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ANALYSIS ON SOME LINEAR SETS

by Robert KAUFMAN

0.

Let F be a compact subset of $(-\infty, \infty)$ and for each integer $N \geq 1$ let $\nu_N \equiv \nu(N; F)$ be the number of intervals $[kN^{-1}, (k+1)N^{-1}]$ meeting F ; F is called *small* provided $\log \nu_N = o(\log N)$. The existence of small sets of « multiplicity » (M_0 -sets in [6I, p. 344]) was proved in 1942 by Salem and used by Rudin [4, VIII]; a program somewhat analogous for locally compact abelian groups was completed by Varopoulos [5].

Does there exist a small set F with the property that both F and (say) $F_2 = \{x^2 : x \in F\}$ are M_0 -sets? The construction of these sets doesn't seem accessible by the method of Rudin and Salem [4], nor by the Brownian motion [3]. In this note an affirmative answer is given to a more general problem.

THEOREM 1. — *Let (h_n) be a sequence of real functions of class $C^1(-\infty, \infty)$ with derivatives $h'_n > 0$. Then there is a small set F with the property that each $h_n(F)$ is an M_0 -set.*

Small sets occur naturally in the construction of independent sets [3, 4, 5]; after the metrical theory of Diophantine approximation a set F is called *metrically independent* if to each integer $N \geq 1$ and each ε in $(0, 1)$ there is a U_0 so that the simultaneous inequalities

$$\left| \sum_{j=1}^N u_j x_j - \nu \right| < U^{-N-\varepsilon}, \quad U = \max(|u_1|, \dots, |u_N|) > U_0 \\ |x_i - x_j| \geq \varepsilon \quad \text{for} \quad 1 \leq i < j \leq N$$

have no solution in integers u_1, \dots, u_N, ν and members x_1, \dots, x_N of F . Compare [1, VII].

Uncountable metrically independent subsets could perhaps be constructed by classical arguments, for example that of Perron [1, p. 79] or Davenport [2].

THEOREM 2. — *The set F determined in Theorem 1 can be required to have the property that each $h_n(F)$ be metrically independent.*

THEOREMS 1a, 2a. — *Theorems 1 and 2 remain true provided each h_n is monotone-continuous and $h'_n > 0$ almost everywhere.*

1.

In the proof of Theorem 1 we require two arrays of independent random variables $(Y_{k,m})$ and $(\xi_{k,m})$ defined on a space (Ω, P) for $1 \leq k < \infty$, $1 \leq m \leq k^6$. Each Y_k is uniformly distributed upon $[0, 1]$ while

$$P\{\xi_{k,m} = 1\} = \pi_k = k^{-1} = 1 - P\{\xi_{k,m} = 0\}.$$

Suppose that f is a measurable function on $(-\infty, \infty)$ and $-1 \leq f \leq 1$, and let $\mu = \pi_k E(f(Y))$; elementary calculations show that

$$E(e^{t\xi_k f(Y_k)} e^{-t\mu}) \leq \exp \frac{1}{2} \pi_k t^2 \exp 0(\pi_k t^3)$$

with an '0' uniform for $-1 \leq f \leq 1$, $-1 \leq t \leq 1$, $0 \leq \pi_k \leq 1$. Hence for any $z > 0$ and $1 > t > 0$

$$P\left\{\left|\sum_m \xi_{k,m} - k^5\right| > zk^5\right\} \leq 2 \exp -zk^5 t \exp \frac{1}{2} k^6 \pi_k t^2 \exp 0(\pi_k k^6 t^3).$$

Choosing $z = t = k^{-2}$ and using $\pi_k = k^{-1}$ we obtain

$$P\left\{\left|\sum_m \xi_{k,m} - k^5\right| \geq k^3\right\} \leq 0(1) \exp -\frac{1}{2} k.$$

Thus

LEMMA 1. — $\sum_{m=1}^{k^6} \xi_{k,m} = k^5 + 0(k^3)$ almost surely in Ω .

