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On the Hausdorff summability of series associated with a Fourier and its allied series


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ON THE HAUSDORFF SUMMABILITY
OF SERIES ASSOCIATED WITH A FOURRIER
AND ITS ALLIED SERIES

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1. Let $S_n$ be the nth partial sum of an infinite series $\sum_1^\infty a_n$ and let

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) S_\nu .$$

(1.1)

Then the sequence $\{t_n\}$ is known as the Hausdorff means of sequence $\{S_n\}$, where $\{\mu_\nu\}$ is a sequence of real or complex numbers and the sequence $\{\Delta^p \mu_\nu\}$ denotes the differences of order $p$.

The series $\sum_1^\infty a_n$ is said to summable by Hausdorff mean to the sum $S$, if $\lim t_n = S$, whenever $S_n \to S$. The necessary and sufficient condition for the Hausdorff summability to be conservative is that the sequence $\{\mu_n\}$ should be a sequence of moment constant, i.e.

$$\mu_n = \int_0^1 x^n d\chi(x), \ n \geq 0 ;$$

where $\chi(x)$ is a real function of bounded variation in $0 \leq x \leq 1$. We may suppose without loss of generality that $\chi(0) = 0$, if also $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then $\mu_n$ is a regular moment constant and the Hausdorff method i.e. $(H, \mu_n)$ is a regular method of summation [2].

If

$$\sum_{n=0}^{\infty} |(t_n - t_{n-1})| < \infty ,$$

(1.2)

then the series $\sum_1^\infty a_n$ is said to be absolutely summable $(H, \mu_n)$ or
summable \(|H, \mu_n|\). It is also known that the Cesàro, Holder and Euler methods of summation are the particular cases of the above method.

2. Let \(f(t)\) be a periodic function with period \(2\pi\) and integrable in the sense of Lebesgue in \((-\pi, \pi)\). Let its Fourier series be

\[
\frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)
\]

and its allied series is

\[
\sum_{1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} B_n(t).
\]

We write

\[
\varphi(t) = \frac{1}{2} \{ f(\theta + t) + f(\theta - t) \},
\]

\[
\psi(t) = \frac{1}{2} \{ f(\theta + t) - f(\theta - t) \}.
\]

Let \(g(x)\) be integrable \(L\) in \((0, 1)\), then for \(\varepsilon > 0\)

\[
g^+_{\varepsilon}(x) = \frac{1}{\Gamma(\varepsilon)} \int_{0}^{x} (x - u)^{\varepsilon-1} g(u) \, du,
\]

\[
g^-_{\varepsilon}(x) = \frac{1}{\Gamma(\varepsilon)} \int_{x}^{1} (u - x)^{\varepsilon-1} g(u) \, du.
\]

Again, let

\[
U_n(t) = \sum_{\nu=1}^{n} e^{i\nu t},
\]

\[
H(n, x, t) = E(n, x, t) + i F(n, x, t)
\]

\[
= \sum_{\nu=0}^{n} \nu^\theta \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} e^{i\nu t}.
\]

The object of this paper is to prove the following:
Theorem 1. — If

i) \( \int_0^t |\phi(u)| \, du = 0(t) \)

ii) \((H, \mu_n)\) is conservative

and

iii) \( \begin{cases} 
    \text{either (a) } \chi(x) = g_{1+\beta+\epsilon}(x) + c, \epsilon > 0 ; \\
    \text{or (b) } \chi(x) = g^+_{1+\beta+\epsilon}(x) + c, \epsilon > 0 ; 
\end{cases} \)

for some \( g(x) \in L (0, 1) ; \)

then the series \( \sum_{n=1}^{\infty} \frac{A_n(t)}{n^{1-\beta}} \), for \( \beta > 0 \) is summable \( |H, \mu_n| \) at \( t = \theta \),

where \( c \) is an absolute constant.

Theorem 2. — If

i) \( \int_0^t |\psi(u)| \, du = 0(t) \)

ii) \((H, \mu_n)\) is conservative

and

iii) \( \begin{cases} 
    \text{either (a) } \chi(x) = g_{1+\beta+\epsilon}(x) + c, \epsilon > 0 ; \\
    \text{or (b) } \chi(x) = g^+_{1+\beta+\epsilon}(x) + c, \epsilon > 0 ; 
\end{cases} \)

for some \( g(x) \in L (0, 1) \),

then the series \( \sum_{n=1}^{\infty} \frac{B_n(t)}{n^{1-\beta}} \), for \( \beta > 0 \) is summable \( |H, \mu_n| \) at \( t = \theta \), where \( c \) is an absolute constant.

It may also be remarked if

\[ \chi(x) = 1 - (1 - x)^{\delta}, \quad \delta > 0 ; \]

the method \((H, \mu_n)\) reduces to the well known Cesàro method of summation of order \( \delta \).

Further if we choose \( \beta \) such that \( \delta > \beta + \epsilon \) then it can be proved that \( \chi(x) - 1 \) is the \((1 + \beta + \epsilon)\)th backward integral of
and $\chi(x)$ is also the $(\varepsilon + \beta + 1)$ th forward integral of

$$
\frac{\delta}{\Gamma(1 - \beta - \varepsilon)} \left\{ x^{-(\beta + \varepsilon)} + (1 - \beta) \int_0^x (1 - \nu)^{\delta - 2}(x - \nu)^{-(\beta + \varepsilon)} d\nu \right\}.
$$

Hence the method $|C, \delta|$ satisfies the hypothesis of our theorem 1 and 2 for $\varepsilon > 0$, $\delta > \beta \geq 0$ and the following theorems of Cheng [1] becomes the corollary of our theorems.

