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ON CONTINUOUS COLLECTIONS OF MEASURES

by **R. M. BLUMENTHAL** and **H. H. CORSON**

1. Introduction.

Let M be a compact space and X a complete metric space. By a probability measure on X we mean a positive regular Borel measure of total mass 1; let P(X) denote the collection of these. Let C(X) denote the set of bounded continuous real valued functions on X, C(M, X) the set of continuous functions from M to X. Let P(X) have the weak topology as functionals on C(X) and give C(M, X) the topology of uniform convergence.

There are many commonly used objects in mathematics which can be viewed as continuous functions from M to P(X). We will first give some examples of these.

Example 1. — Any continuous T from M to X may be viewed in this way, considering $x \in X$ as a point mass.

Example 2. — If T is a continuous function from M into a finite dimensional simplex or simplicial complex, then T may be viewed as a continuous map from M to P(X) where X is the set of vertices under the discrete topology, since there is an obvious choice for the measure corresponding to T(t).

Example 3. — Let X be compact, and let T be a positive linear operator from C(X) to C(M) such that T(1) = 1. Then the Riesz representation theorem implies that there is a unique continuous map from M to P(X) corresponding to T; in fact it is just the adjoint of T. Conversely, any continuous map from M to P(X) corresponds to such an operator. Note that T is multiplicative if and only if T corresponds to a function from M into X, as in Example 1.

Example 4. — Let X be a partition of unity of M and give X the discrete topology. Define T from M to P(X) by $T(t)g = g(t)$ for g in X. Then T is continuous. This is similiar to the situation immediately following the proof of the Lemma in Section 2.

Before giving the last, basic example we need more notation : Let Y and X be complete metric, and let π be a continuous function from Y into X. Then π induces a mapping, also denoted by π, from P(Y) to P(X) and defined by $\pi\mu(E) = \mu(\pi^{-1}(E))$. Also, for $t \in M$ denote simply by t the mapping $f \to f(t)$ from C(M, X) into X. Hence we see that $\mu \to t\mu$ from P(C(M, X)) to P(X) is defined and continuous for any X by taking Y to be C(M, X) and t to be π in the definition of $\mu \to \pi\mu$ above.

Example 5. — For a fixed $\mu \in P(C(M, X))$, the mapping $t \to t\mu$ is continuous from M to P(X).

In this paper we show that any continuous map from M to P(X) is of the form given in Example 5, provided that M is totally disconnected. As we see from Example 3, this establishes an integral representation theorem which generalizes the theorem of F. Riesz for compact metric X: For any such X and any positive linear operator T from C(X) to C(M) with $T(1) = 1$ and M totally disconnected, there is a regular Borel measure μ on the space of multiplicative operators under the strong operator topology such that $Tf(t) = \int Sf(t)\, d\mu(S)$. Obviously the Riesz theorem is the case where M is a space with just one element.

The most obvious shortcomings of this statement are first that M is very special (although it is clear that the restriction on M is essential for a conclusion in this generality) and second that it is not clear which of the μ on C(M, X) are to be preferred, since several of them can give rise to the same mapping from M into P(X). As far as the connectivity of M goes, see e.g. [3] for a discussion of its significance.

2.

THEOREM. — *Let* M *be compact and totally disconnected and let* X *be a complete metric space. Then for each continuous function* T *from* M *into* $P(X)$ *there is a* μ *in* $P(C(M, X))$ *such that* $t\mu = T(t)$ *for all* t *in* M.

We will first prove a lemma which treats a special case and provides some additional information for use in the general case. Before stating it we need one more piece of notation. Let X and Y be complete metric spaces and $\pi : Y \to X$ a continuous mapping. Then π induces a continuous mapping from $C(M, Y)$ to $C(M, X)$ by $(\pi\varphi)(t) = \pi(\varphi(t))$, $t \in M$, $\varphi \in C(M, Y)$. We denote this mapping by π also. If μ is a measure on $C(M, Y)$ then it is simply an exercise in unscrambling the notation to check that the measures $t\pi\mu$ and $\pi t\mu$ on X are the same. In fact,

$$t\pi\mu(E) = \pi\mu\{f \in C(M, X) : tf = f(t) \in E\}$$
$$= \mu\{f \in C(M, Y) : \pi f(t) \in E\},$$

and

$$\pi t\mu(E) = t\mu\{y \in Y : \pi y \in E\} = \mu\{f \in C(M, Y) : \pi tf = \pi f(t) \in E\}.$$

LEMMA. — *Let* X *and* Y *be discrete spaces and let* π *be a continuous mapping from* Y *onto* X. *Let* T *be a continuous mapping from* M *into* $P(Y)$ *and let* μ *be a measure on* $C(M, X)$ *such that* $t\mu = \pi T(t)$. *Then there is a measure* ν *in* $P(C(M, Y))$ *such that* (1) $t\nu = T(t)$ *for all* t *and* (2) $\pi\nu = \mu$.