A sequence of random measures λ_k is now determined as

follows: for any function g on $(-\infty, \infty)$

$$\int g d\lambda_k = k^{-2}g(0) + k^{-5} \sum_m \xi_{n,m} g(e^{-k \log^2 k} Y_{k,m}).$$

Thus in every instance $\lambda_k \geq 0$ and $\|\lambda_k\| \geq k^{-2}$; moreover $\|\lambda_k\| = 1 + O(k^{-2})$ almost surely. Because $\sum e^{-k \log^2 k} < \infty$ the convolution $\lambda = \pi * \lambda_k$ converges, and F is defined to be its closed support. F is contained in at most

$$\prod_{j=1}^k [j^5 + O(j^3)] = e^{O(k \log k)}$$

intervals of length $e^{-k \log^2 k}$.

Because $(k+1) \log^2(k+1)/k \log^2 k \rightarrow 1$, this is sufficient to obtain

LEMMA 2. — F is almost surely a small set.

LEMMA 3. — Let $h \in C^1(-\infty, \infty)$ and $h' > 0$; let (c_m) , (u_m) , (v_m) be sequences of real numbers such that

$$|c_m| + |v_m| = O(1) \quad \text{and} \quad |u_m v_m| \rightarrow \infty.$$

Then

$$\lim_{m \rightarrow \infty} \int_0^1 \exp iu_m h(c_m + v_m t) dt = 0.$$

Proof. — Let g denote the C^1 function inverse to h , and let $v_m > 0$. The integral is transformed to

$$J = \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \cdot v_m^{-1} dy,$$

where $\alpha_m = h(c_m)$, $\beta_m = h(v_m + c_m)$. A further substitution $y = y_1 + \pi u_m^{-1}$ yields

$$J = \frac{1}{2} \int_{\alpha_m}^{\beta_m} g'(y) \exp iu_m y \cdot v_m^{-1} dy \\ - \frac{1}{2} \int_{\alpha_m - \pi u_m^{-1}}^{\beta_m - \pi u_m^{-1}} g'(y + \pi u_m^{-1}) \exp iu_m y \cdot v_m^{-1} dy.$$

This tends to 0 because $\beta_m - \alpha_m = O(v_m)$ and $v_m^{-1} u_m^{-1} = o(1)$.

Proof of Theorem 1. — We show that for each function h_n $\lim_{u \rightarrow \infty} \int \exp iuh_n(s) \lambda(ds) = 0$, almost surely. Then $h_n(F)$ is an

M_0 -set; because $h_n(F)$ is compact it is enough to prove

$$\lim_{r \rightarrow \infty} \int \exp ir^{\frac{1}{3}} h_n(s) \lambda(ds) = 0, \quad r = 1, 2, 3, \dots$$

To each integer $r \geq 3$ we attach the integer $k(r)$ defined by $k(r) \leq \log^{\frac{1}{3}} r < k(r) + 1$ and write $\lambda'_k = \prod_{j \neq k} * \lambda_j$. Then

$$\int \exp ir^{\frac{1}{3}} h_n(s) \lambda(ds) = \iint \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds) \lambda'_k(d\omega).$$

For each real number ω in the support of λ'_k let $m(\omega)$ be the expected value of $\int \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds)$. Then

$$\left| \int \exp ir^{\frac{1}{2}} h_n(s) \lambda(ds) \right| \leq \int \left| \int \exp ir^{\frac{1}{2}} h_n(s + \omega) \lambda_k(ds) - m(\omega) \right| \lambda'_k(d\omega) + \|\lambda'_k\| \max |m(\omega)|.$$

The second integral, say I, can be handled by Jensen's inequality and the estimates at the beginning of 1. Let $-1 < t < 1$ and $\Phi(x) = e^{tx}$. Then

$$\mathbb{E}(\Phi(\|\lambda'_k\|^{-1} k^5 \text{Re I})) \leq 2 \exp \frac{1}{2} k^5 t^2 0 (\exp k^5 t^3).$$

