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## THEOREMS OF KREIN-MILMAN TYPE FOR CERTAIN CONVEX SETS OF FUNCTIONS AND OPERATORS

by Robert R. PHELPS

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Let  $X$  be a compact Hausdorff space and  $E$  a real [or complex] locally convex Hausdorff vector space. Denote by  $C(X, E)$  the real [or complex] linear space of all continuous functions from  $X$  to  $E$ , provided with the topology of uniform convergence. (Thus, a typical neighborhood of  $0$  has the form

$$\{f \in C(X, E) : \sup_x p(f(x)) \leq 1\}$$

where  $p$  is a continuous seminorm on  $E$ .)

For any subset  $A$  of  $E$  we let

$$C(X, A) = \{f \in C(X, E) : f(X) \subset A\}.$$

It follows that if  $B$  is a bounded closed convex subset of  $E$ , then  $C(X, B)$  is a bounded closed convex subset of  $C(X, E)$ . We denote by  $\text{exts}$  the set of extreme points of a given convex set  $S$ ; it is readily verified that

$$C(X, \text{ext } B) \subset \text{ext } C(X, B).$$

[It is known [2, p. 755] that this inclusion can be proper, even for four dimensional  $E$ . There are also examples where  $\text{ext } C(X, B)$  is empty for every  $X$ ; for instance, if  $E = c_0$  in the norm topology and  $B$  is its unit ball.] The main purpose of this note is to exhibit conditions under which the set  $C(X, B)$  will be the closed convex hull  $\overline{c_0} C(X, \text{ext } B)$  of this subset of extreme points.

[Note that, even for one dimensional  $B$ , the set  $C(X, B)$  need not be compact, so the Krein-Milman theorem does not apply.] Our main result was proved in two special cases in [6] (Theorems 2.1 and 4.1), where applications were made to various convex sets of bounded (or of compact) linear operators from a Banach space into  $C(X)$ . The more general result of the present note may be applied to analogous sets of *weakly* compact operators. We give one such application, as well as two results which were overlooked in [6].

As in [6], the problem is handled in two steps. First, we consider a condition (D) (below) on a pair of spaces  $(X, A)$ , with  $X$  compact Hausdorff and  $A$  bounded in  $E$ , which implies that

$$C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A).$$

[This formulation was first considered by G. Seever [7].] We then apply this to bounded closed convex subsets  $B$  of  $E$  such that (with  $A = \text{ext } B$ ), the pair  $(X, A)$  satisfies condition (D) and  $B = \overline{\text{co}} A$ .

**DEFINITION.** — *A pair of Hausdorff spaces  $(X, A)$  is said to satisfy condition (D) if the following holds for each  $n > 0$ :*

*Given nonempty open sets  $U_1, U_2, \dots, U_n$  in  $A$  and pairwise disjoint nonempty compact sets  $K_1, K_2, \dots, K_n$  in  $X$ , there exists  $f \in C(X, A)$  such that  $f(K_i) \subset U_i$ ,  $i = 1, 2, \dots, n$ .*

Condition (D) is a sort of density property for the subspace  $C(X, A)$  in the space  $A^X$  of all functions from  $X$  to  $A$ . Indeed, condition (D) implies that  $C(X, A)$  is dense in the pointwise topology on the space  $A^X$ , while density of  $C(X, A)$  in the compact-open topology implies condition (D). As noted in [6], if  $X$  is a totally disconnected compact space, then  $(X, A)$  satisfies condition (D) for any  $A$ . On the other hand, if  $A$  is arcwise connected (or even only « almost arcwise connected » [6]), then  $(X, A)$  satisfies (D) for any compact  $X$ .

**THEOREM 1.** — *Let  $E$  be a real or complex locally convex Hausdorff vector space,  $X$  a compact Hausdorff space and  $A$  a bounded subset of  $E$ . If  $(X, A)$  satisfies condition (D), then*

$$C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A).$$

The proof of the theorem depends on the following two technical lemmas.

