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SOME IMBEDDING PROPERTIES OF HILBERT SUBSPACES IN TOPOLOGICAL VECTOR SPACES (1)

by Eberhard GERLACH

Introduction.

In Chap. III of [3] the writer showed that any proper functional Hilbert space $\mathfrak{H}$ of analytic functions on a domain $D \subset \mathbf{C}^n$, with reproducing kernel $K(z, \zeta)$, is Hilbert-Schmidt expansible. That is, given $\mathfrak{H}$, a dense Hilbert-Schmidt subspace Φ of $\mathfrak{H}$ was found such that $K(\circ, \zeta) \in \Phi$ for all $\zeta \in D$. Consequently the generalized eigenvectors for any selfadjoint operator in $\mathfrak{H}$ could be regarded as functions on D, and they were moreover elements of the proper functional Hilbert space Φ^* ; one had the "rigging" $\Phi \subset \mathfrak{H} \subset \Phi^*$ of $\mathfrak{H}$. Showing analyticity of the generalized eigenfunctions [3] (and also [4]) – actually of all the functions in Φ^* – then amounted to proving that the function $\zeta \to K(\cdot, \zeta)$ is analytic not only from D into $\mathfrak{H}$ but also from D into Φ.

Subsequently K. Maurin [7] used another approach to the question of regularity of generalized eigenfunctions. In general, instead of a rigging $\Phi \subset \mathfrak{H} \subset \Phi^*$ with Hilbert spaces and Hilbert-Schmidt imbeddings, one may use a nuclear space L which has

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dense and continuous imbedding into $\mathfrak{H}$: $L \subset \mathfrak{H} \subset L^*$ (2). For instance, if $\mathcal{E}(D)$ is the space of all infinitely differentiable functions on a domain $D \subset \mathbf{R}^n$ with the usual topology of uniform convergence of all derivatives on compact subsets of D, and $\mathfrak{H}$ is a Hilbert subspace of $\mathcal{E}(D)$, the rigging $L \subset \mathfrak{H} \subset L^*$ with nuclear L is available, where L^* is the closure of $\mathfrak{H}$ in $\mathcal{E}(D)$ and $L = L^{**}$; consequently the generalized eigenfunctions of any selfadjoint operator in $\mathfrak{H}$ are infinitely differentiable. If moreover $\mathfrak{H}$ lies in the nullspace of a (closed) hypoelliptic differential operator T $- \mathfrak{H} \subset N = \{f \in \mathcal{E}(D) : Tf = 0\}$ $-$ then also the generalized eigenfunctions φ lie in N, i.e. $T\varphi = 0$. Maurin [7] presents results of this type when $\mathfrak{H}$ is a weighted L^2-space contained in $\mathcal{E}(D)$, and he formulates them on manifolds ; in the case of holomorphicity he discusses differential forms as well as functions. In further work [8] he considers Hilbert spaces of harmonic functions (in the axiomatic sense of H. Bauer [2]) with an L^2-norm over the boundary and finds that the generalized eigenfunctions are also harmonic. In [9] he extends the holomorphicity results to functions defined on complex spaces(3).

From this second approach to the regularity of generalized eigenfunctions and related problems, we may abstract a general procedure which is presented in section 2 below. Section 3 contains a few supplements to § 9 of [11]. In the final section 4 we state results on regularity of generalized eigenfunctions. In section 1 we collect the notations and basic results to be used.

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(2) In the "general theory of abstract eigenfunction expansions", rigging by means of a nuclear space L has been used by many authors. One advantage of this is that for a given selfadjoint operator A on a separable Hilbert space $\mathcal{H}$, one can construct a dense, nuclear subspace L of $\mathcal{H}$ such that $AL \subset L$ and A is continuous on L ; then spectral theory for A can be developed further than in the general case. From various other points of view, a rigging by Hilbert spaces seems to be more useful.

