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Y. K. KWON
LEO SARIO
BERTRAM WALSH

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BEHAVIOR OF BIHARMONIC FUNCTIONS ON WIENER'S AND ROYDEN'S COMPACTIFICATIONS

by Y.K. KWON, L. SARIO, and B. WALSH

In the theory of bending of thin plates the biharmonic functions play an important role ; their local properties have been studied by several authors (cf. Bergman-Schiffer [1], Vekua [9], Garabedian [3]). The main purpose of the present paper is to establish some global properties of biharmonic functions in terms of Wiener's and Royden's compactifications of a smooth Riemannian manifold (see also Nakai-Sario [4], [5]). For notation and terminology we refer the reader to the monograph Sario-Nakai [7].

1. On a smooth Riemannian manifold R of dimension $n \geqslant 2$, consider the Laplace-Beltrami operator

$$\Delta u = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, g^{ij} \frac{\partial u}{\partial x^j} \right)$$

where $x = (x^1, \ldots, x^n)$ is the local coordinate, (g^{ij}) the inverse matrix of the fundamental metric tensor (g_{ij}), and g the determinant of (g_{ij}).

A C^4-function u on R satisfying the equation

$$\Delta^2 u = \Delta\Delta u = 0$$

is called a *biharmonic function* on R. In view of a theorem of de Rham [6, p. 149], every biharmonic function is smooth. We denote by W(R) the family of all biharmonic functions on R.

As an example of a simple biharmonic function we give the function v of the following

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LEMMA. — *If the volume of* R *is finite and* R *is hyperbolic, then the function*

$$\nu(x) = \int_{\mathrm{R}} g(x, y)\, dy$$

is biharmonic on R. *Here* $g(x, y)$ *is the harmonic Green's function on* R *with singularity at* y, *and* dy *the volume element of* R.

Proof. — For an arbitrary $x \in \mathrm{R}$ choose a real $\alpha = \alpha_x > 0$ so large that the set

$$\mathrm{A} = \{y \in \mathrm{R} \mid g(x, y) > \alpha\}$$

is relatively compact in R. If $g_{\mathrm{A}}(x, y)$ is the Green's function of A, then

$$0 \leqslant \nu(x) - \int_{\mathrm{A}} g_{\mathrm{A}}(x, y)\, dy = \int_{\mathrm{A}} [g(x, y) - g_{\mathrm{A}}(x, y)]\, dy +$$

$$+ \int_{\mathrm{R}-\mathrm{A}} g(x, y)\, dy \leqslant \alpha \operatorname{vol}(\mathrm{A}) + \alpha \operatorname{vol}(\mathrm{R} - \mathrm{A}) = \alpha \operatorname{vol}(\mathrm{R}) .$$

Since $\int_{\mathrm{A}} g_{\mathrm{A}}(x, y)\, dy \leqslant \int_{\mathrm{A}} g(x, y)\, dy < \infty$, $\nu(x)$ is well-defined on R. In view of the fact that $\Delta\nu = -1$ (see Theorem 3 below), we can draw the desired conclusion.

We remark in passing that the finiteness of the volume of R is not necessary for $\nu(x)$ to be defined on R. In fact, take

$$\mathrm{R} = \{x = (x^1, x^2) \mid |x|^2 = (x^1)^2 + (x^2)^2 < 1\}$$

with the metric tensor $g_{ij} = (1 - r)^{-1}\, \delta_{ij}$, where $r = \sqrt{(x^1)^2 + (x^2)^2}$. It is easy to see that $\operatorname{vol}(\mathrm{R}) = \infty$ and

$$\nu(x) \leqslant \frac{1}{\varepsilon} \log \frac{\varepsilon + 2}{\varepsilon} + \frac{1}{2}\, \varepsilon \log(2 + \varepsilon) + 1 - \log \varepsilon < \infty$$

for any $x \in \mathrm{R}$ and $0 < 2\varepsilon < 1 - |x|$.

2. *Throughout the following discussion we assume that the manifold* R *has finite volume.* Let N(R) be the Wiener algebra, which consists of bounded continuous harmonizable functions on R, and $\mathrm{N}_\delta(\mathrm{R})$ the Wiener potential subalgebra, i.e. the subfamily of functions $f \in \mathrm{N}(\mathrm{R})$ whose harmonic projections πf on R vanish identically (cf. Sario-Nakai [7]).

