S. K. Bajpai

On the mean values of an entire function represented by Dirichlet series

Annales de l’institut Fourier, tome 21, n° 2 (1971), p. 31-34

<http://www.numdam.org/item?id=AIF_1971__21_2_31_0>
ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES

by S. K. BAJPAI

1. Let \( f(s) \) be an entire function of the complex variable \( s = \sigma + it \) defined everywhere in the complex plane by absolutely convergent Dirichlet series

\[
f(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n}
\]

where \( 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty \) as \( n \to \infty \).

As usual, the symbols \( M(\sigma, f) \), \( \mu(\sigma, f) \) and \( v(\sigma, f) \) denote the maximum modulus, the maximum term and the rank of the maximum term respectively for \( f(s) \), and can be found in [8]. We define

\[
A_k(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k \, dt
\]

and

\[
W_{\gamma, \delta, k}(\sigma) = \lim_{T \to \infty} \frac{e^{-\delta \sigma}}{2T} \int_{-T}^{T} \int_{-T}^{T} |f(x + it)|^k e^{\delta x} \, dx \, dt; \quad \delta > 0.
\]

The mean values for \( k = 2 \) and \( y = 0 \) were defined by Hadamard [2] and also by Kamthan ([3], [4]) who has obtained the following main results for \( k = 2 \) and \( y = 0 \)

\[
\lim_{\sigma \to \infty} \inf_{\lambda} \frac{\log \log A_k(\sigma)}{\sigma} = \rho
\]

(1.2)

\[
\lim_{\sigma \to \infty} \inf_{\lambda} \frac{\log \log W_{\gamma, \delta, k}(\sigma)}{\sigma} = \rho
\]

(1.3)

Juneja [6] also proved (1.3) by using a different technique for \( k = 2 \) and \( y = 0 \). In [7] another extension of (1.3)
S. K. BAJPAI

has been obtained for \( k = 2 \) and \( y = -\infty \). Kamthan [5] has further attempted to establish (1.2) for every \( k > 0 \), but his arguments of the proof are vague. Thus, in the present paper, we restrict ourselves to functions of finite Ritt-order \( \rho \), establish (1.2) and extend (1.3) for every \( k > 0 \) by a different technique.

2. We have

**Theorem 1.** — If \( f(s) \), defined by Dirichlet series (1.1), be an entire function of finite Ritt-order \( \rho \), lower order \( \lambda \), type \( T \) and lower type \( t \), then, for \( 0 < k < \infty \)

\[
\rho = \lim_{\sigma \to \infty} \sup_{\sigma} \log \log A_{k}(\sigma) = \lim_{\sigma \to \infty} \inf_{\sigma} \frac{\log \log W_{\lambda, k}(\sigma)}{\sigma}
\]

and

\[
kT = \lim_{\sigma \to \infty} \sup_{\sigma} \log A_{k}(\sigma) = \lim_{\sigma \to \infty} \inf_{\sigma} \frac{\log W_{\lambda, k}(\sigma)}{e^{\sigma}}
\]

First we prove the following:

**Lemma.** — Let \( f(s) \), given by (1.1), be an entire function of finite Ritt-order \( \rho \), then, for \( 0 < k < \infty \)

\[
\log A_{k}(\sigma) \sim \log W_{\lambda, k}(\sigma) \sim k \log M(\sigma, f).
\]

**Proof.** — It is well known

\[
|a_{n}|e^{\sigma \lambda_{n}} = \left| \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it)e^{-\lambda_{n}t} dt \right|
\]

First, let \( k \geq 1 \), then by Holder’s inequality, we have

\[
|a_{n}|e^{\sigma \lambda_{n}} \leq C_{k} \left[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^{k} dt \right]^{1/k}
\]

where

\[
C_{k} = \begin{cases} 
1 & \text{if } k = 1 \\
4 \left[ \frac{1}{2} \frac{k}{2(k-1)} \right]^{k-1} & \text{if } k > 1.
\end{cases}
\]
Further, from (2.2) we have
\[\{|a_n|e^{\sigma_n}\}^k \leq C_k \lim_{T=\infty} \frac{1}{2T} \int_{-T}^{T} |f(x + it)|^k \, dt\]
Hence, for \(y \leq 0\)
\[\left(\frac{|a_n|e^{\sigma_n}}{k \lambda_n + \delta}\right)^k - \frac{e^{-\delta|\sigma_n|} |a_n|e^{(k \lambda_n + \delta)y}}{k \lambda_n + \delta} \leq C_k W_y, \delta, k(\sigma) \leq \frac{C_k}{\delta} M^k(\sigma, f)\]
The case when \(0 < k < 1\) may be treated as following:
\[\left[\frac{|a_n|e^{\sigma_n}}{k \lambda_n + \delta}\right]^{k+1} - \frac{e^{-\delta|\sigma_n|} |a_n|e^{(k+1) \lambda_n + \delta)y}}{k \lambda_n + \delta} \leq C_{k+1} \lim_{T=\infty} \frac{1}{2T} \int_{-T}^{T} |f(x + it)|^{k+1} \, dt\]
Also, from (2.1), for \(0 < k < 1\), we have
\[\left[\frac{|a_n|e^{\sigma_n}}{(k+1) \lambda_n + \delta}\right]^{k+1} - \frac{e^{-\delta|\sigma_n|} |a_n|e^{(k+1) \lambda_n + \delta)y}}{(k+1) \lambda_n + \delta} \leq C_{k+1} \lim_{T=\infty} \frac{1}{2T} \int_{-T}^{T} e^{\delta|\sigma_n|} |f(x + it)|^{k+1} \, dt \, dx\]
\[\leq C_{k+1} M(\sigma, f) W_{y, \delta, k}(\sigma) \leq \frac{1}{\delta} C_{k+1} M^{k+1}(\sigma, f)\]
Since \(f(s)\) is of finite Ritt-order \(\rho\), it follows that ([1], p. 719), ([9], p. 265)
\[(2.6) \quad \log \mu(\sigma, f) \sim \log M(\sigma, f)\]
and
\[\log \log \mu(\sigma, f) \sim \log \lambda(\sigma, f)\]
Hence (2.2), (2.3), (2.4), (2.5) and (2.6) together imply the lemma. Proof of theorem 1, follows immediately from the above lemma.

Theorem 2. — If \(f(s)\) is an entire function of finite Ritt-order \(\rho\) and is defined by (1.1) then
\[\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \{W_{y, \delta, k}(\sigma, f')/W_{y, \delta, k}(\sigma, f)\}}{\sigma} = k\rho\]
provided \(k\) is an even integer.
For \(k = 2\), this result is due to Juneja [6] and for \(k\) even it follows with the use of Minkowski’s inequality and the
fact that $W_{r,\delta,k}(\sigma)$ is an increasing convex function of $\sigma$
for $k$ even.

In the end author wishes to express his sincere thanks to Professor R. S. L. Srivastava for his kind guidance. The author is also thankful to the referee for his suggestions.

REFERENCES


Manuscrit reçu le 24 avril 1970.

S. K. Bajpai,
Department of Mathematics,
Indian Institute of Technology,
Kanpur 16, India.