Proof. — Consider a positive regular Borel measure θ on $C(M, Y)$ having mass $\leqslant 1$ (but perhaps not a probability measure) and such that (1) $t\theta \leqslant T(t)$ and (2) $\pi\theta \leqslant \mu$ (an inequality between positive measures means set-wise inequality). Let A denote the set of all such measures. Of course, A is non-empty, for example the 0 measure is in A, and we will now show that there is a non-zero measure in A. Indeed $C(M, X)$ is a discrete space and so there is an element $f \in C(M, X)$ such that $\mu(\{f\}) = \varepsilon > 0$. Suppose that f

takes on values $x_1, \ldots, x_n$ on sets $M_1, \ldots, M_n$ respectively and let $Y_i = \pi^{-1}(x_i)$, $1 \leqslant i \leqslant n$. For each $y \in Y$ the function $t \to T(t)(y)$ is continuous, and for each i $\sum_{y \in Y_i} T(t)(y) \geqslant \varepsilon$ for all $t \in M_i$. Since M_i is compact and totally disconnected, each point has arbitrarily small open and closed neighborhoods [2, page 20]. Hence we may find a finite cover $\mathfrak{U}_i$ of M_i and a $\delta_i > 0$ such that each element of $\mathfrak{U}_i$ is open and closed and for $U \in \mathfrak{U}_i$ there is a $y_U \in Y_i$ such that $T(t)(y_U) \geqslant \delta_i$ for all $t \in U$. In fact we may clearly choose $\mathfrak{U}_i$ to be a partition of M_i. For each i and each $U \in \mathfrak{U}_i$ define $g(t) = y_U$ for $t \in U$. If δ is the minimum of the δ_i then $\delta > 0$ and $T(t)(g(t)) \geqslant \delta$ for all $t \in M$. If we let θ be the measure putting mass $(\delta \wedge \varepsilon)$ at the point $g \in C(M, Y)$ then θ is non-zero and is an element of A. Now we return to the proof of the lemma. The set A is inductively ordered: indeed if K is a totally ordered subset of A and we take $\mu_1 \leqslant \mu_2 \leqslant \cdots$ from K such that $\lim_n \mu_n(C(M, Y)) = \sup_{\mu \in K} \mu(C(M, Y))$ then $\alpha = \lim_n \mu_n$ is an element of A and $\alpha \geqslant \rho$ for every $\rho \in K$. Let θ be a maximal element of A. If θ has total mass 1 then $t\theta = T(t)$ and $\pi\theta = \mu$. If θ has mass $\eta < 1$ then we may apply the first part of the proof to the mapping

$$T'(t) = (T(t) - t\theta)/1 - \eta$$

and measure $\mu' = (\mu - \pi\theta)/1 - \eta$. This will yield a strictly positive measure θ' with $t\theta' \leqslant T'(t)$ and $\pi\theta' \leqslant \mu'$ and then $\theta + (1 - \eta)\theta'$ will be an element of A strictly exceeding θ. This completes the proof.

Now we return to the proof of the theorem. For each $n \geqslant 0$ let F_n be a partition of unity on X subordinate to a cover of diameter less than $1/n$. Give F_n the discrete topology. We take F_0 to be the trivial partition consisting of the function 1. Let X_n be the subspace of $F_0 \times \ldots \times F_n$ consisting of all $(g_0, \ldots, g_n)$ such that $g_0 g_1 \ldots g_n$ is not identically 0. Let $\pi_n : X_n \to X_{n-1}$ be the mapping that sends $(g_0, \ldots, g_{n-1}, g_n)$ to $(g_0, \ldots, g_{n-1})$. Let

$$G = \{(g_0, g_1, \ldots) \in F_0 \times F_1 \times \ldots : (g_0, \ldots, g_n \in X_n) \text{ for all } n\},$$

and let $\tilde{G} = C(M, G)$. Then $\tilde{G}$ is simply

$$\{(f_0, f_1, \ldots) \in C(M, F_0) X \ldots : (f_0, \ldots, f_n) \in C(M, X_n) \text{ for all } n\}.$$

G and $\tilde{G}$ are closed subsets of ΠF_i and $\Pi C(M, F_i)$ respectively.