THEOREM. – *Let* v *be an element of the Wiener potential subalgebra* $N_\delta(R)$ *such that* Δv *is bounded. Then*

$$v(x) = -\int_R g(y, x)\, \Delta v(y)\, dy$$

on R.

Proof. – For a regular exhaustion $\{R_m\}$ of R, let $g_m(y, x)$ be the Green's function on R_m and $\{B_l\}$ a sequence of parametric balls about $x \in R_1$ such that $\overline{B}_l \subset R_1$ and the sequence $\{B_l\}$ shrinks down to x as $l \to \infty$. Then the Green's formula yields

$$\int_{\partial(R_m - B_l)} v(y) * dg_m(y, x) - g_m(y, x) * dv(y) =$$
$$= -\int_{R_m - B_l} g_m(y, x)\, \Delta v(y)\, dy\,.$$

On the other hand

$$\int_{\partial(R_m - B_l)} g_m(y, x) * dv(y) = -\int_{\partial B_l} g_m(y, x) * dv(y) \to 0$$
$$\text{as} \quad l \to \infty,$$

and

$$\int_{R_m - B_l} g_m(y, x)\, \Delta v(y)\, dy \to \int_{R_m} g_m(y, x)\, \Delta v(y)\, dy \quad \text{as} \quad l \to \infty\,.$$

In view of $\int_{\partial B_l} v(y) * dg_m(y, x) \to -v(x)$ as $l \to \infty$, we obtain

$$v(x) = -\int_{\partial R_m} v(y) * dg_m(y, x) - \int_{R_m} g_m(y, x)\, \Delta v(y)\, dy$$

for all $x \in R_m$.

Consider $v_m \in H(R_m)$ such that $v_m \equiv v$ on $R - R_m$. Then we may assume that the sequence $\{v_m\}$ converges to zero uniformly on compact subsets of R (cf. Sario-Nakai [7]), and

$$v_m(x) = -\int_{\partial R_m} v(y) * dg_m(y, x)$$

on R_m. Thus Lebesgue's dominated convergence theorem yields

$$v(x) = -\int_R g(y, x)\, \Delta v(y)\, dy$$

on R as desired.

3. For the sake of completeness we include the proof of the following well-known theorem, which establishes a right inverse of the Laplace-Beltrami operator on bounded smooth functions :

THEOREM. – *For any function* $f \in C^\infty(R) \cap B(R)$,

$$\Delta_x \int_R g(x\,,y) f(y)\, dy = -f(x)$$

for all $x \in R$.

Proof. – Fix a point $x_0 \in R$, and construct a function h defined on a neighborhood U of x_0 in R such that $\Delta h = f$ on U (de Rham [6, p. 151]). Choose an open neighborhood V of x_0 with $\overline{V} \subset U$, and a function $\varphi \in C_0^\infty(R)$ with $\varphi \,|\, V \equiv 1$, $\varphi \,|\, R - U \equiv 0$, and $0 \leqslant \varphi \leqslant 1$. Then $\Delta(h\varphi) = f$ on V, and $h\varphi \in N_\delta(R)$ with its obvious extension to R.

By Theorem 2 we have

$$(h\varphi)(x) = -\int_R g(x\,,y)\, \Delta(h\varphi)(y)\, dy$$

on R. In particular on V,

$$\begin{aligned} h(x) = (h\varphi)(x) &= -\int_R g(x\,,y)\, \Delta(h\varphi)(y)\, dy \\ &= -\int_R g(x\,,y)\, [\Delta(h\varphi)(y) - f(y)]\, dy - \int_R g(x\,,y) f(y)\, dy\,. \end{aligned}$$

Moreover

$$f(x_0) = (\Delta h)(x_0) = -\left[\Delta_x \int_R g(x\,,y) f(y)\, dy\right]_{x=x_0}$$

as asserted, because the first integral on the right is harmonic on V (cf. Constantinescu-Cornea [2, p. 15]).

Remark. – The boundedness of $\Delta\nu$ in Theorem 2 and of f in Theorem 3 was used only to assure the existence of their Green's potentials. Although these potentials do exist under milder conditions we do not intend to seek the most general statements.