Choosing $t = k^{-\frac{1}{2}}$ we observe

$$\begin{aligned} \mathbb{P}\left\{ |\text{Re I}| > \|\lambda'_k\| k^{-\frac{1}{2}} \right\} &= \mathbb{P}\left\{ \Phi(\|\lambda'_k\|^{-1} k^5 \text{Re I}) > \exp k^4 \right\} \\ &\leq 2 \exp \frac{1}{2} k^4 \exp 0(k^{7/2}) \exp -k^4. \end{aligned}$$

This is the general term of a convergent series, inasmuch as $k = k(r) > -1 + \log^{\frac{1}{3}} r$. Thus, almost surely in Ω , for $r > r_0$

$$\left| \text{Re} \int \exp ir^{\frac{1}{2}} h_n(s) \lambda(ds) \right| \leq k^{-\frac{1}{2}} \|\lambda'_k\| + \|\lambda'_k\| \max |m(\omega)|$$

and of course a similar statement holds for the imaginary part of the integral. Now

$$|m(\omega)| \leq k^{-2} + \left| \int_0^1 \exp ir^{\frac{1}{2}} h_n(e^{-k \log^2 kt} + \omega) dt \right|$$

with $\omega = 0(1)$ and $k = k(r)$. To apply Lemma 3 we must

verify $r^{\frac{1}{2}} e^{-k \log^2 k} \rightarrow \infty$ but this is plain from $k(r) < \log^{\frac{1}{3}} r$. Because $\max_k \|\lambda'_k\| < \infty$ almost surely, the proof of Theorem 1 is complete.

2.

Theorem 2 requires the construction of a random function φ in $C^\infty(-\infty, \infty)$. Let ψ be a function in $C^\infty(-\infty, \infty)$ with the properties

- (i) $\psi = 0$ on $[-\infty, -2]$, $\psi = 3$ on $[2, \infty]$,
(ii) $\psi' \geq 0$, and $\psi' > 1$ on $(-1, 1)$.

Let (a_p) be a sequence of real numbers such that every real number belongs to infinitely many of the intervals $(a_p - p^{-1}, a_p + p^{-1})$. Finally, let (Z_p) be a sequence of independent random variables on (Ω, P) , uniformly distributed upon $[0, 1]$. We define

$$\varphi(x) = \sum_{p=1}^{\infty} e^{-p\psi} (p^{-1}Z_p + p^{\frac{1}{2}}(x - a_p)) + x.$$

To each compact set F and number $\delta > 0$ there are numbers q_1 and q_2 so that

$$q_1 \geq 4, \quad q_1^{\frac{1}{2}} \delta \geq 5, \quad \bigcup_{p=q_1}^{q_2} (a_p - p^{-1}, a_p + p^{-1}) \supseteq F.$$

THEOREM 3. — *Let F be a small set and $h \in C^1(-\infty, \infty)$, $h' > 0$; then $h\varphi(F)$ is almost surely metrically independent.*

For each integer $U \geq 1$ we can choose a subset $S(N, U)$ of R^N so that every point in F^N has distance $< U^{-3N}$ from some point in $S(N, U)$, while $\text{card } S(N, U) \leq v^N(NU^{3N}; F)$.

