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On the Hausdorff summability of series associated with a Fourier and its allied series


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ON THE HAUSDORFF SUMMABILITY
OF SERIES ASSOCIATED WITH A FOURRIER
AND ITS ALLIED SERIES

by B. L. GUPTA

1. Let $S_n$ be the nth partial sum of an infinite series $\sum_1^\infty a_n$ and let

$$t_n = \sum_{\nu=0}^n \binom{n}{\nu} (\Delta^n \nu \mu_\nu) S_\nu .$$

(1.1)

Then the sequence $\{t_n\}$ is known as the Hausdorff means of sequence $\{S_n\}$, where $\{\mu_\nu\}$ is a sequence of real or complex numbers and the sequence $\{\Delta^p \mu_\nu\}$ denotes the differences of order $p$.

The series $\sum_1^\infty a_n$ is said to summable by Hausdorff mean to the sum $S$, if $\lim_{n \to \infty} t_n = S$, whenever $S_n \to S$. The necessary and sufficient condition for the Hausdorff summability to be conservative is that the sequence $\{\mu_\nu\}$ should be a sequence of moment constant, i.e.;

$$\mu_\nu = \int_0^1 x^n d\chi(x), \ n \geq 0 ;$$

where $\chi(x)$ is a real function of bounded variation in $0 \leq x \leq 1$. We may suppose without loss of generality that $\chi(0) = 0$, if also $\chi(1) = 1$ and $\chi(+0) = \chi(0) = 0$, so that $\chi(x)$ is continuous at the origin, then $\mu_\nu$ is a regular moment constant and the Hausdorff method i.e. $(H, \mu_\nu)$ is a regular method of summation [2].

If

$$\sum_{n=0}^\infty |(t_n - t_{n-1})| < \infty ,$$

(1.2)

then the series $\sum_1^\infty a_n$ is said to be absolutely summable $(H, \mu_\nu)$ or
summable $|H, \mu_n|$. It is also known that the Cesàro, Holder and Euler methods of summation are the particular cases of the above method.

2. Let $f(t)$ be a periodic function with period $2\pi$ and integrable in the sense of Lebesgue in $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t)$$

and its allied series is

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt) = \sum_{n=0}^{\infty} B_n(t).$$

We write

$$\varphi(t) = \frac{1}{2} \{f(\theta + t) + f(\theta - t)\},$$

$$\psi(t) = \frac{1}{2} \{f(\theta + t) - f(\theta - t)\}.$$

Let $g(x)$ be integrable $L$ in $(0, 1)$, then for $\varepsilon > 0$

$$g^{+}_{\varepsilon}(x) = \frac{1}{\Gamma(\varepsilon)} \int_{0}^{x} (x - u)^{\varepsilon-1} g(u) \, du,$$

$$g^{-}_{\varepsilon}(x) = \frac{1}{\Gamma(\varepsilon)} \int_{x}^{1} (u - x)^{\varepsilon-1} g(u) \, du.$$

Again, let

$$U_n(t) = \sum_{\nu=0}^{n} e^{i\nu t},$$

$$H(n, x, t) = E(n, x, t) + i F(n, x, t)$$

$$= \sum_{\nu=0}^{n} \nu^\theta \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} e^{i\nu t}.$$

The object of this paper is to prove the following:
THEOREM 1. — If 

i) \( \int_0^t |\varphi(u)| \, du = 0(t) \)

ii) \((H, \mu_n)\) is conservative

and

iii) \[
\begin{cases}
  \text{either (a)} & \chi(x) = g_{1+\beta+\varepsilon}(x) + c, \quad \varepsilon > 0 ; \\
  \text{or (b)} & \chi(x) = g_{1+\beta+\varepsilon}^+(x) + c, \quad \varepsilon > 0 ;
\end{cases}
\]

for some \( g(x) \in L(0,1) \);

then the series \( \sum_{n=1}^{\infty} \frac{A_n(t)}{n^{1-\beta}} \), for \(|\beta| > 0\) is summable \( |H, \mu_n| \) at \( t = \theta \), where \( c \) is an absolute constant.

