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ON THE MEAN VALUES OF AN ENTIRE FUNCTION REPRESENTED BY DIRICHLET SERIES

by S. K. BAJPAI

1. Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ defined everywhere in the complex plane by absolutely convergent Dirichlet series

$$(1.1) \qquad f(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$$

where $0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$ as $n \to \infty$.

As usual, the symbols $M(\sigma, f)$, $\mu(\sigma, f)$ and $\nu(\sigma, f)$ denote the maximum modulus, the maximum term and the rank of the maximum term respectively for $f(s)$, and can be found in [8]. We define

$$A_k(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k \, dt$$

and

$$W_{y,\delta,k}(\sigma) = \lim_{T\to\infty} \frac{e^{-\delta\sigma}}{2T} \int_{y}^{\sigma} \int_{-T}^{T} |f(x + it)|^k e^{\delta x} \, dx \, dt; \qquad \delta > 0.$$

The mean values for $k = 2$ and $y = 0$ were defined by Hadamard [2] and also by Kamthan ([3], [4]) who has obtained the following main results for $k = 2$ and $y = 0$

$$(1.2) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log A_k(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

$$(1.3) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log W_{y,\delta,k}(\sigma)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}$$

Juneja [6] also proved (1.3) by using a different technique for $k = 2$ and $y = 0$. In [7] another extension of (1.3) has been obtained for $k = 2$ and $y = -\infty$. Kamthan [5] has further attempted to establish (1.2) for every $k > 0$, but his arguments of the proof are vague. Thus, in the present paper, we restrict ourselves to functions of finite Ritt-order ρ, establish (1.2) and extend (1.3) for every $k > 0$ by a different technique.

2. We have

THEOREM 1. — *If $f(s)$, defined by Dirichlet series* (1.1), *be an entire function of finite Ritt-order ρ, lower order λ, type* T *and lower type t, then, for* $0 < k < \infty$

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log A_k(\sigma)}{\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log W_{y,\delta,k}(\sigma)}{\sigma}$$

and

$$\begin{matrix}kT \\ kt\end{matrix} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log A_k(\sigma)}{e^{\rho\sigma}} = \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log W_{y,\delta,k}(\sigma)}{e^{\rho\sigma}}$$

First we prove the following:

LEMMA. — *Let $f(s)$, given by* (1.1), *be an entire function of finite Ritt-order ρ, then, for* $0 < k < \infty$

$$\log A_k(\sigma) \sim \log W_{y,\delta,k}(\sigma) \sim k \log M(\sigma, f).$$

Proof. — It is well known

$$(2.1) \qquad |a_n| e^{\sigma\lambda_n} = \left| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(\sigma + it) e^{-i\lambda_n t}\, dt \right|$$

First, let $k \geqslant 1$, then by Holder's inequality, we have

$$(2.2) \qquad |a_n| e^{\sigma\lambda_n} \leqslant C_k \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^k\, dt \right]^{1/k}$$

where

$$C_k = \begin{cases} 1 \text{ if } k = 1 \\ 4 \left[\dfrac{\Gamma\left(\dfrac{1}{2} + \dfrac{k}{2(k-1)}\right)}{\sqrt{\pi}\,\Gamma\left(1 + \dfrac{k}{2(k-1)}\right)} \right]^{\frac{k-1}{k}} \text{ if } k > 1. \end{cases}$$

Further, from (2.2) we have

$$\{|a_n|e^{x\lambda_n}\}^k \leqslant C_k^k \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x+it)|^k\, dt$$

Hence, for $y \leqslant 0$

$$(2.3)\quad \frac{\{|a_n|e^{\sigma\lambda_n}\}^k}{k\lambda_n+\delta} - \frac{e^{-\delta\sigma}|a_n|^k e^{(k\lambda_n+\delta)y}}{k\lambda_n+\delta} \leqslant C_k^k W_{y,\delta,k}(\sigma) \leqslant \frac{C_k^k}{\delta} M^k(\sigma, f)$$

The case when $0 < k < 1$ may be treated as following:

$$(2.4)\quad |a_n|e^{\sigma\lambda_n} \leqslant C_{k+1}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^{k+1}\,dt\right\}^{\frac{1}{k+1}}$$
$$\leqslant C_{k+1}\{M(\sigma, f)A_k(\sigma)\}^{\frac{1}{k+1}}$$

Also, from (2.1), for $0 < k < 1$, we have

$$(2.5)\quad \frac{\{|a_n|e^{\sigma\lambda_n}\}^{k+1}}{(k+1)\lambda_n+\delta} - \frac{e^{-\delta\sigma}|a_n|^{k+1}e^{((k+1)\lambda_n+\delta)y}}{(k+1)\lambda_n+\delta}$$
$$\leqslant C_{k+1}^{k+1}\lim_{T\to\infty}\frac{e^{-\delta\sigma}}{2T}\int_0^{\sigma}\int_{-T}^{T} e^{x\delta}|f(x+it)|^{k+1}\,dt\,dx$$
$$\leqslant C_{k+1}^{k+1}M(\sigma, f)W_{y,\delta,k}(\sigma) \leqslant \frac{1}{\delta}C_{k+1}^{k+1}M^{k+1}(\sigma, f)$$

Since $f(s)$ is of finite Ritt-order ρ, it follows that ([1], p. 719), ([9], p. 265)

$$(2.6)\qquad \log\mu(\sigma, f) \sim \log M(\sigma, f)$$

and

$$\log\log\mu(\sigma, f) \sim \log\lambda_{\nu(\sigma,f)}$$

Hence (2.2), (2.3), (2.4), (2.5) and (2.6) together imply the lemma. Proof of theorem 1, follows immediately from the above lemma.

THEOREM 2. — *If $f(s)$ is an entire function of finite Ritt-order ρ and is defined by* (1.1) *then*

$$\lim_{\sigma\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ \frac{\log\{W_{y,\delta,k}(\sigma, f')/W_{y,\delta,k}(\sigma, f)\}}{\sigma} = \frac{k\rho}{k\lambda}$$

provided k is an even integer.

For $k = 2$, this result is due to Juneja [6] and for k even it follows with the use of Minkowski's inequality and the fact that $W_{\gamma,\delta,k}(\sigma)$ is an increasing convex function of σ for k even.

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