LEMMA 1. — Suppose that  $L$  is a continuous linear functional on  $C(X, E)$ . Then there exists a continuous seminorm  $p$  on  $E$  and a regular Borel positive measure  $\mu$  on  $X$  such that  $\mu(X) \leq 1$  and

$$|L(f)| \leq \int_X p(f(x)) d\mu(x) \quad (\text{for each } f \in C(X, E)).$$

LEMMA 2. — Suppose that  $X$  and  $A$  are as described in the statement of the theorem. Given a continuous seminorm  $p$  on  $E$ ,  $\varepsilon > 0$ , a regular Borel probability measure  $\mu$  on  $X$  and  $f \in C(X, \overline{\text{co}} A)$ , there exist  $g \in \overline{\text{co}} C(X, A)$  and a compact subset  $K \subset X$  such that

$$p(g(x) - f(x)) < \varepsilon \quad \text{for } x \in K \quad \text{and} \quad \mu(X \setminus K) < \varepsilon.$$

Assuming that these lemmas have been proved, the theorem follows readily. Indeed, since  $C(X, A) \subset C(X, \overline{\text{co}} A)$  and since the latter is closed and convex, we have

$$\overline{\text{co}} C(X, A) \subset C(X, \overline{\text{co}} A).$$

To show equality, it suffices to show that for each  $\varepsilon > 0$ , each  $L \in C(X, E)^*$  and each  $f \in C(X, \overline{\text{co}} A)$ , there exists  $g \in \overline{\text{co}} C(X, A)$  with

$$\text{Re } L(g) > \text{Re } L(f) - \varepsilon.$$

Choose  $p$  and  $\mu$  according to Lemma 1, and let  $M = \sup\{p(a) : a \in A\}$ . Choose  $K \subset X$  and  $g \in \overline{\text{co}} C(X, A)$  according to Lemma 2, with  $\varepsilon$  replaced by  $\varepsilon/2(M + 1)$ . It follows that

$$\text{Re } L(f) - \text{Re } L(g) \leq |L(f - g)| \leq \int_X p(f(x) - g(x)) d\mu(x).$$

The integral on the right is the sum of the integral over  $K$  and the integral over  $X \setminus K$ . From Lemma 2, the first summand is at most  $\varepsilon/2(M + 1)$ , while the second is at most  $M\varepsilon/2(M + 1)$ , hence the total is at most  $\varepsilon$ .

We now turn to the proof of Lemma 1. Since  $L$  is continuous on  $C(X, E)$  it is bounded in absolute value by 1 on a

neighborhood of the form

$$\{f \in C(X, E) : p(f(x)) \leq 1, x \in X\},$$

where  $p$  is a continuous seminorm on  $E$ . Thus,

$$(*) \quad |L(f)| \leq \sup\{p(f(x)) : x \in X\}, \quad f \in C(X, E).$$

Let  $N$  denote the closed subspace  $p^{-1}(0)$  and consider the space  $E/N$ , normed by the quotient norm  $\|\cdots\|$  defined by  $p$ . Let  $\varphi$  denote the quotient map from  $E$  into  $F = E/N$ ; the composition  $f \rightarrow \varphi \circ f$  defines a linear mapping of  $C(X, E)$  into  $C(X, F)$  which satisfies

$$\|\varphi(f(x))\| = p(f(x))$$

for all  $f \in C(X, E)$ ,  $x \in X$ . The space  $C(X, F)$  has the norm

$$\|g\| = \sup\{\|g(x)\| : x \in X\}.$$

It follows from (\*) that the formula  $J(\varphi \circ f) = L(f)$  defines a continuous linear functional  $J$  of norm at most 1 on the subspace  $\varphi \circ C(X, E)$  of  $C(X, F)$ , and we can extend  $J$  to a functional of norm at most 1 on all of  $C(X, F)$ . At this point we could apply known results, which represent  $C(X, F)^*$  in terms of dominated vector valued measures [4, p. 387], but we prefer to use the following direct (and simple) proof which was kindly furnished us by Dr. Erik Thomas. Let us define, for  $h \in C(X)$ ,  $h \geq 0$ ,

$$(**) \quad \mu(h) = \sup \{ |J(g)| : g \in C(X, F), \\ \|g(x)\| \leq h(x) \text{ for } x \text{ in } X \}$$