Beginning with an inequality

$$\left| \sum_{j=1}^N u_j h\varphi(y_j) - \nu \right| < U^{-N-\epsilon}, \quad |h\varphi(y_j) - h\varphi(y_i)| > \epsilon \quad (i \neq j)$$

we conclude first that $|y_i - y_j| > \eta$ for some fixed $\eta > 0$. Let (z_1, \dots, z_n) be the member of $S(N, U)$ associated to

(y_1, \dots, y_n) . Then

$$(1) \quad \left| \sum_{j=1}^N u_j h\varphi(z_j) - \nu \right| < U^{-N-\varepsilon} + 0(U \cdot U^{-3N}),$$

$$|z_i - z_j| > \eta - 2U^{-3N}.$$

For large U we can find $\delta < \eta - 2U^{-3N}$ and corresponding numbers q_1, q_2 . Let $q_1 \leq p \leq q_2$, $|z_i - a_p| < p^{-1}$.

$$\left| p^{-1}Z_p + p^{\frac{1}{2}}(z_i - a_p) \right| < p^{-1} + p^{-\frac{1}{2}} < 1,$$

$$|p^{-1}Z_p + p^{\frac{1}{2}}(Z_j - a_p)| > p^{\frac{1}{2}}\delta - p^{-1} - p^{-\frac{1}{2}} > 4, \quad \text{when } j \neq i.$$

Therefore $\frac{\partial}{\partial Z_p} \sum_{j=1}^N u_j h\varphi(Z_j) = u_i \frac{\partial}{\partial Z_p} h\varphi(Z_i)$ exceeds $\alpha|u_i|$ in modulus, with an $\alpha > 0$ independent of u_1, \dots, u_n . Hence the probability of the inequality (1) is $0(U^{-1} \cdot U^{-N-\varepsilon})$ for each (z_1, \dots, z_N) . The requirement $U = \max(|u_1|, \dots, |u_n|)$ determines $0(U^{N-1})$ N -tuples and plainly $\nu = 0(U)$. Because F is a small set $\nu^N(NU^{3N}; F) = U^{o(1)}$ as $U \rightarrow \infty$. Theorem 3 follows from this and $\Sigma U^{-1-\varepsilon} U^{o(1)} < \infty$.

Proof of Theorem 2. — Here we use the fact that F and φ depend on independent σ -fields. F is almost surely small, whence each $h_n\varphi(F)$ is almost surely metrically independent, by Theorem 3. By Theorem 1, each $h_n\varphi(F)$ is almost surely an M_0 -set and Theorem 2 is proved.

3.

Proof of Theorems 1a and 2a. — According to a theorem of Marcinkiewicz [6II, pp. 73-77], to each $\delta > 0$ there exist functions g_n in $C^1(-\infty, \infty)$ so that

$$m(h_n \neq g_n) < \delta n^{-2}, \quad n = 1, 2, 3, \dots$$

At almost all points of density of the set $(h_n = g_n)$, $g'_n = h'_n > 0$. Passing to a perfect subset of the set $(g'_n > 0, g'_n = h'_n, g_n = h_n)$, we can find a \tilde{g}_n in $C^1(-\infty, \infty)$ such that

$$m(h_n \neq \tilde{g}_n) < 2\delta n^{-2}, \quad n = 1, 2, 3, \dots,$$

$\tilde{g}'_n > 0$ everywhere.

We observe next that to each $\varepsilon > 0$ there is a constant $B(\varepsilon)$ so that for all Borel sets S

$$\int_{\Omega} \lambda(S) dP \leq \varepsilon + B(\varepsilon)m(S).$$

Thus to each $\varepsilon > 0$ we can choose functions \tilde{g}_n by Marcinkiewicz' theorem, so that

$$P\{\lambda(x: \tilde{g}_n\varphi(x) \neq \tilde{h}_n\varphi(x) \text{ for some } n) > \varepsilon\} < \varepsilon.$$

In proving this implication it must be observed that φ and λ are stochastically independent and $\varphi' > 1$. Writing G for the inner set in the last inequality, we know that $h_n\varphi(G' \cap F) = \tilde{g}_n\varphi(G' \cap F)$ is almost surely metrically independent and that $h_n\varphi(G' \cap F)$ is almost surely an M_0 -set, if only $\lambda(G' \cap F) > 0$; and this holds for $\|\lambda\| > \varepsilon$ excepting an event of probability $< \varepsilon$. Thus Theorems 1a and 2a are derived from Theorems 1 and 2.

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