(3) The results of [7-9] are formulated for (generalized eigenfunctions of) commuting families of bounded normal operators. But they may equally well be formulated for any commutative von Neumann algebra of operators on the given Hilbert space $\mathcal{H}$ or, equivalently, for any selfadjoint operator in $\mathcal{H}$ (since any commutative von Neumann algebra is equal to the algebra of bounded (Borel) functions of some selfadjoint operator A).

1. Notation and definitions.

Throughout this paper, E will denote a locally convex Hausdorff topological vector space which is moreover assumed to be boundedly complete([4]). By E^* we denote the anti-dual $\overline{E'} = \overline{E}'$ (cf. [11]), consisting of all continuous conjugate-linear functionals on E. The space E is said to be *dually nuclear* [10] if E^* with the strong topology is nuclear. In several of our examples we shall make use of the fact that a nuclear F-space or DF-space is reflexive and dually nuclear (theorems 12 and 13 in section 4.4 of [10]).

A *Hilbert subspace* $\mathcal{H}$ of E is a Hilbert space $\mathcal{H}$ with continuous imbedding into E. By a *Hilbert-Schmidt subspace* $\mathcal{K}$ of E we understand a Hilbert space $\mathcal{K}$ with the following property : there exists a Hilbert subspace $\mathcal{H}$ of E such that $\mathcal{K}$ is a linear subspace of $\mathcal{H}$ for which the imbedding $\mathcal{K} \subset \mathcal{H}$ is of Hilbert-Schmidt type. All Hilbert spaces in this paper are tacitly assumed to be separable.

If $\mathcal{H}$ is a Hilbert subspace of E with injection j, and θ the canonical isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$, then the (*Schwartz*) *kernel* of $\mathcal{H}$ rel. E is the (weakly continuous) map $H = j\theta j^*$, where $j^* : E^* \to \mathcal{H}^*$ is the adjoint map to j [11].

If X is a set, we denote the space C^X of complex valued functions on X, with the topology of pointwise convergence, by G_X. A proper functional Hilbert space $\mathcal{H}$ on X is simply a Hilbert subspace of G_X. If S is the Schwartz kernel of $\mathcal{H}$ rel. G_X then the (*Aronszajn*) *reproducing kernel* A of $\mathcal{H}$ is given by $A(\cdot, x) = S\delta_{(x)}$, where $\delta_{(x)}$ is the Dirac measure at the point x. The reproducing formula simply says $f(x) = (f, A(\circ, x))$ for all $f \in \mathcal{H}$, $x \in X$. (Cf. § 9 of [11]).

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{G}$ a Hilbert subspace of it with injection j and Schwartz kernel $j\theta j^*$. Then $J = j\theta j^* \theta^{-1}$ (more precisely $j\theta_{\mathcal{G}} j^* \theta_{\mathcal{H}}^{-1}$) is selfadjoint in $\mathcal{H}$; its square root $J^{1/2}$ regarded as an operator $\mathcal{H} \to \mathcal{G}$ is the canonical partial isometry of $\mathcal{H}$ onto $\mathcal{G}$. We shall occasionally refer to J as the *Hilbert kernel* of $\mathcal{G}$ rel. $\mathcal{H}$. The imbedding j is Hilbert-Schmidt if and only if J is nuclear in $\mathcal{H}$.

([4]) "Quasi-complete" in Bourbaki terminology ; every closed bounded subset of E is complete.

2. Hilbert subspaces of dually nuclear spaces.

PROPOSITION 1. – *Let the space* E *be dually nuclear, and* $\mathcal{H}$ *a Hilbert subspace of* E. *Then the generalized eigenvectors of any self-adjoint operator in* $\mathcal{H}$ *lie in* E. *Consequently, if the elements of* E *have a certain property, this property is inherited by the generalized eigenvectors.*

Proof. – Since E is dually nuclear, so is every linear subspace of E with the induced topology (Proposition 5.1.2 in [10]). Let L be the closure of $\mathcal{H}$ in E. Then L^* is nuclear and $\mathcal{H}$ dense in L. Hence the adjoint $j^* : L^* \to \mathcal{H}^*$ of the given injection $j : \mathcal{H} \to L$ is also injective. If θ is the canonical isomorphism $\mathcal{H}^* \to \mathcal{H}$, we have the rigging

$$L^* \xrightarrow{\theta j^*} \mathcal{H} \xrightarrow{j^{**}} L^{**} \tag{1}$$

with nuclear L^*. By Proposition 4.4.11 in [10], every boundedly complete nuclear or dually nuclear locally convex space is semireflexive. Thus $L^{**} = L$ and $j^{**} = j$ at least algebraically.

