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**ON THE MEAN VALUES OF AN ENTIRE FUNCTION  
 REPRESENTED BY DIRICHLET SERIES**

by **S. K. BAJPAI**

1. Let  $f(s)$  be an entire function of the complex variable  $s = \sigma + it$  defined everywhere in the complex plane by absolutely convergent Dirichlet series

$$(1.1) \quad f(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$$

where  $0 \leq \lambda_0 < \lambda_1 < \dots < \lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

As usual, the symbols  $M(\sigma, f)$ ,  $\mu(\sigma, f)$  and  $\nu(\sigma, f)$  denote the maximum modulus, the maximum term and the rank of the maximum term respectively for  $f(s)$ , and can be found in [8]. We define

$$A_k(\sigma) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^k dt$$

and

$$W_{y, \delta, k}(\sigma) = \lim_{T \rightarrow \infty} \frac{e^{-\delta\sigma}}{2T} \int_y^\sigma \int_{-T}^T |f(x + it)|^k e^{\delta x} dx dt; \quad \delta > 0.$$

The mean values for  $k = 2$  and  $y = 0$  were defined by Hadamard [2] and also by Kamthan ([3], [4]) who has obtained the following main results for  $k = 2$  and  $y = 0$

$$(1.2) \quad \lim_{\sigma \rightarrow \infty} \sup \frac{\log \log A_k(\sigma)}{\sigma} = \frac{\rho}{\lambda}$$

$$(1.3) \quad \lim_{\sigma \rightarrow \infty} \sup \frac{\log \log W_{y, \delta, k}(\sigma)}{\sigma} = \frac{\rho}{\lambda}$$

Juneja [6] also proved (1.3) by using a different technique for  $k = 2$  and  $y = 0$ . In [7] another extension of (1.3)

has been obtained for  $k = 2$  and  $y = -\infty$ . Kamthan [5] has further attempted to establish (1.2) for every  $k > 0$ , but his arguments of the proof are vague. Thus, in the present paper, we restrict ourselves to functions of finite Ritt-order  $\rho$ , establish (1.2) and extend (1.3) for every  $k > 0$  by a different technique.

2. We have

**THEOREM 1.** — *If  $f(s)$ , defined by Dirichlet series (1.1), be an entire function of finite Ritt-order  $\rho$ , lower order  $\lambda$ , type  $T$  and lower type  $t$ , then, for  $0 < k < \infty$*

$$\frac{\rho}{\lambda} = \lim_{\sigma \rightarrow \infty} \sup \frac{\log \log A_k(\sigma)}{\sigma} = \lim_{\sigma \rightarrow \infty} \sup \frac{\log \log W_{y, \delta, k}(\sigma)}{\sigma}$$

and

$$\frac{kT}{kt} = \lim_{\sigma \rightarrow \infty} \sup \frac{\log A_k(\sigma)}{e^{\rho\sigma}} = \lim_{\sigma \rightarrow \infty} \sup \frac{\log W_{y, \delta, k}(\sigma)}{e^{\rho\sigma}}$$

First we prove the following:

**LEMMA.** — *Let  $f(s)$ , given by (1.1), be an entire function of finite Ritt-order  $\rho$ , then, for  $0 < k < \infty$*

$$\log A_k(\sigma) \sim \log W_{y, \delta, k}(\sigma) \sim k \log M(\sigma, f).$$

*Proof.* — It is well known

$$(2.1) \quad |a_n| e^{\sigma \lambda_n} = \left| \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(\sigma + it) e^{-i \lambda_n t} dt \right|$$

First, let  $k \geq 1$ , then by Holder's inequality, we have

$$(2.2) \quad |a_n| e^{\sigma \lambda_n} \leq C_k \left[ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^k dt \right]^{1/k}$$

where

$$C_k = \begin{cases} 1 & \text{if } k = 1 \\ 4 \left[ \frac{\Gamma\left(\frac{1}{2} + \frac{k}{2(k-1)}\right)}{\sqrt{\pi} \Gamma\left(1 + \frac{k}{2(k-1)}\right)} \right]^{\frac{k-1}{k}} & \text{if } k > 1. \end{cases}$$

Further, from (2.2) we have

$$\{|a_n|e^{x\lambda_n}\}^k \leq C_k^k \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x + it)|^k dt$$

Hence, for  $y \leq 0$

$$(2.3) \quad \frac{\{|a_n|e^{\sigma\lambda_n}\}^k}{k\lambda_n + \delta} - \frac{e^{-\delta\sigma}|a_n|^k e^{(k\lambda_n + \delta)y}}{k\lambda_n + \delta} \leq C_k^k W_{y, \delta, k}(\sigma) \leq \frac{C_k^k}{\delta} M^k(\sigma, f)$$

The case when  $0 < k < 1$  may be treated as following:

$$(2.4) \quad |a_n|e^{\sigma\lambda_n} \leq C_{k+1} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(\sigma + it)|^{k+1} dt \right\}^{\frac{1}{k+1}} \\ \leq C_{k+1} \{M(\sigma, f)A_k(\sigma)\}^{\frac{1}{k+1}}$$

Also, from (2.1), for  $0 < k < 1$ , we have

$$(2.5) \quad \frac{\{|a_n|e^{\sigma\lambda_n}\}^{k+1}}{(k+1)\lambda_n + \delta} - \frac{e^{-\delta\sigma}|a_n|^{k+1} e^{((k+1)\lambda_n + \delta)y}}{(k+1)\lambda_n + \delta} \\ \leq C_{k+1}^{k+1} \lim_{T \rightarrow \infty} \frac{e^{-\delta\sigma}}{2T} \int_0^\sigma \int_{-T}^T e^{x\delta} |f(x + it)|^{k+1} dt dx \\ \leq C_{k+1}^{k+1} M(\sigma, f) W_{y, \delta, k}(\sigma) \leq \frac{1}{\delta} C_{k+1}^{k+1} M^{k+1}(\sigma, f)$$

Since  $f(s)$  is of finite Ritt-order  $\rho$ , it follows that ([1], p. 719), ([9], p. 265)

$$(2.6) \quad \log \mu(\sigma, f) \sim \log M(\sigma, f)$$

and  $\log \log \mu(\sigma, f) \sim \log \lambda_{\nu(\sigma, f)}$

Hence (2.2), (2.3), (2.4), (2.5) and (2.6) together imply the lemma. Proof of theorem 1, follows immediately from the above lemma.

**THEOREM 2.** — *If  $f(s)$  is an entire function of finite Ritt-order  $\rho$  and is defined by (1.1) then*

$$\lim_{\sigma \rightarrow \infty} \sup \frac{\log \{W_{y, \delta, k}(\sigma, f')/W_{y, \delta, k}(\sigma, f)\}}{\sigma} = \frac{k\rho}{k\lambda}$$

*provided  $k$  is an even integer.*

For  $k = 2$ , this result is due to Juneja [6] and for  $k$  even it follows with the use of Minkowski's inequality and the

fact that  $W_{y,\delta,k}(\sigma)$  is an increasing convex function of  $\sigma$  for  $k$  even.

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