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# ON THE HAUSDORFF SUMMABILITY OF SERIES ASSOCIATED WITH A FOURRIER AND ITS ALLIED SERIES

by B. L. GUPTA

1. Let $S_n$ be the nth partial sum of an infinite series $\sum_1^\infty a_n$ and let

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) S_\nu . \quad (1.1)$$

Then the sequence $\{t_n\}$ is known as the Hausdorff means of sequence $\{S_n\}$, where $\{\mu_\nu\}$ is a sequence of real or complex numbers and the sequence $\{\Delta^p \mu_\nu\}$ denotes the differences of order $p$.

The series $\sum_1^\infty a_n$ is said to summable by Hausdorff mean to the sum S, if lim $t_n \to S$, whenever $S_n \to S$. The necessary and sufficient condition for the Hausdorff summability to be conservative is that the sequence $\{\mu_n\}$ should be a sequence of moment constant, i.e. ;

$$\mu_n = \int_0^1 x^n \, d\chi(x), \ n \geqslant 0 ;$$

where $\chi(x)$ is a real function of bounded variation in $0 \leqslant x \leqslant 1$. We may suppose without loss of generality that $\chi(0) = 0$, if also $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then $\mu_n$ is a regular moment constant and the Hausdorff method i.e. $(H, \mu_n)$ is a regular method of summation [2].

If

$$\sum_{n=0}^{\infty} |(t_n - t_{n-1})| < \infty , \quad (1.2)$$

then the series $\sum_1^\infty a_n$ is said to be absolutely summable $(H, \mu_n)$ or summable $|H, \mu_n|$. It is also known that the Cesàro, Holder and Euler methods of summation are the particular cases of the above method.

2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue in $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

and its allied series is

$$\sum_{1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{1}^{\infty} B_n(t) .$$

We write

$$\varphi(t) = \frac{1}{2} \{ f(\theta + t) + f(\theta - t) \} ,$$

$$\psi(t) = \frac{1}{2} \{ f(\theta + t) - f(\theta - t) \} .$$

Let $g(x)$ be integrable L in $(0, 1)$, then for $\varepsilon > 0$

$$g_\varepsilon^+(x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x (x - u)^{\varepsilon - 1} g(u) \, du ,$$

$$g_\varepsilon^-(x) = \frac{1}{\Gamma(\varepsilon)} \int_x^1 (u - x)^{\varepsilon - 1} g(u) \, du .$$

Again, let

$$U_n(t) = \sum_{\nu=1}^{n} e^{i\nu t} ,$$

$$H(n, x, t) = E(n, x, t) + i F(n, x, t)$$

$$= \sum_{\nu=0}^{n} \nu^\beta \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} e^{i\nu t} .$$

The object of this paper is to prove the following :

THEOREM 1. – *If*

i) $\int_0^t |\varphi(u)|\, du = 0(t)$

ii) $(H, \mu_n)$ *is conservative*

*and*

iii) *either* (a) $\chi(x) = g^{-}_{1+\beta+\varepsilon}(x) + c,\ \varepsilon > 0$ ;
*or* (b) $\chi(x) = g^{+}_{1+\beta+\varepsilon}(x) + c,\ \varepsilon > 0$ ;

*for some* $g(x) \in L\ (0, 1)$ ;

*then the series* $\sum_{n=1}^{\infty} \frac{A_n(t)}{n^{1-\beta}}$, *for* $1 > \beta \geqslant 0$ *is summable* $|H, \mu_n|$ *at* $t = \theta$, *where c is an absolute constant.*

THEOREM 2. – *If*

i) $\int_0^t |\psi(u)|\, du = 0(t)$

ii) $(H, \mu_n)$ *is conservative*

*and*

iii) *either* (a) $\chi(x) = g^{-}_{1+\beta+\varepsilon}(x) + c,\ \varepsilon > 0$ ;
*or* (b) $\chi(x) = g^{+}_{1+\beta+\varepsilon}(x) + c,\ \varepsilon > 0$ ;

*for some* $g(x) \in L\ (0, 1)$,

*then the series* $\sum_{n=1}^{\infty} \frac{B_n(t)}{n^{1-\beta}}$, *for* $1 > \beta \geqslant 0$ *is summable* $|H, \mu_n|)$ *at* $t = \theta$, *where c is an absolute constant.*

It may also be remarked if

$$\chi(x) = 1 - (1 - x)^{\delta}, \quad \delta > 0 ;$$

the method $(H, \mu_n)$ reduces to the well known Cesàro method of summation of order $\delta$.