Remarks. – 1) As it stands, generalized eigenvectors of self-adjoint operators in $\mathcal{H}$, regarded as limits of vectors in $\mathcal{H}$, exist only in the topology of L^{**} which may be coarser than that of L.

2) The injection $j : \mathcal{H} \to L$ is both weakly and strongly continuous (and dense). To start with, j^* is just weakly continuous. Let $\mathfrak{A} = \{ j(A) : A \text{ bounded in } \mathcal{H}\}$, then all sets in $\mathfrak{A}$ are bounded in L, and j^* is continuous for the topology of $\mathfrak{A}$-convergence on L^* and the strong topology on $\mathcal{H}^*$; hence $j^* : L_b^* \to \mathcal{H}^*$ (the subscript b denotes strong topology) is also continuous. Thus all maps in the rigging (1) are strongly continuous.

The result of Proposition 1 may be improved to a Hilbert-Schmidt rigging as follows.

THEOREM 2. – *Let* E *be a boundedly complete Hausdorff locally convex space. Then every Hilbert subspace* $\mathcal{H}$ *of* E *for which the closure* L *of* $\mathcal{H}$ *in* E *is dually nuclear*([5]), *is a Hilbert-Schmidt subspace*

([5]) Equivalently : every Hilbert subspace $\mathcal{H}$ of E which is dually nuclear in the topology induced by E.

of L. *Consequently, if* Ψ *is a Hilbert space with continuous dense imbeddings* $\mathcal{H} \subset \Psi \subset L$ *the first of which is of Hilbert-Schmidt type, and* A *any selfadjoint operator in* $\mathcal{H}$ *with spectral measure* μ, *then* μ*-almost all generalized eigenvectors of* A *are approximated in the norm of* Ψ *by elements in* $\mathcal{H}$.

Proof. – Let $\mathcal{H}$ be a Hilbert subspace of E, L its closure in E, and j the imbedding of $\mathcal{H}$ into L. Then the adjoint j^* is an injection of the nuclear space L_b^* into $\mathcal{H}^*$ which is strongly continuous (Remark 2) above). By a fundamental property of nuclear spaces, this continuous map j^* may be factored through another Hilbert space Φ, $L_b^* \xrightarrow{\varphi} \Phi \xrightarrow{J} \mathcal{H}^*$ and $j^* = J\varphi$, so that J is Hilbert-Schmidt. We know that $j^* L^* = L^*$ is dense in $\mathcal{H}^*$. Hence Φ, φ and J may be chosen so that J is a dense injection of Φ into $\mathcal{H}^*$. Set $\Psi = \Phi^*$ in the chain of dense continuous imbeddings

$$L^* \to \Phi \xrightarrow{J} \mathcal{H}^* \xrightarrow{\theta} \mathcal{H} \xrightarrow{J^*} \Phi^* \to L \ ,$$

where J^* is Hilbert-Schmidt. The Hilbert-Schmidt rigging

$$\Phi \xrightarrow{\theta J} \mathcal{H} \xrightarrow{J^*} \Phi^* = \Psi$$

may now be used to do spectral theory in $\mathcal{H}$.

3. Hilbert subspaces in spaces of smooth functions.

Let X be a set now and E continuously imbedded in G_X. If $\mathcal{H}$ is a Hilbert subspace of E, then $\mathcal{H}$ must trivially have a reproducing kernel. For instance the Hilbert spaces of harmonic functions, considered in [8], have reproducing kernels.

