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A hereditary property in locally convex spaces


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A HEREDITARY PROPERTY
IN LOCALLY CONVEX SPACES (1)

by Manuel VALDIVIA

J. Dieudonné has given in [1] the two following theorems:
1) If $F$ is a subspace, of finite codimension, of a barrelled space $E$, then $F$ is a barrelled space.

2) If $F$ is a subspace, of finite codimension, of a bornological space, then $F$ is a bornological space.

In this paper we give a theorem analogous to the previous ones, but using infrabarrelled spaces instead of barrelled or bornological spaces. So we shall prove the following theorem:
If $F$ is a subspace, of finite codimension, of an infrabarrelled space $E$, then $F$ is an infrabarrelled space.

Let $K$ be the field of real or complex numbers. Let $E$ be a locally convex topological vector space over the field $K$. If $\mathcal{B}$ is the family of all the absolutely convex, bounded and closed sets of $E$, we denote with $E_B, B \in \mathcal{B}$, the linear hull of $E$ with the seminorm associated to $B$. Let $\mathcal{T}$ be the topology on $E$, so that $E[\mathcal{T}]$ is the inductive limit of the family $\{E_B : B \in \mathcal{B}\}$.

**Theorem.** — Let $F$ be a subspace of $E$, with finite codimension. If $U$ is a closed, bornivorous and absolutely convex set of $F$, then there exists in $E$ an $U'$, closed, bornivorous and absolutely convex set, such that $U' \cap F = U$.

In particular, if $E$ is an infrabarrelled space, then $F$ is also an infrabarrelled space.

**Proof.** — Clearly, the $\mathcal{T}$-topology is finer than the initial one on $E$. On the other hand, for every bounded set $A$, there exists a set $B \in \mathcal{B}$, such that $A \subseteq B$. Hence $A$ is a bounded

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set of $E_B$, therefore $A$ is a bounded set of $E[\mathcal{C}]$. That is, the bounded sets of $E$ and those of $E[\mathcal{C}]$ are the same.

We denote with $F[\mathcal{C}]$ the subspace $F$, equipped with the topology induced by $\mathcal{C}$. Since $E[\mathcal{C}]$ is the inductive limit of seminormed spaces, it is a bornological space and, according to theorem 2), $F[\mathcal{C}]$ is a bornological space. Hence, $U$ is a closed neighborhood of $0$ in $F[\mathcal{C}]$.

Clearly, it is sufficient to prove the theorem in the case of $F$ being a vector subspace of $E$, with codimension one. So that we suppose that $F$ is so.

Two cases are possible:

1° $F[\mathcal{C}]$ being dense in $E[\mathcal{C}]$. Let $\overline{U}$ and $\overline{U}^*$ be the closures of $U$ in $E$ and $E[\mathcal{C}]$ respectively. Since $U$ is a neighborhood of $0$ in $F[\mathcal{C}]$, then $\overline{U}^*$ is a neighborhood of $0$ in $E[\mathcal{C}]$, hence $\overline{U}^*$ is a bornivorous set in the same space.

Furthermore, $\overline{U} \supset \overline{U}^*$, then $\overline{U}$ is a bornivorous set in $E$. We can take $U' = \overline{U}$, then $U'$ is a closed, bornivorous and absolutely convex set of $E$, such that $U' \cap F = U$.

2° $F[\mathcal{C}]$ being closed in $E[\mathcal{C}]$. If $U = \overline{U}$, we take a vector $x$ such that $x \in E$ and $x \in F$. Let $C$ be the balanced hull of the set $\{x\}$, then $U + C$ is a closed set in $E$ and $U + C$ is a neighborhood of $0$ in $E[\mathcal{C}]$, therefore, $U + C$ is bornivorous in $E$. If we take $U' = U + C$ the theorem is satisfied.

If $U \neq \overline{U}$, $U$ is absorbing in $E$, hence there exists an element $z \in \overline{U}$ such as $z \in F$. Let $D$ be the balanced hull of $\{z\}$. $U + D$ is a neighbourhood of $0$ in $E[\mathcal{C}]$, hence it is bornivorous in $E$. Furthermore $\overline{U} = U$ and $\overline{U} = D$, then $2\overline{U} = \overline{U} + D$, hence $\overline{U}$ is bornivorous in $E$. If we take $\overline{U} = U'$ the theorem is satisfied.

BIBLIOGRAPHY


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