**Theorem.** — The series $\sum \frac{A_n(t)}{n^{1-\beta}}$ for $0 \leq \beta < 1$ is summable $|C, \delta|$ for $\delta > \beta$, at the point $\theta$, whenever i) of theorem 1 holds and similarly the series $\sum \frac{B_n(t)}{n^{1-\beta}}$, for $0 \leq \beta < 1$, is summable $|C, \delta|$, for $\delta > \beta$, at the point $\theta$, whenever i) of theorem 2 holds.

3. For the proof of the theorems, we require the following lemmas.

**Lemma 1.** — Uniformly in $0 < t \leq \pi$

$$
|U_n(t)| \leq \frac{k}{t}. \quad (3.1)
$$

This can be easily proved.

**Lemma 2.** — If $g(x)$ and $h(x)$ be Lebesgue integrable in $(0, 1)$, then for $\varepsilon > 0$

$$
\int_0^1 g^+(x) h(x) \, dx = \int_0^1 g(x) h^+(x) \, dx. \quad (3.2)
$$

This is known [3].

**Lemma 3.** — Uniformly in $0 \leq x \leq 1$

$$
\int_0^x H(n, \nu, t) \, dv = 0 \left( \frac{n^{\beta-1}}{t} \right) \quad (3.3)
$$
Lemma 4. - Let $\beta > 0$ $\varepsilon > 0$ and fixed, then for $\beta + \varepsilon < 1$

$$\int_0^x (x - u)^{\beta + \varepsilon - 1} H(n, u, t) \, du = 0 \left( \frac{n-\varepsilon}{t^{\beta + \varepsilon}} \right) \quad (3.4)$$

uniformly in $0 \leq x \leq 1$ and similarly

$$\int_x^1 (u - x)^{\beta + \varepsilon - 1} \times H(n, u, t) \, du = 0 \left( \frac{n-\varepsilon}{t^{\beta + \varepsilon}} \right) \quad (3.5)$$

The lemma 3 and 4 are due to Tripathy [4].

Proof of Theorem 1. - If $t_n$ and $u_n$ denote the Hausdorff means of $\sum \frac{A_n(\theta)}{n^{1-\beta}}$ and the sequence $\{ n A_n(\theta) \}$ then for $n \geq 1$

$$u_n = n(t_n - t_{n-1})$$

Hence, from (1.2) the series $\sum \frac{A_n(\theta)}{n^{1-\beta}}$ is summable $|H, \mu_n|$, if

$$I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{\nu=1}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) v^\beta A_\nu(\theta) \right| < \infty.$$ 

Since $(H, \mu_n)$ is conservative, we have

$$I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n\nu} v^\beta A_\nu(\theta) \right|$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n\nu} v^\beta \int_0^\pi \varphi(t) \cos \nu t \, dt \right|$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n\nu} v^\beta \cos \nu t \, dt \right|$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\pi |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n\nu} v^\beta \cos \nu t \, dt \right|$$

$$= I_1 + I_2 \text{, say.}$$

Since

$$|H(n, x, t)| \leq n^\beta \left| \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n\nu} \right|$$

$$= n^\beta$$
We have

\[ I_1 = 0(1) \sum_{n=1}^{\infty} \frac{1}{n} n^\beta \int_0^1 |\varphi(t)| \, dt \int_0^1 |d\chi(x)| \]

\[ = 0(1) \sum_{n=1}^{\infty} \frac{1}{n^{1-\beta}} \cdot \frac{1}{n} \]

\[ = 0(1). \]

Without loss of generality, we can suppose that \( \beta + \varepsilon < 1 \), if a) \( \chi(x) = g_{1+\beta+\varepsilon}(x) + c \), then

\[ I_2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\pi |\varphi(t)| \cdot \left| \int_0^1 g_{\varepsilon+\beta}(x) E(n, x, t) \, dx \right| \, dt \]

\[ = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\pi |\varphi(t)| \cdot \left| \int_0^1 g(x) E^+_{\beta+\varepsilon}(n, x, t) \, dx \right| \, dt. \]

Since

\[ E^+_{\beta+\varepsilon}(n, x, t) = \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^\pi (x - u)^{\beta+\varepsilon-1} E(n, u, t) \, du \]

\[ = \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^\pi (x - u)^{\beta+\varepsilon-1} I_m H(n, u, t) \, du \]

\[ = 0 \left( \frac{1}{n^{\varepsilon+\beta+\varepsilon}} \right), \text{ by lemma-4}. \]

Therefore

\[ I_2 \leq \int_0^1 |g(x)| \, dx \sum_{n=1}^{\infty} \frac{1}{n} \int_1^\pi |\varphi(t)| \cdot 0 \left( \frac{1}{n^{\varepsilon+\beta+\varepsilon}} \right) \, dt \]

\[ = \int_0^1 |g(x)| \, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \int_1^\pi \frac{|\varphi(t)|}{t^{\beta+\varepsilon}} \, dt \]

\[ = \int_0^1 |g(x)| \, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \{ 0(1) + 0(n^{\beta+\varepsilon-1}) \} \]

\[ = 0(1) \int_0^1 |g(x)| \, dx \]

\[ = 0(1). \]
If b) \( \chi(x) = g_{1+\beta}e^x(x) + c \), then proceeding in a similar way as in case a) and using estimate (4.5) of lemma 4, it can be proved that

\[ I_2 = 0(1). \]

This completes the proof of theorem-1.

If we use the condition i) of Theorem-2 instead of the condition i) of theorem-1, we can prove that the series \( \sum B_n(\theta) \frac{\mu_n}{n^{1-\beta}} \) is summable \( |H, \mu_n| \).

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