Conversely one may pose the following question. Let $\mathcal{H}$ be a Hilbert subspace of G_X, with reproducing kernel A, and suppose E is continuously imbedded in G_X. Under what conditions on A is $\mathcal{H}$ also a Hilbert subspace of E ? When is $\mathcal{H}$ compactly imbedded in E ? Necessary as well as sufficient conditions were given by L. Schwartz [11] in a number of concrete cases. In particular we mention the following (Propositions 24 and 25 in [11]).

i) X is a locally compact Hausdorff space and E the space $C^0(X)$ of all continuous functions on X, with the topology of uniform convergence on compacts.

ii) X is a domain in Euclidean $\mathbf{R}^n$ and E the space $C^m(X)$ of functions possessing continuous derivatives of all orders $\leqslant m$, with the topology of uniform convergence on compacts for all these derivatives ; $1 \leqslant m \leqslant \infty$.

We shall make use of these propositions (pp. 191-201 in [11]) and provide some supplements to them.

Let X and E be as in i) or ii) above, and let N be a closed subspace of E. Then the proper functional Hilbert space $\mathcal{H}$ on X, with reproducing kernel A, is a Hilbert subspace of N if and only if it is a Hilbert subspace of E and in addition $A(\cdot, x) \in N$ for all $x \in X$. In particular, if X is a domain in $\mathbf{C}^k = \mathbf{R}^{2k}$ and $N(\subset C^0(X))$ the space $\mathfrak{A}(X)$ of holomorphic functions on X – or more generally in any $\mathbf{R}^n$, the nullspace of a hypoelliptic linear differential operator with constant coefficients – Proposition 24 of [11] yields a complete description of Hilbert subspaces of N. Similarly, if X is locally compact and N is a harmonic space on X (in the sense of [2]), we obtain a characterization of Hilbert subspaces of this harmonic space. (In all these examples N is a nuclear Fréchet space and hence dually nuclear).

In the case ii), for any linear differential operator P with C^∞-coefficients and order $\leqslant m$, it turns out that $P_2 A(\cdot, x) \in \mathcal{H}$ for all $x \in X$ (the subscript 2 indicates that the derivatives are applied to the second set of variables of A), and $\overline{P}h(x) = (h, P_2 A(\cdot, x))_{\mathcal{H}}$ for $h \in \mathcal{H}$ where $\overline{P}$ is obtained from P by taking the complex conjugate of all coefficients (cf. p. 199-200 in [11]). As it stands, this is the result of applying P to the function $x \to A(\cdot, x)$ in either the pointwise or distribution sense. It will be useful to have the derivatives $D_2^p A(\cdot, x)$ of A (as limits of the corresponding difference expressions) existing in the norm of $\mathcal{H}$, and also the functions $x \to D_2^p A(\cdot, x)$ continuous in the norm of $\mathcal{H}$, $|p| \leqslant m$. To this end one may use the results of [11] (p. 199-201, as well as Proposition 9 ter (p. 152), bearing in mind that if $\mathcal{B}$ is a Banach space and $E = C^m(X)$ as described above, $\mathcal{B} \check{\otimes}_\varepsilon E = \mathcal{B} \hat{\otimes}_\varepsilon E = C^m(X, \mathcal{B})$ is the space of functions $X \to \mathcal{B}$ which possess continuous derivatives of all orders $\leqslant m$ in the *norm* of $\mathcal{B}$, endowed with the topology of uniform norm-convergence of all these derivatives on compacts in X). For our purposes we are interested in direct proofs for strong derivatives of $A(\cdot, x)$.

Our basic tool is the proof of Proposition 4 in [4]. By imitating that proof we obtain the following.