THEOREM 2. — If 

i) \( \int_0^t |\psi(u)| \, du = 0(t) \)

ii) \((H, \mu_n)\) is conservative

and

iii) \[
\begin{cases}
  \text{either (a)} & \chi(x) = g_{1+\beta+\varepsilon}(x) + c, \quad \varepsilon > 0 ; \\
  \text{or (b)} & \chi(x) = g_{1+\beta+\varepsilon}^+(x) + c, \quad \varepsilon > 0 ;
\end{cases}
\]

for some \( g(x) \in L(0,1) \),

then the series \( \sum_{n=1}^{\infty} \frac{B_n(t)}{n^{1-\beta}} \), for \(|\beta| > 0\) is summable \( |H, \mu_n| \) at \( t = \theta \), where \( c \) is an absolute constant.

It may also be remarked if 

\( \chi(x) = 1 - (1 - x)^\delta , \quad \delta > 0 \); 

the method \((H, \mu_n)\) reduces to the well known Cesàro method of summation of order \( \delta \).

Further if we choose \( \beta \) such that \( \delta > \beta + \varepsilon \) then it can be proved that \( \chi(x) - 1 \) is the \((1 + \beta + \varepsilon)\)th backward integral of
\[
- \frac{\Gamma(1 + \delta)}{\Gamma(\delta - \beta - \varepsilon)} (1 - x)^{\delta - \beta - \varepsilon - 1}
\]
and \(x(x)\) is also the \((\varepsilon + \beta + 1)\) th forward integral of

\[
\frac{\delta}{\Gamma(1 - \beta - \varepsilon)} \left\{ x^{-(\beta + \varepsilon)} + (1 - \beta) \int_0^x (1 - \nu)^{\delta - 2}(x - \nu)^{-(\beta + \varepsilon)} d\nu \right\}.
\]

Hence the method \(|C, \delta|\) satisfies the hypothesis of our theorem 1 and 2 for \(\varepsilon > 0, \delta > \beta \geq 0\) and the following theorems of Cheng [1] becomes the corrollary of our theorems.

**Theorem.** — The series \(\sum A_n(t) / n^{1-\beta}\) for \(0 \leq \beta < 1\) is summable \(|C, \delta|\) for \(\delta > \beta\), at the point \(\theta\), whenever i) of theorem 1 holds and similarly the series \(\sum B_n(t) / n^{1-\beta}\), for \(0 \leq \beta < 1\), is summable \(|C, \delta|\), for \(\delta > \beta\), at the point \(\theta\), whenever i) of theorem 2 holds.

3. For the proof of the theorems, we require the following lemmas.

**Lemma 1.** — Uniformly in \(0 < t \leq \pi\)

\[
|U_n(t)| \leq \frac{k}{t}.
\]  
(3.1)

This can be easily proved.

**Lemma 2.** — If \(g(x)\) and \(h(x)\) be Lebesgue integrable in \((0, 1)\), then for \(\varepsilon > 0\)

\[
\int_0^1 g^+(x) h(x) \, dx = \int_0^1 g(x) h^-(x) \, dx.
\]  
(3.2)

This is known [3].

**Lemma 3.** — Uniformly in \(0 \leq x \leq 1\)

\[
\int_0^x H(n, \nu, t) \, dv = 0 \left( \frac{n^{\beta-1}}{t} \right)
\]  
(3.3)
Lemma 4. — Let \( \beta > 0 \) \( \varepsilon > 0 \) and fixed, then for \( \beta + \varepsilon < 1 \)

\[
\int_{0}^{x} (x - u)^{\beta + \varepsilon - 1} H(n, u, t) \, du = 0 \left( \frac{n^{-\varepsilon}}{t^{\beta + \varepsilon}} \right) \quad (3.4)
\]

uniformly in \( 0 \leq x \leq 1 \) and similarly

\[
\int_{x}^{1} (u - x)^{\beta + \varepsilon - 1} \times H(n, u, t) \, du = 0 \left( \frac{n^{-\varepsilon}}{t^{\beta + \varepsilon}} \right). \quad (3.5)
\]

The lemma 3 and 4 are due to Tripathy [4].