Let T be a continuous mapping from M into P(X). Then for each n, T induces a continuous mapping T_n of M into $P(X_n)$ by the rule $T_n(t)[(g_0, \ldots, g_n)] = \int g_0 g_1 \ldots g_n \, dT(t)$. Clearly, $\pi_n T_n(t) = T_{n-1}(t)$. When $n = 0$ we have of course the trivial measure μ_0 putting mass 1 on the one point of $C(M, X_0)$ so that $t\mu_0 = T_0(t)$ for all t. Consequently by repeatedly applying the lemma we obtain a sequence of measures $\mu_n \in P(C(M, X_n))$ such that $t\mu_n = T_n(t)$ for all t and $\pi_n \mu_n = \mu_{n-1}$ for all n. By Kolomogorov's Consistency Theorem [1, Th. 5, 11, page 120] there is a measure $\tilde{\mu}$ in $P(\tilde{G})$ such that $P_n \tilde{\mu} = \mu_n$ for all n. (Here P_n stands for the natural projection of G onto X_n or for any of its other interpretations as a mapping of continuous functions or of measures.)

Since X is complete we can define a continuous function $\varphi : G \to X$ by taking $\varphi(g_0, g_1, \ldots)$ to be the unique point $x \in X$ such that $x \in \operatorname{supp}(g_n)$ for all n. As usual φ may be regarded also as a continuous function from $\tilde{G}$ to $C(M, X)$. Let $\mu = \varphi \tilde{\mu}$ so that $\mu \in P(C(M, X))$. We will complete the proof by showing that $t\mu = T(t)$ for all t. Let K be a closed subset of X. Write K_n for $\{(g_0, \ldots, g_n) \in X_n$ such that $\operatorname{supp}\ (g_0 g_1 \ldots g_n) \cap K \neq \varnothing\}$ and h_n for $\Sigma g_0 \ldots g_n$, the sum being over all $(g_0, \ldots, g_n) \in K_n$. Then $h_n \to I_K$ boundedly as $n \to \infty$ and $P_n^{-1}(K_n)$ decreases to $\varphi^{-1}(K)$. Since $t\varphi\tilde{\mu} = \varphi t \tilde{\mu}$ and $t\mu_n = P_n t \tilde{\mu}$ we have

$$\begin{aligned} T(t)(K) &= \lim_n \int h_n \, dT(t) \\ &= \lim_n \sum_{K_n} \int (g_0 \ldots g_n) \, dT(t) \\ &= \lim_n t\mu_n(K_n) = \lim_n t\tilde{\mu}(P_n^{-1}(K_n)) \\ &= t\tilde{\mu}(\varphi^{-1}(K)) = \varphi t \tilde{\mu}(K) = t\varphi\tilde{\mu}(K) = t\mu(K). \end{aligned}$$

A measure on a metric space is determined by its values on closed sets, so the proof is complete.

3.

In this section we will give two corollaries. The first is simply Prohorov's theorem on tightness of compact sets of measures.

COROLLARY 1. — *Let* X *be complete metric and* K *a compact subset of* $P(X)$. *Then for every* $\varepsilon > 0$ *there is a compact subset* K_ε *of* X *such that* $\nu(K_\varepsilon) \geqslant 1 - \varepsilon$ *for every* $\nu \in K$.

Proof. — There is a totally disconnected compact space M and a continuous mapping $T: M \rightarrow P(X)$ such that $T(M) = K$. Such an M may be constructed by letting M be the Stone-Cech compactification of the *set* K under the discrete topology. The identity map extends continuously over M, and it is a simple, well known exercise to check that M has the required properties. Let μ be a measure in $P(C(M, X))$ such that $t\mu = T(t)$ for all $t \in M$. By the regularity of μ there is a compact subset L_ε of $C(M, X)$ such that $\mu(L_\varepsilon) \geqslant 1 - \varepsilon$. Then $K_\varepsilon = \{f(t) : f \in L_\varepsilon, t \in M\}$ satisfies the conclusion of the theorem.

As stated above, it is not possible to find many measures on $C(M, X)$ if M is not totally disconnected. The reason for this is that there are not many functions from M to X that are continuous. However, our theorem gives some information in this situation, if we allow more functions.

In fact, let $\mathfrak{R}(M)$ be any collection of functions on M, and let rM be the set M with the weakest topology such that each $f \in \mathfrak{R}(M)$ is continuous. Suppose that the Stone-Cech compactification βrM is totally disconnected. If T is a continuous function from rM to PX such that $T(rM)$ is contained in a compact subset of PX, then T may be extended over βrM and our theorem gives a measure on $C(\beta rM, X)$. However, $C(\beta rM, X)$ may be considered as a collection of functions from M to X, and by a suitable choice of $\mathfrak{R}(M)$ one gets results such as the next corollary.

COROLLARY 2. — *Let* I *denote the unit interval, and let* T *be a continuous function from* I *to* $P(X)$ *or more generally a right continuous function from* I *to* PX *such that* $T(I)$

is contained in a compact subset of $P(X)$. *Then there is a regular Borel probability measure* μ *on the space of right continuous functions from* I *to* X *under the uniform topology such that* $t\mu = T(t)$ *for all* t *in* I.

Proof. — Pick $\mathcal{R}(I)$ to be the right continuous functions on I, and proceed as above.

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