PROPOSITION 3. – *Let $\mathcal{B}$ be a Banach space with dual (or anti-dual) $\mathcal{B}'$ and* X *a domain in* $\mathbf{R}^n$; $1 \leqslant m < \infty$. *If* $\varphi : X \to \mathcal{B}'$ *is any function which belongs to* $C^m(X)$ *weak*, i.e.* $x \to \langle f , \varphi(x) \rangle$ *is in* $C^m(x)$ *for all* $f \in \mathcal{B}$, *then* $\varphi \in C^{m-1}(X , \mathcal{B}')$. *The same holds with both* C^m *and* C^{m-1} *replaced by* C^∞, *as well as* C^ω *(real-analytic functions).*

Proposition 3 also remains true if both C^m and C^{m-1} are replaced by $C^{m,\alpha}$, where $C^{m,\alpha}(X)$ denotes the space of all functions in $C^m(X)$ whose derivatives of order m are Hölder continuous with exponent α. (The exponent $\alpha \in]0 , 1]$ may even vary on different compacts $K \subset X$). To obtain this, one uses the following simple result.

LEMMA 4. – *Let* $\{X , d\}$ *be a compact metric space,* $\mathcal{B}'$ *the dual of some Banach space, and* $\varphi : X \to \mathcal{B}'$ *Hölder continuous with exponent* α *in the weak* topology of* $\mathcal{B}'$. *Then* φ *is* α*-Hölder continuous in the norm of* $\mathcal{B}'$.

Proof. – By hypothesis there exist constants C_f so that for each $f \in \mathcal{B}$, $|\langle f , \varphi(x) - \varphi(y) \rangle| \leqslant C_f d^\alpha(x , y)$ or

$$|\langle f , \frac{\varphi(x) - \varphi(y)}{d^\alpha(x , y)} \rangle| \leqslant C_f$$

for all $x \neq y$ in X. The uniform boundedness theorem then provides a constant C so that

$$\|\varphi(x) - \varphi(y)\| \leqslant C d^\alpha(x , y)$$

for all x, y.

As we are interested in triples of Hilbert spaces

$$\Phi \subset \mathcal{H} \subset \Phi^* \subset E \subset G_X$$

for the purpose of spectral theory in $\mathcal{H}$, we would like to find the reproducing kernel of $\mathcal{H}$ belonging to some regularity class, in the stronger norm of the space Φ. The following lemma is the first step.

LEMMA 5. – *Let* $\mathcal{H}$ *be any Hilbert subspace of* G_X , *with reproducing kernel* A, *and* Φ *a Hilbert subspace of* $\mathcal{H}$ *with dense imbedding ;*

thus there are the dense continuous imbeddings $\Phi \subset \mathcal{H} \subset \Phi^*$, *the pairing of* Φ *and* Φ^* *realized via the scalar product of* $\mathcal{H}$. *Then* $A(\cdot, x) \in \Phi$ *for all* $x \in X$ *if and only if also* Φ^* *is a Hilbert subspace of* G_X. *When this is the case, the reproducing kernel of* Φ^* *taken in* Φ *is precisely* $A(\cdot, \cdot)$, *and for any subset* Y *of* X *on which every* $\varphi \in \Phi^*$ *is bounded, the function* $x \to \|A(\cdot, x)\|_\Phi$ *is bounded on* Y.

Proof. – Suppose $A(\cdot, x) \in \Phi$ for all $x \in X$. Then we may identify each $\varphi \in \Phi^*$ with a function $x \to \widetilde{\varphi}(x) = (\varphi, A(\cdot, x))_{\Phi^*, \Phi}$ (the correspondence $\varphi \to \widetilde{\varphi}$ is clearly injective), and

$$|\widetilde{\varphi}(x)| \leqslant \|\varphi\|_{\Phi^*} \|A(\cdot, x)\|_\Phi$$

for each fixed x. Hence Φ^* is a proper functional Hilbert space on X.

Conversely, let Φ^* be a Hilbert subspace of G_X. Then its reproducing kernel C taken in Φ, say, is expressed by

$$\varphi(x) = (\varphi, C(\cdot, x))_{\Phi^*, \Phi} = (\varphi, C(\cdot, x))_{\mathcal{H}} ,$$

but for all f in the dense subspace $\mathcal{H}$ of Φ^* we have

$$f(x) = (f, A(\cdot, x))_{\mathcal{H}} .$$

Thus $C = A$ and $A(\cdot, x) \in \Phi$ for all $x \in X$.