Proof of Theorem 1. — If \( t_n \) and \( u_n \) denote the Hausdorff means of \( \sum \frac{A_n(\theta)}{n^{1-\beta}} \) and the sequence \( \{ n A_n(\theta) \} \) then for \( n \geq 1 \)

\[
u_n = n(t_n - t_{n-1}).
\]

Hence, from (1.2) the series \( \sum_{n=1}^{\infty} \frac{A_n(\theta)}{n^{1-\beta}} \) is summable \( |H, \mu_n| \), if

\[
I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{\nu=1}^{n} \binom{n}{\nu} (\Delta^{n-\nu} \mu_\nu) v^\beta A_\nu(\theta) \right| < \infty.
\]

Since \( (H, \mu_n) \) is conservative, we have

\[
I = \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{1} d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} v^\beta A_\nu(\theta) \right| \]

\[
= 2 \pi \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{1} d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} v^\beta \int_{0}^{\pi} \varphi(t) \cos \nu t \, dt \right|
\]

\[
\leq \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_{0}^{1} d\chi(x) \sum_{\nu=1}^{n} \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} v^\beta \cos \nu t \, dt \right|
\]

\[
= I_1 + I_2, \text{ say.}
\]

Since

\[
|H(n, x, t)| \leq n^\theta | \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu (1 - x)^{n-\nu} = n^\theta,
\]
We have

\[
I_1 = 0(1) \sum_{n=1}^{\infty} \frac{1}{n} n^\beta \int_0^1 |\varphi(t)| \, dt \int_0^1 |d\chi(x)|
= 0(1) \sum_{n=1}^{\infty} \frac{1}{n^{1-\beta}} \cdot \frac{1}{n}
= 0(1).
\]

Without loss of generality, we can suppose that \(\beta + \varepsilon < 1\), if

\[\chi(x) = g_{1+\beta+\varepsilon}(x) + c,\]  

then

\[
I_2 = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\pi |\varphi(t)| \cdot \left| \int_0^1 g_{\varepsilon+\beta}(x) E(n, x, t) \, dx \right| \, dt
= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\pi |\varphi(t)| \cdot \left| \int_0^1 g(x) E_{\beta+\varepsilon}(n, x, t) \, dx \right| \, dt .
\]

Since

\[
E_{\beta+\varepsilon}(n, x, t) = \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^x (x - u)^{\beta + \varepsilon - 1} E(n, u, t) \, du
= \frac{1}{\Gamma(\beta + \varepsilon)} \int_0^x (x - u)^{\beta + \varepsilon - 1} I_m H(n, u, t) \, du
= 0\left(\frac{1}{n^{\varepsilon/2}}\right), \text{ by lemma-4.}
\]

Therefore

\[
I_2 \leq \int_0^1 \left| g(x) \right| \, dx \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\pi |\varphi(t)| \cdot 0\left(\frac{1}{n^{\varepsilon/2}}\right) \, dt
= \int_0^1 \left| g(x) \right| \, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \int_0^\pi |\varphi(t)| \cdot \frac{1}{t^{\beta + \varepsilon}} \, dt
= \int_0^1 \left| g(x) \right| \, dx \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(0(1) + 0(n^{\beta + \varepsilon - 1})\right)
= 0(1) \int_0^1 \left| g(x) \right| \, dx
= 0(1).
\]
If b) \( x = g^+ + e(x) + c \), then proceeding in a similar way as in case a) and using estimate (4.5) of lemma 4, it can be proved that

\[ I_2 = O(1). \]

This completes the proof of theorem-1.

If we use the condition i) of Theorem-2 instead of the condition i) of theorem-1, we can prove that the series \( \sum \frac{B_n(\theta)}{n^{1-\beta}} \) is summable \([H, \mu_n]\).

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