4. For a bounded measurable function f on R, set

$$(\Gamma f)(x) = -\int_R g(x\,,y) f(y)\, dy$$

on R. It is easy to see that Γf is harmonizable on R. If f belongs to the family B(R) of bounded continuous functions, then Γf is continuous and therefore the operator

$$\Gamma : \mathrm{B(R)} \to \mathrm{N(R)}$$

is well-defined whenever $\Gamma 1$ is bounded.

Set $\mathrm{WBB}_\Delta(\mathrm{R}) = \{u \in \mathrm{W(R)} \mid u,\ \Delta u \text{ are bounded}\}$, and denote by HB(R) the class of bounded harmonic functions on R.

THEOREM. – *Let* $\Gamma 1$ *be bounded on* R. *Then the decomposition*

$$\mathrm{WBB}_\Delta(\mathrm{R}) = \mathrm{HB(R)} \oplus \Gamma\mathrm{HB(R)}$$

is valid.

Proof. – In view of Theorem 3 it is easily seen that

$$\mathrm{HB(R)} + \Gamma\mathrm{HB(R)} \subseteq \mathrm{WBB}_\Delta(\mathrm{R})$$

when $\Gamma 1$ is bounded. Since every function in ΓHB(R) vanishes on the Wiener harmonic boundary Δ_N, the maximum principle for HB-functions yields

$$\mathrm{HB(R)} \cap \Gamma\mathrm{HB(R)} = \{0\}\,.$$

Thus it remains to show that every $u \in \mathrm{WBB}_\Delta(\mathrm{R})$ has the desired decomposition.

Let $\pi : \mathrm{N(R)} \to \mathrm{HB(R)}$ be the harmonic projection (cf. Sario-Nakai [7]). By Theorem 3, the function

$$u(x) - (\pi u)(x) - [\Gamma\Delta(u - \pi u)](x)$$

is a bounded harmonic function on R. Furthermore it vanishes on the Wiener harmonic boundary Δ_N and therefore on R. Thus

$$u = \pi u + \Gamma\Delta u$$

on R as desired.

COROLLARY. – *Suppose* $\Gamma 1$ *is bounded on* R. *Then for any* $m \geqslant 1$, dim $\mathrm{WBB}_\Delta(\mathrm{R}) = 2m$ *if and only if the cardinality of the Wiener harmonic boundary* Δ_N *of* R *is* m.

Proof. – It is known that the cardinality of Δ_N is m if and only if dim HB(R) $= m$.

Let $\{u_1, \ldots, u_m\}$ be a basis of the space HB(R). In view of Theorems 3 and 4, it is seen that the set $\{u_1, \ldots, u_m; \Gamma u_1, \ldots, \Gamma u_m\}$ forms a basis for the space $\mathrm{WBB}_\Delta(\mathrm{R})$.

Throughout the rest of our discussion we shall assume that the function $\Gamma 1$ *is bounded on* R.

5. Denote by $\mathrm{NN}_\Delta(\mathrm{R})$ the family of functions f on R with f, $\Delta f \in \mathrm{N}(\mathrm{R})$, and by $\mathrm{N}_\delta \mathrm{N}_{\delta\Delta}(\mathrm{R})$ the family of functions g on R with g, $\Delta g \in \mathrm{N}_\delta(\mathrm{R})$.

THEOREM. – *The following biharmonic decomposition of the class* $\mathrm{NN}_\Delta(\mathrm{R})$ *is valid :*

$$\mathrm{NN}_\Delta(\mathrm{R}) = \mathrm{WBB}_\Delta(\mathrm{R}) \oplus \mathrm{N}_\delta \mathrm{N}_{\delta\Delta}(\mathrm{R}) .$$

Proof. – Note that for any $v \in \mathrm{N}_\delta(\mathrm{R})$ with Δv bounded,

$$v(x) = -\int_{\mathrm{R}} g(x, y)\, \Delta v(y)\, dy = (\Gamma \Delta v)(x)$$

on R (Theorem 2).