It is straightforward to verify that  $\mu(h) < \infty$ , that  $\mu(\lambda h) = \lambda \mu(h)$  for  $\lambda > 0$ , and that  $\mu(h_1 + h_2) \geq \mu(h_1) + \mu(h_2)$  if  $h_1, h_2 \geq 0$  are in  $C(X)$ . The reverse inequality follows easily once we have the following fact: If  $h = h_1 + h_2$  ( $h_i \geq 0$ ) and  $\|g(x)\| \leq h(x)$  for all  $x$  in  $X$ , then there exists  $g_1, g_2$  in  $C(X, F)$  such that  $g = g_1 + g_2$  and  $\|g_i(x)\| \leq h_i(x)$ ,  $i = 1, 2$  and  $x \in X$ . Indeed, let  $V = \{x \in X : \|g(x)\| > 0\}$  and for  $x$  in  $V$  let

$$\alpha_1(x) = \min(1, h_1(x)/\|g(x)\|), \quad \alpha_2(x) = 1 - \alpha_1(x).$$

If we define  $g_i(x) = \alpha_i(x)g(x)$  for  $x \in V$ ,  $= 0$  for  $x \in X \setminus V$ ,

then  $g = g_1 + g_2$ ,  $\|g_i(x)\| \leq h_i(x)$  and  $\|g_i(x)\| \leq \|g(x)\|$  ( $x \in X$ ,  $i = 1, 2$ ). (The last inequality shows that each  $g_i$  is continuous.) Thus,  $\mu$  is additive, non negative and positive homogeneous on the positive cone in  $C(X)$ , hence can be considered as an integral with respect to a finite positive regular Borel measure, say  $\mu$ , on  $X$ . Furthermore, from (\*\*) it is obvious that  $|J(g)| \leq \int \|g(x)\| d\mu(x)$  for all  $g \in C(X, F)$  and that  $\mu(1) = \|J\| \leq 1$ . Finally, for  $f \in C(X, E)$  we have  $|L(f)| = |J(\varphi \circ f)| \leq \int \|\varphi(f(x))\| d\mu(x) = \int p(f(x)) d\mu(x)$ , which completes the proof of Lemma 1.

We next give the proof of Lemma 2. For  $x \in X$ , let

$$V_x = \{y \in X : p(f(x) - f(y)) < \varepsilon/3\};$$

this is an open neighborhood of  $x$ , and we can choose  $x_1, \dots, x_n$  such that the collection  $\{V_{x_1}, \dots, V_{x_n}\}$  covers  $X$ , and such that no proper subcollection covers  $X$ . An easy induction argument, using the regularity of  $\mu$ , shows that we can find another cover  $\{V_1, \dots, V_n\}$  of open sets  $V_i$  such that  $V_i \subset V_{x_i}$  and such that  $\mu(D) < \varepsilon$ , where  $D = \cup \{V_i \cap V_j : i, j = 1, 2, \dots, n; i \neq j\}$ . Let

$$K_i = V_i \setminus \cup \{V_j : j \neq i\} = X \setminus \cup \{V_j : j \neq i\}, i = 1, 2, \dots, n.$$

Then each  $K_i$  is compact, nonempty and  $K_i \cap K_j$  is empty if  $i \neq j$ . Furthermore, if  $K = \cup K_i$ , then  $K$  is compact and  $X \setminus K \subset D$ , hence  $\mu(X \setminus K) < \varepsilon$ . Now, for each  $i = 1, 2, \dots, n$  we have  $f(x_i) \in \overline{\text{co}} A$ , hence we can find  $u_i \in \text{co} A$ , with  $p[u_i - f(x_i)] < \varepsilon/3$ , of the following form:

$$u_i = \sum_{k=1}^{m_i} \lambda_{ik} a_{ik}, \quad \{a_{ik}\}_{k=1}^{m_i} \subset A, \quad \lambda_{ik} > 0, \quad \sum_{i=1}^{m_i} \lambda_{ik} = 1$$

where each  $\lambda_{ik}$  is a rational number,  $k = 1, 2, \dots, m_i$ . We can assume that the numbers  $\lambda_{ik}$  have a common denominator  $Q > 0$ , so by allowing repetitions of the points  $a_{ik}$  and by relabelling, we have

$$u_i = Q^{-1} \sum_{k=1}^Q b_{ik}, \quad \{b_{ik}\}_{k=1}^Q \subset A, \quad i = 1, 2, \dots, n.$$

By property (D), for each  $k = 1, 2, \dots, Q$ , we can choose

$g_k \in C(X, A)$  such that

$$g_k(K_i) \subset \{\nu \in E : p(\nu - b_{ik}) < \varepsilon/3\}.$$

Let  $g = Q^{-1} \sum_{k=1}^Q g_k$ , so that  $g \in \text{co } C(X, A)$ .