Finally, let Φ^* be a proper functional Hilbert space as described, and let $Y \subset X$ be such that each $\varphi \in \Phi^*$ is bounded on Y,

$$M_{Y,\varphi} \geqslant |\varphi(y)| = |(\varphi, A(\cdot, x))_{\Phi^*, \Phi}| , \quad x \in Y .$$

By the uniform boundedness theorem there is a constant M_Y such that $\|A(\cdot, x)\|_\Phi \leqslant M_Y$ for all $x \in Y$.

COROLLARY. – *In the case* i) *above, if* $\mathcal{H}$ *is a Hilbert subspace of* $E = C^0(X)$, *then* Φ^* *is a Hilbert subspace of* $E = C^0(X)$ *if and only if the function* $x \to \|A(\cdot, x)\|_\Phi$ *is bounded on each compact subset of* X.

Remarks. – 1) When $\mathcal{H}$ is a Hilbert subspace of $E = C^0(X)$, there is in general no guarantee for existence of a (dense) Hilbert subspace Φ of $\mathcal{H}$ (preferably with Hilbert-Schmidt imbedding) for which also $\Phi^* \subset E$. The results of § 2 show that such Φ can be found at least whenever $\mathcal{H}$ with the topology induced by $E = C^0(X)$ is dually nuclear.

2) We shall leave aside consideration of questions similar to those in Lemma 5 in cases when Φ is a Hilbert subspace of $\mathcal{H}$ not dense in $\mathcal{H}$, or when one has Hilbert subspaces $\Phi \subset \Psi \subset \mathcal{H}$ of $\mathcal{H}$, each dense in $\mathcal{H}$, but Φ a closed subspace of Ψ.

PROPOSITION 6. — *Let* X *be a domain in* $\mathbf{R}^n$, *and* E *one of the following spaces* : $C^m(X)$ $(1 \leqslant m < \infty)$, $C^{m,a}(X)$, $C^\infty(X)$, $C^\omega(X)$, *with the usual topology*(6). *Let* $\mathcal{H}$ *be a Hilbert subspace of* E *and* Φ *a dense Hilbert subspace of* $\mathcal{H}$ *with continuous imbeddings*

$$\Phi \subset \mathcal{H} \subset \Phi^* \subset E .$$

Then the reproducing kernel $A(\cdot, x)$ *of* $\mathcal{H}$ *belongs to* E *in the norm of* Φ, *more precisely, the function* $x \to A(\cdot, x) \in \Phi$ *is of class* C^{m-1}, $C^{m,a}$, C^∞ *or real-analytic, respectively, in the norm topology of* Φ.

The proof is immediate from Proposition 3.

So far we could not prove continuity of the function $x \to A(\cdot, x)$ or existence and continuity of its m-th order derivatives, respectively, in the norm topology when $\mathcal{H} \subset C^m(X)$ and $m = 0$ or $1 \leqslant m < \infty$. We now attend to these cases.

When $E = C^0(X)$ where X is locally compact, and $\mathcal{H}$ is a Hilbert subspace of E, then $\mathcal{H}$ is compactly imbedded in E if and only if the reproducing kernel function $x \to A(\cdot, x)$ is continuous with respect to the norm of $\mathcal{H}$ (proof of Proposition 24 in [11]). Similarly, if $\Phi \subset \mathcal{H} \subset \Phi^* \subset C^0(X)$, then Φ^* has compact imbedding in $C^0(X)$ if and only if $x \to A(\cdot, x) \in \Phi$ is continuous for the norm of Φ. In particular this is the case when $\mathcal{H}$ (or Φ^*) is dually nuclear in the topology induced by $C^0(X)$.

We turn to the remaining case, $C^m(X)$ with $1 \leqslant m < \infty$.