Let $f \in \mathrm{NN}_\Delta(\mathrm{R})$. By the direct sum decomposition

$$\mathrm{N}(\mathrm{R}) = \mathrm{HB}(\mathrm{R}) \oplus \mathrm{N}_\delta(\mathrm{R}) ,$$

$f = u_1 + v_1$ for $u_1 \in \mathrm{HB}(\mathrm{R})$ and $v_1 \in \mathrm{N}_\delta(\mathrm{R})$. Thus the above remark yields $v_1 = \Gamma \Delta v_1$ on R. Since $\Delta v_1 = \Delta f = u_2 + v_2$ for some $u_2 \in \mathrm{HB}(\mathrm{R})$ and $v_2 \in \mathrm{N}_\delta(\mathrm{R})$, $v_1 = \Gamma u_2 + \Gamma v_2$ on R and therefore

$$f = (u_1 + \Gamma u_2) + \Gamma v_2$$

on R.

To show the uniqueness of the decomposition, suppose that $u \in \mathrm{WBB}_\Delta(\mathrm{R}) \cap \mathrm{N}_\delta \mathrm{N}_{\delta\Delta}(\mathrm{R})$. Since $\Delta u \in \mathrm{HB}(\mathrm{R}) \cap \mathrm{N}_\delta(\mathrm{R})$, $\Delta u \equiv 0$ on R and therefore $u \in \mathrm{HB}(\mathrm{R}) \cap \mathrm{N}_\delta(\mathrm{R}) = \{0\}$ as desired.

This completes the proof of the theorem.

6. We turn to the integral representation of the WBB_Δ-functions on R.

Let $\mathrm{P}(x, t)$ be the harmonic kernel on $\mathrm{R} \times \Delta_\mathrm{N}$ with $\mathrm{P}(x_0, t) \equiv 1$, and μ the harmonic measure on Δ_N centered at the fixed point $x_0 \in \mathrm{R}$.

As immediate consequences of Theorem 4 we state the following results.

THEOREM. – *Every* $u \in \mathrm{WBB}_\Delta(\mathrm{R})$ *has the integral representation*

$$u(x) = \int_{\Delta_N} \mathrm{P}(x, t) f(t)\, d\mu(t) - \iint_{\mathrm{R} \times \Delta_N} g(x, y)\, \mathrm{P}(y, t)\, \Delta u(t)\, d\mu(t)\, dy$$

on R.

THEOREM. – *Let f and h be bounded μ-measurable functions on* Δ_N. *Then the function*

$$u(x) = \int_{\Delta_N} \mathrm{P}(x, t) f(t)\, d\mu(t) - \iint_{\mathrm{R} \times \Delta_N} g(x, y)\, \mathrm{P}(y, t)\, h(t)\, d\mu(t)\, dy$$

is biharmonic on R. *If f and h are continuous at* $t_0 \in \Delta_N$, *then* $\lim_{x \in \mathrm{R}, x \to t_0} u(x) = f(t_0)$ *and* $\lim_{x \in \mathrm{R}, x \to t_0} \Delta u(x) = h(t_0)$.

7. A function $u \in \mathrm{WBB}_\Delta(\mathrm{R})$ is called WBB_Δ*-minimal* on R if $u \not\equiv 0$, $u \geqslant 0$, $\Delta u \leqslant 0$, and for any $v \in \mathrm{WBB}_\Delta(\mathrm{R})$ with $0 \leqslant v \leqslant u$, there exists a constant c_v with $v = c_v u$ on R.

The WBB_Δ-minimal functions have the following characterization in terms of Δ_N.

THEOREM. – *If u is* WBB_Δ*-minimal on* R, *then there exists an isolated point $t \in \Delta$ such that either* $u_1(x) = \mathrm{P}(x, t)\, \mu(t)$, $u_2 = 0$, or $u_1 = 0$, $u_2(x) = \mathrm{P}(x, t)\, \mu(t)$ *where $u_1 = \pi u$ and $u_2 = -\Delta u$.*

Proof. – By the proof of Theorem 4, we have $u = u_1 - \Gamma u_2$ on R. Since $0 \leqslant -\Gamma u_2 \leqslant u$, the WBB_Δ-minimality of u yields $-\Gamma u_2 = c_1 u = c_1 u_1 - c_1 \Gamma u_2$ for some constant c_1. Since $\Gamma u_2 = 0$ on Δ_N, $c_1 u_1 = 0$ on Δ_N and therefore on R in view of the maximum principle for HB-functions. If $c_1 = 0$, $u_2 = 0$ and u_1 is a HB-minimal function. If $c_1 \neq 0$, then $u_1 = 0$ and $-\Gamma u_2$ is WBB_Δ-minimal.