Suppose that  $x \in K$ ; then  $x \in K_i$  for some  $i$  and

$$\begin{aligned} p[g(x) - u_i] &= p[Q^{-1} \sum g_k(x) - Q^{-1} \sum b_{ik}] \\ &\leq Q^{-1} \sum p[g_k(x) - b_{ik}] < \varepsilon/3. \end{aligned}$$

Since  $K_i \subset V_i \subset V_{x_i}$ , we have  $p[f(x) - f(x_i)] < \varepsilon/3$ . Thus,

$$\begin{aligned} p[g(x) - f(x)] &\leq p[g(x) - u_i] + p[u_i - f(x_i)] \\ &\quad + p[f(x_i) - f(x)] < \varepsilon, \end{aligned}$$

which completes Lemma 2.

**COROLLARY 1.** — *Suppose that  $B$  is a bounded closed convex subset of the locally convex space  $E$ , and that  $X$  is a compact Hausdorff space. Let  $A \subset \text{ext } B$ . If  $B = \overline{\text{co}} A$  and if  $(X, A)$  satisfies condition (D), then*

$$\overline{\text{co}} C(X, A) = C(X, B);$$

*in particular, the latter set is the closed convex hull of its extreme points.*

The hypothesis in Corollary 1 that  $B = \overline{\text{co}} A$  is obviously a necessary one for the conclusion; indeed, since

$$C(X, A) \subset C(X, \overline{\text{co}} A)$$

and since the latter is closed and convex, it contains  $\overline{\text{co}} C(X, A)$ . Thus, if  $C(X, B) = \overline{\text{co}} C(X, A)$ , then  $C(X, B) \subset C(X, \overline{\text{co}} A)$ , whence  $B = \overline{\text{co}} A$ .

In general, condition (D) is not a necessary one for the validity of the equality  $C(X, \overline{\text{co}} A) = \overline{\text{co}} C(X, A)$ . Consider, for instance,  $X = [0, 1]$ ,  $E = \mathbb{C}$  (complex plane) and

$$A = \{z : |z| < 1/4\} \cup \{z : 3/4 < |z| < 1\}.$$

Then  $\overline{\text{co}} A = \{z : |z| \leq 1\}$  is compact, and the above equality holds, but it is easily seen that  $C(X, A)$  is not even pointwise dense in  $A^X$ . If, however,  $A$  is the set of extreme points of  $\overline{\text{co}} A$  — this is the situation we are mainly interested in — then there is a partial converse to Theorem 1.

**THEOREM 2.** — *If  $B$  is a compact convex subset of the locally convex space  $E$  and  $A = \text{ext } B$  (so  $B = \overline{\text{co}} A$ ), then the equality*

$$\overline{\text{co}} C(X, A) = C(X, \overline{\text{co}} A)$$

*implies that  $C(X, A)$  is pointwise dense in  $A^X$ .*

We omit the proof, since it, closely parallels that of Theorem 3.1 of [6], in which  $E$  is a dual Banach space (in the weak\* topology) and  $B$  is its unit ball. The same argument works in the general case, using the fact that each extreme point of  $B$  has a neighborhood base in  $B$  consisting of « slices » [3, p. 108].