Further if we choose $\beta$ such that $\delta > \beta + \varepsilon$ then it can be proved that $\chi(x) - 1$ is the $(1 + \beta + \varepsilon)$ th backward integral of

$$- \frac{\Gamma(1 + \delta)}{\Gamma(\delta - \beta - \varepsilon)} (1 - x)^{\delta - \beta - \varepsilon - 1}$$

and $\chi(x)$ is also the $(\varepsilon + \beta + 1)$ th forward integral of

$$\frac{\delta}{\Gamma(1 - \beta - \varepsilon)} \left\{ x^{-(\beta + \varepsilon)} + (1 - \beta) \int_0^x (1 - v)^{\delta - 2} (x - v)^{-(\beta + \varepsilon)} dv \right\} .$$

Hence the method $|C, \delta|$ satisfies the hypothesis of our theorem 1 and 2 for $\varepsilon > 0$, $\delta > \beta \geqslant 0$ and the following theorems of Cheng [1] becomes the corrollary of our theorems.

THEOREM. – *The series* $\sum \frac{A_n(t)}{n^{1-\beta}}$ *for* $0 \leqslant \beta < 1$ *is summable* $|C, \delta|$ *for* $\delta > \beta$, *at the point* $\theta$, *whenever* i) *of theorem* I *holds and similarly the series* $\sum_1^\infty \frac{B_n(t)}{n^{1-\beta}}$, *for* $0 \leqslant \beta < 1$, *is summable* $|C, \delta|$, *for* $\delta > \beta$, *at the point* $\theta$, *whenever* i) *of theorem 2 holds.*

3. For the proof of the theorems, we require the following lemmas.

LEMMA 1. – *Uniformly in* $0 < t \leqslant \pi$

$$|U_n(t)| \leqslant \frac{k}{t} \cdot \tag{3.1}$$

This can be easily proved.

LEMMA 2. – *If* $g(x)$ *and* $h(x)$ *be Lebesgue integrable in* $(0, 1)$, *then for* $\varepsilon > 0$

$$\int_0^1 g_\varepsilon^+(x)\, h(x)\, dx = \int_0^1 g(x)\, h_\varepsilon^-(x)\, dx . \tag{3.2}$$

This is known [3].

LEMMA 3. – *Uniformly in* $0 \leqslant x \leqslant 1$

$$\int_0^x H(n, v, t)\, dv = 0\left(\frac{n^{\beta - 1}}{t}\right) \tag{3.3}$$

LEMMA 4. — *Let* $\beta \geqslant 0$ $\varepsilon > 0$ *and fixed, then for* $\beta + \varepsilon < 1$

$$\int_0^x (x-u)^{\beta+\varepsilon-1} \, \mathrm{H}(n,u,t)\, du = 0\left(\frac{n^{-\varepsilon}}{t^{\beta+\varepsilon}}\right) \tag{3.4}$$

*uniformly in* $0 \leqslant x \leqslant 1$ *and similarly*

$$\int_x^1 (u-x)^{\beta+\varepsilon-1} \times \mathrm{H}(n,u,t)\, du = 0\left(\frac{n^{-\varepsilon}}{t^{\beta+\varepsilon}}\right) \cdot \tag{3.5}$$

The lemma 3 and 4 are due to Tripathy [4].