PROPOSITION 7. — *Let the chain of Hilbert subspaces*

$$\Psi \subset \Phi \subset \mathcal{H} \subset \Phi^* \subset \Psi^*$$

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(6) On the space $C^\omega(X)$ of real-analytic functions on X we use the inductive or projective topology ; these two agree in our case and make $C^\omega(X)$ a complete nuclear and dually nuclear space. Cf. Theorem 1.2 in [6] ; in Martineau's notation we are dealing with $H_{l,X}(C^n) = H_{p,X}(C^n) = C^\omega(X)$. Proposition 4 of [4] amounts to saying that, if the proper functional Banach space $\mathcal{B}$ is linearly contained in $C^\omega(X)$, it is also continuously imbedded.

of $C^m(X)$ *be given, with the imbedding* $\Psi \subset \Phi$ *compact. Then the function* $x \to A(\cdot, x)$ (A *the reproducing kernel for the given chain*) *is of class* $C^m(X)$ *in the norm of* Φ.

Proof. – For any $|p| \leqslant m$, the derivatives $D_2^p A(\cdot, x)$ exist in the weak sense, and for each $f \in \Psi^*$, $(f, D_2^p A(\cdot, x)) = D^p f(x)$ is the limit of the corresponding difference expressions. Thus the difference expressions for $D_2^p A(\cdot, x)$ converge in the weak topology of Ψ, and as the imbedding $\Psi \subset \Phi$ is compact, they converge to $D_2^p A(\cdot, x)$ in the norm of Φ. Similarly it is seen that $x \to D_2^p A(\cdot, x)$ is continuous with respect to the norm of Φ.

4. Regularity properties of generalized eigenfunctions.

PROPOSITION 8. – *Let* X *be a domain in* $\mathbf{R}^n$ *and* P *any linear differential operator with real (for simplicity) coefficients which are of class* C^∞ *or real-analytic, and* $\mathcal{H}$ *a Hilbert subspace of* $E = C^\infty(X)$ *or* $E = C^\omega(X)$, *respectively. If* $Pf = 0$ *for all* $f \in \mathcal{H}$, *then for any Hilbert space* Φ *densely imbedded in* $\mathcal{H}$ *with* $\Phi \subset \mathcal{H} \subset \Phi^* \subset E$, ($P_2 A(\cdot, x) = 0$ *and hence*) $P\varphi = 0$ *for all* $\varphi \in \Phi^*$.

Proof. – $Pf = 0$ for all $f \in \mathcal{H}$ implies $P_2 A(\cdot, x) = 0$. But as $P_2 A(\cdot, x)$ exists in the norm of Φ, we may transfer derivatives and find $P\varphi(x) = (P\varphi, A(\cdot, x))_{\Phi^*, \Phi} = (\varphi, P_2 A(\cdot, x))_{\Phi^*, \Phi} = 0$ for all $\varphi \in \Phi^*$, $x \in X$.

Remarks. – 1) This proposition extends to the case "$Pf(x) = 0$ for all $x \in Y$" for any subset Y of X.

2) An alternative proof of Proposition 8 would proceed as follows. The operator P is continuous on E, and its restriction to $\mathcal{H}$ is a closed operator in $\mathcal{H}$ (note that for arbitrarily fixed $\varphi \in E$, the function $\varphi(x) f(x)$ with $f \in \mathcal{H}$ may fail to be in $\mathcal{H}$). P has closed nullspace N in E, and hence $\mathcal{H} \subset N$ implies $\Phi^* \subset N$.

As a corollary to all the foregoing considerations, the theorem below now follows at once. It was announced, without proof, by N. Aronszajn [1].

THEOREM 9. – *Let* X *be a domain in* $\mathbf{R}^n$ *and* $\mathcal{H}$ *a proper functional Hilbert space of functions of class* C^∞ [*real-analytic, or complex analytic with* $\mathbf{R}^n = \mathbf{C}^m$ *and* $2m = n$]. *Then for any selfadjoint operator* A *in* $\mathcal{H}$ *with spectral measure* μ, μ*-almost all generalized eigenfunctions of* A *are also of class* C^∞ [*real or complex analytic, respectively*]. *Moreover, if all functions in* $\mathcal{H}$ *satisfy some linear differential equations with coefficients of class* C^∞ [*with real or complex analytic coefficients*], *then the generalized eigenfunctions satisfy the same equations.*

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