in \text{Spec}(A) : |\hat{f}(x)| \leq |f|_K \text{ for all } f \in A \}.
\]

We claim that \( \pi \) determines a homeomorphism from \( \text{Hull}(K) \) onto \( P(K) \), the polynomially convex hull of \( K \) in \( \mathbb{C}^n \).

Firstly we take a point \( z \) in \( P(K) \). If \( / \in A \) then \( \lim |P_n(z) - P_m(z) |_{\mathcal{O}_K} = 0 \), so that \( \lim P_n(z) \) exists. So if we define \( L(f) = \lim P_n(z) \), where \( (P_n) \) are polynomials such that \( \lim |f - P_n|_K = 0 \), then we see that \( L \) is a continuous complex-valued homomorphism on \( A \). Hence \( L(f) = \hat{f}(x) \) for some point \( x \) in \( \text{Hull}(K) \) and here \( \pi(x) = z \) holds. So we have proved that \( \pi \) maps \( \text{Hull}(K) \) onto \( P(K) \). Suppose next that \( x_1, x_2 \in \text{Hull}(K) \) are such that \( \pi(x_1) = \pi(x_2) = z \) for some \( z \) in \( \mathbb{C}^n \). Since \( |P(z)| = |\hat{P}(x_i)| \leq |P|_K \) for all polynomials it follows that \( z \in P(K) \). Suppose now that \( f \in A \) and let \( \lim |f - P_n|_K = 0 \) for some polynomials \( P_n \). Since \( x_i \in \text{Hull}(K) \) it follows that \( \lim |f(x_i) - \hat{P}_n(x_i)| \leq \lim |f - P_n|_K = 0 \). Hence

\[
\hat{f}(x_i) = \lim \hat{P}_n(x_i) = \lim P_n(z)
\]

holds, so it follows that \( \hat{f}(x_1) = \hat{f}(x_2) \). Since this holds for all \( f \) in \( A \) we conclude that \( x_1 = x_2 \) in \( \text{Spec}(A) \). Notice that we also have proved that \( \pi(x_1) \in P(K) \) here. Together with the previous result it follows that \( \pi(\text{Hull}(K)) = P(K) \) and if \( z \in P(K) \) then the set \( \pi^{-1}(z) \cap \text{Hull}(K) \) consists of one point.

Next a topological consideration will show that there exists an open neighborhood \( U \) of \( \text{Hull}(K) \) in \( \text{Spec}(A) \) such that the restriction of \( \pi \) to the set \( U \) maps \( U \) homeomorphically onto the open set \( \pi(U) \) in \( \mathbb{C}^n \). For if no such \( U \) exists then we can find a sequence \( (x_n, y_n) \) of pairs of points in \( \text{Spec}(A) \) for which \( \pi(x_n) = \pi(y_n) \), while \( x_n \) and \( y_n \) both converge to the set \( \text{Hull}(K) \). Since we already know
that \( \pi \) maps \( \text{Hull}(K) \) homeomorphically onto \( P(K) \) it follows that \( \text{Hull}(K) \) is a compact subset of \( \text{Spec}(A) \). So we can pass to a subsequence and assume that \( \lim x_n = x_0 \) and that \( \lim y_n = y_0 \) both exist in \( \text{Spec}(A) \). Clearly \( x_0 \) and \( y_0 \) belong to \( \text{Hull}(K) \). We also have \( \pi(x_0) = \lim \pi(x_n) = \pi(y_n) = \pi(y_0) \). Hence \( x_0 = y_0 \) follows, and now we have derived a contradiction because \( \pi \) is a local homeomorphism in a neighborhood of \( x_0 \).

So we have proved that there exists an open set \( U \) in \( \text{Spec}(A) \) such that \( \text{Hull}(K) \) is contained in \( U \) while \( \pi \) maps \( U \) homeomorphically onto the open set \( \pi(U) \) in \( \mathbb{C}^n \). Suppose now that \( f \in A \). Then we define a function \( f_0 \) on \( \pi(U) \) as follows. If \( z \in \pi(U) \) we put \( f_0(z) = \hat{f}(x(z)) \), where \( x(z) \) is the unique point in the set \( \pi^{-1}(z) \cap U \). The properties of \( \pi \) show that \( f_0 \) is analytic on \( \pi(U) \). Clearly \( P(K) \) is contained in \( \pi(U) \) so we can choose a compact polynomially convex set \( S \), where \( P(K) \) is contained in the interior of \( S \) while \( S \) is contained in \( \pi(U) \).

Then the Oka-Weil Theorem shows that if \( f \in A \) then there exists a sequence \( (P_n) \) of polynomials for which \( \lim |P_n - f_0|_S = 0 \).

Now we can use the set \( S \) above to finish the proof. Let \( f \in A \) so that \( f_0 \) is determined on \( \pi(U) \). If \( z \in K \) we see that

\[
f_0(z) = \hat{f}(x(z)) = f(z),
\]

where \( f(z) \) denotes the original value of the function \( f \) considered as an element of \( O(W) \). We claim that there exists an open set \( U_1 \) in \( \mathbb{C}^n \), where \( K \subset U_1 \subset (\pi(U) \cap W) \) and for which \( f_0(z) = f(z) \) holds for all \( f \) in \( A \) and for all \( z \in U_1 \).

For let \( z_0 \) be a point in \( K \). Then \( j(z_0) \in U \) so we can choose an open neighborhood \( \Delta \) of \( j(z_0) \) in \( \text{Spec}(A) \) such that \( \Delta \) is contained in the set \( j(W) \cap U \). It follows that if \( z \in \pi(\Delta) \) and if \( x(z) \) is the point in \( U \) for which \( \pi(x(z)) = z \), then \( x(z) = j(z) \) must hold. Hence \( f_0(z) = \hat{f}(x(z)) = \hat{f}(j(z)) = f(z) \). Since \( \pi(\Delta) \) is an open set in \( \mathbb{C}^n \) here we can cover \( K \) by a finite union of such open sets and obtain the set \( U_1 \).

(1) The proof gives a sharper version of the Main Theorem as follows. Let \( f \in O(W) \) be such that \( f \) together with all its derivatives can be uniformly approximated by polynomials on \( K \). Then \( f \) can be uniformly approximated on \( U_0 \) by polynomials.
Finally we put $U_0 = U_1 \cap \text{int}(S)$, so that $U_0$ is an open neighborhood of $K$ in $\mathbb{C}^n$. If now $f \in A$ then we can determine a sequence $(P_n)$ of polynomials for which $\lim |P_n - f_0|_S = 0$. It follows that $\lim |P_n - f|_{U_0} = 0$, so that $U_0$ gives the required neighborhood in the Main Theorem.

BIBLIOGRAPHY


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