We now consider some applications of the foregoing results to spaces of linear operators. Suppose that  $M$  is a real (resp. complex) Banach space and let  $C(X)$  denote the real (resp. complex) continuous functions on the compact Hausdorff space  $X$ . The space  $\mathcal{L}(M, C(X))$  (or simply  $\mathcal{L}$ ) of all bounded linear operators from  $M$  into  $C(X)$  is linearly isomorphic to the space  $C(X, E)$ , where  $E = M_w^*$  is the space  $M^*$  in its weak\* topology [5, p. 490]. The correspondence between an operator  $T$  in  $\mathcal{L}$  and a function  $f$  in  $C(X, E)$  is defined by

$$(Tm)(x) = \langle m, f(x) \rangle, \quad (x \in X, m \in M).$$

Moreover,  $\|T\| = \sup \{ \|f(x)\| : x \in X \} = \|f\|$ .

Thus, the unit ball  $\mathcal{U}$  of  $\mathcal{L}$  may be identified with the subset  $C(X, U^*)$  of  $C(X, E)$ , where  $U^*$  is the unit ball of  $M^*$ . This correspondence was used in [6] to obtain various corollaries to Theorem 1, which was proved there for this particular choice of  $E$ . Similarly, the subspace

$$\mathcal{L}_c = \mathcal{L}_c(M, C(X))$$

of all compact operators in  $\mathcal{L}$  can be identified with the subspace  $C(X, M_n^*)$ , of  $C(X, E)$ , where  $M_n^*$  is  $M^*$  in its norm topology [5], and Theorem 1 was also proved in [6] for this case. It is readily verified that the uniform topology on  $C(X, E)$  carries over (under the correspondence indicated above) to the strong operator topology on  $\mathcal{L}$ , and that in  $C(X, M_n^*)$  the uniform topology is the norm topology (norm defined as above) and this identifies on  $\mathcal{L}_c$  with the norm (or « uniform operator ») topology. The fact that Theorem 1



was proved for arbitrary  $E$  allows us to consider the case where  $E = M_w^*$ , the space  $M^*$  in its weak (i.e.  $\sigma(M^*, M^{**})$ ) topology. Under the above correspondence,  $C(X, M_w^*)$  is exactly the space  $\mathcal{L}_{wc} = \mathcal{L}_{wc}(M, C(X))$  of all weakly compact operators from  $M$  into  $C(X)$ . The topology induced on  $\mathcal{L}_{wc}$  by the uniform topology on  $C(X, M_w^*)$  is not one of the usual « operator » topologies, but is easily seen to be between the strong operator and norm topologies on  $\mathcal{L}_{wc}$ .

We will denote by  $\mathcal{U}$ ,  $\mathcal{U}_c$  and  $\mathcal{U}_{wc}$  the unit ball of  $\mathcal{L}$ ,  $\mathcal{L}_c$  and  $\mathcal{L}_{wc}$  respectively. These are, of course, the same as the sets  $C(X, U^*)$ ,  $C(X, U_n^*)$  and  $C(X, U_w^*)$ . An operator which corresponds to an element  $f$  of one of these sets such that  $f(X) \subset \text{ext } U^*$  is called a *nice* (resp. nice compact, nice weakly compact) operator. They are of course, extreme points of the sets  $\mathcal{U}$ ,  $\mathcal{U}_c$  and  $\mathcal{U}_{wc}$  respectively.

The next result is almost a direct application of Corollary 1 to the ball of weakly compact operators. The main point is to account for the difference between the two topologies involved.

**PROPOSITION 1.** — *Let  $M$  and  $C(X)$  be as above, and let  $U^*$  be the unit ball of  $M^*$ . Suppose that there is a subset  $A \subset \text{ext } U^*$  such that:*

- (i) *The pair  $(X, A_w)$  satisfies condition (D).*
- (ii)  *$U^*$  is the norm closed convex hull of  $A$ .*

*Then the unit ball  $\mathcal{U}_{wc}$  of  $\mathcal{L}_{wc}$  is the strong operator closed convex hull of the nice weakly compact operators.*

*Proof.* — Hypotheses (i) and (ii) allow us to apply Corollary 1 to obtain the equality  $C(X, U_w^*) = \overline{\text{co}} C(X, A_w)$ , where the closure is in the uniform topology of  $C(X, M_w^*)$ . Since  $C(X, M_w^*) \subset C(X, M_w^{**})$ , the uniform topology on the latter space induces a topology on  $C(X, M_w^*)$  which is weaker than the original; we will call it the « strong » topology since it corresponds exactly to the strong operator topology on  $\mathcal{L}_{wc}$ . Thus, we want to show that  $C(X, U_w^*)$  is the strong closed convex hull of  $C(X, A_w)$ , since the latter is clearly a subset of the nice weakly compact operators. But it is easily verified that (since  $U^*$  is weak\* closed)  $C(X, U_w^*)$  is strongly closed in  $C(X, M_w^*)$ , hence contains the strong closure of  $\text{co } C(X, A_w)$ , which in turn contains  $\overline{\text{co}} C(X, A_w) = C(X, U_w^*)$ .