*Proof of Theorem 1.* — If $t_n$ and $u_n$ denote the Hausdorff means of $\sum \frac{\mathrm{A}_n(\theta)}{n^{1-\beta}}$ and the sequence $\{n\,\mathrm{A}_n(\theta)\}$ then for $n \geqslant 1$

$$u_n = n(t_n - t_{n-1}) \,.$$

Hence, from (1.2) the series $\sum\limits_{n=1}^{\infty} \frac{\mathrm{A}_n(\theta)}{n^{1-\beta}}$ is summable $|\mathrm{H}, \mu_n|$, if

$$\mathrm{I} = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{\nu=1}^{n} \binom{n}{\nu} (\Delta^{n-\nu}\mu_\nu)\, \nu^\beta \mathrm{A}_\nu(\theta) \right| < \infty \,.$$

Since $(\mathrm{H}, \mu_n)$ is conservative, we have

$$\mathrm{I} = \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \nu^\beta \mathrm{A}_\nu(\theta) \right|$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \nu^\beta \int_0^\pi \varphi(t) \cos \nu t \, dt \right|$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\frac{1}{n}} |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \nu^\beta \cos \nu t \right| dt$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \left| \int_0^1 d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \nu^\beta \cos \nu t \right| dt$$

$$= \mathrm{I}_1 + \mathrm{I}_2 \,, \text{ say.}$$

Since

$$|\mathrm{H}(n,x,t)| \leqslant n^\beta \left| \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1-x)^{n-\nu} \right.$$

$$= n^\beta$$

We have

$$I_1 = 0(1) \sum_{n=1}^{\infty} \frac{1}{n} n^{\beta} \int_0^{\frac{1}{n}} |\varphi(t)|\, dt \int_0^1 |d\chi(x)|$$

$$= 0(1) \sum_{n=1}^{\infty} \frac{1}{n^{1-\beta}} \cdot \frac{1}{n}$$

$$= 0(1)\,.$$

Without loss of generality, we can suppose that $\beta + \varepsilon < 1$, if a) $\chi(x) = g^{-}_{1+\beta+\varepsilon}(x) + c$, then

$$I_2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot \left| \int_0^1 g^{-}_{\varepsilon+\beta}(x)\, E(n, x, t)\, dx \right| dt$$

$$= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot \left| \int_0^1 g(x)\, E^{+}_{\beta+\varepsilon}(n, x, t)\, dx \right| dt\,.$$

Since

$$E^{+}_{\beta+\varepsilon}(n, x, t) = \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^x (x-u)^{\beta+\varepsilon-1}\, E(n, u, t)\, du$$

$$= \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^x (x-u)^{\beta+\varepsilon-1}\, I_m H(n, u, t)\, du$$

$$= 0\left(\frac{1}{n^{\varepsilon} t^{\beta+\varepsilon}}\right), \text{ by lemma-4.}$$

Therefore

$$I_2 \leqslant \int_0^1 |g(x)|\, dx \sum_{n=1}^{\infty} \frac{1}{n} \int_{\frac{1}{n}}^{\pi} |\varphi(t)| \cdot 0\left(\frac{1}{n^{\varepsilon} t^{\beta+\varepsilon}}\right) dt$$

$$= \int_0^1 |g(x)|\, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \int_{\frac{1}{n}}^{\pi} \frac{|\varphi(t)|}{t^{\beta+\varepsilon}}\, dt$$

$$= \int_0^1 |g(x)|\, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \{0(1) + 0(n^{\beta+\varepsilon-1})\}$$

$$= 0(1) \int_0^1 |g(x)|\, dx$$

$$= 0(1)\,.$$

If b) $\chi(x) = g^{+}_{1+\beta+\varepsilon}(x) + c$, then proceeding in a similar way as in case a) and using estimate (4.5) of lemma 4, it can be proved that

$$I_2 = 0(1) .$$

This completes the proof of theorem-1.

If we use the condition i) of Theorem-2 instead of the condition i) of theorem-1, we can prove that the series $\sum \frac{B_n(\theta)}{n^{1-\beta}}$ is summable $|H, \mu_n|$.

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