The fact that in hypothesis (ii) above we used the norm closure instead of the weak closure (which Corollary 1 would have allowed) is no loss in generality, of course, since the set involved is convex.

Recall that a real or complex Banach space  $M$  is said to be *smooth* if for each point  $x \in S(M) = \{x \in M : \|x\| = 1\}$  there exists a unique functional  $f_x$  in the unit sphere  $S(M^*)$  of  $M^*$  such that  $\operatorname{Re} f_x(x) = 1$ . This is equivalent to Gateaux differentiability of the norm (at each nonzero point), and the functional  $f_x$  is the Gateaux differential of the norm at  $x$ .

**THEOREM 3.** — *Let  $M$  be a real or complex Banach space and  $X$  a compact Hausdorff space. In the real case, we assume that  $\dim M > 1$ .*

(a) *If  $M$  is smooth, then  $\mathcal{U}$  is the strong operator closed convex hull of the nice operators.*

(b) *If the norm in  $M$  is Fréchet differentiable at each nonzero point, then  $\mathcal{U}_{wc}$  [resp.  $\mathcal{U}_c$ ] is the strong operator [resp. norm] closed convex hull of its nice operators.*

*Proof.* — (a) It is well known (and easily proved) that if  $M$  is smooth, then the map  $x \rightarrow f_x$  defined above is continuous from  $S(M)$  in its norm topology into  $S(M^*)$  in its weak\* topology. It is readily verified that  $U^*$  is the weak\* closed convex hull of the image  $A$  of  $S(M)$  under this map, and that  $A \subset \operatorname{ext} U^*$ . [In fact,  $A$  is known [1] to be norm dense in  $S(M^*)$ .] Since  $S(M)$  is arcwise connected (in the real case this assertion obviously requires that  $\dim M > 1$ ), the set  $A$  is arcwise connected in the weak\* topology. Thus,  $(X, A)$  satisfies condition (D) so Corollary 1 yields the desired conclusion.

(b) The Fréchet differentiability of the norm in  $M$  implies that the derivative map  $x \rightarrow f_x$  defined above is continuous from the norm topology on  $S(M)$  into the norm topology on  $S(M^*)$ . With the same notation as in (a), the set  $A$  is norm arcwise connected and norm dense in  $S(M^*)$ , hence  $U^*$  is the norm closed convex hull of  $A$  and Proposition 1 [resp. Corollary 1] applies.

In the case when  $M = C(X)$  for some compact Hausdorff space  $X$ , it is possible to obtain necessary and sufficient

conditions on  $X$  and  $Y$  that  $\mathcal{U}_{wc} \subset \mathcal{L}_{wc}(C(X), C(Y))$  be the strong operator closed convex hull of the nice weakly compact operators. These conditions are the same as those in Theorem 4.6 of [6], and the methods for obtaining them are essentially the same. (We don't know, in this case, whether every extreme element of  $\mathcal{U}_{wc}$  is a nice operator.) Similar results hold in the real case for the set of positive normalized weakly compact operators.

The following problem arises in the context of Corollary 1: Suppose that  $C(X, B) = \overline{\text{co}} \text{ ext } C(X, B)$ . Must  $\text{ext } B$  be nonempty?

[*Note added in proof:* J. Lindenstrauss (private communication) has answered this question in the negative by showing that there exists a normed linear space  $E$ , a nonempty convex closed and bounded subset  $B \subset E$  and a nonempty compact Hausdorff space  $X$  such that  $\text{ext } B$  is empty, but  $C(X, B) = \overline{\text{co}} \text{ ext } C(X, B)$ .]

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