It remains to show that u_2 is HB-minimal whenever $-\Gamma u_2$ is WBB_Δ-minimal. Let $w \in \mathrm{HB}(\mathrm{R})$ be such that $0 \leqslant w \leqslant u_2$. Since $-\Gamma$ is a positive operator, we have $0 \leqslant -\Gamma w \leqslant -\Gamma u_2$ and therefore

$$\Gamma(w - c u_2) = 0$$

on R for some constant c. The assertion follows by virtue of Theorem 3.

8. Finally we turn to the study of biharmonic functions in connection with the Dirichlet integrals of these functions and their Laplacians.

First we establish :

THEOREM. – *For a function* $f \in C^\infty(R) \cap B(R)$,

$$D(\Gamma f) = \iint_{R \times R} g(x\,,\,y)\, f(x)\, f(y)\, dx\, dy\ .$$

Proof. – Let $\{R_m\}$ be a regular exhaustion of R and $g_m(x\,,\,y)$ the Green's function for R_m. Define $g_m(x\,,\,y) = 0$ on

$$(R - R_m) \times R_m \cup R_m \times (R - R_m)\ .$$

Set

$$v_m(x) = -\int_R g_m(x\,,\,y)\, f(y)\, dy\ .$$

Then $v_m = 0$ on $R - R_m$ and the Green's formula yields

$$0 = \int_{\partial R_m} v_m(x) * dv_m(x) = D_R(v_m) + \int_R v_m(x)\, \Delta v_m(x)\, dx\ .$$

Since $\Delta v_m(x) = f(x)$ on R_m, we have

$$\begin{aligned} D_R(v_m) &= -\int_R v_m(x)\, f(x)\, dx \\ &= \iint_{R \times R} g_m(x\,,y)\, f(y)\, f(x)\, dx\, dy \leqslant \|f\|_\infty^2 \iint_{R \times R} g(x\,,y)\, dx\, dy. \end{aligned}$$

Therefore we may assume that the sequence $\{D_R(v_m)\}_m$ converges. By Fatou's lemma and Lebesgue's convergence theorem we obtain

$$D_R(\Gamma f) \leqslant \varliminf_{m \to \infty} D_R(v_m) = \iint_{R \times R} g(x\,,y)\, f(x)\, f(y)\, dx\, dy < \infty\ .$$

On the other hand $\Delta(\Gamma f - v_m) = 0$ on R_m and $v_m = 0$ on ∂R_m. Let $h_m \in H(R_m)$ be such that $h_m \mid \partial R_m = \Gamma f \in M_\delta(R)$. Here $M_\delta(R)$ is the Royden potential subalgebra which consists of limits of uniformly bounded functions in the Royden algebra $M(R)$ with compact supports which converge uniformly in compact subsets and in the

Dirichlet norm. The sequence $\{h_m\}$ converges to zero uniformly on compact subsets of R and $D_R(h_m) \to 0$ as $m \to \infty$ (cf. Sario-Schiffer-Glasner [8]). Since

$$D_R(\Gamma f - h_m) = D_R(v_m) = \iint_{R \times R} g_m(x, y) f(x) f(y)\, dx\, dy \ ,$$

we have

$$\lim_{m \to \infty} D_R(\Gamma f - h_m) = \iint_{R \times R} g(x, y) f(x) f(y)\, dx\, dy \ .$$

In view of

$$|\sqrt{D_R(\Gamma f)} - \sqrt{D_R(\Gamma f - h_m)}| \leqslant \sqrt{D_R(h_m)} \to 0 \ ,$$

we conclude that

$$D_R(\Gamma f) = \iint_{R \times R} g(x, y) f(x) f(y)\, dx\, dy \ .$$

9. Let $WCC_\Delta(R)$ be the family of all biharmonic functions $u \in WBB_\Delta(R)$ such that u and Δu are Dirichlet-finite.

By virtue of the above theorem we have a counterpart of Theorem 4 :

THEOREM. – *The decomposition*

$$WCC_\Delta(R) = HBD(R) \oplus \Gamma HBD(R)$$

is valid.

COROLLARY. – *For any* $m \geqslant 1$, $\dim WCC_\Delta(R) = 2m$ *if and only if the cardinality of the Royden harmonic boundary* Δ_M *of* R *is* m.

Let $MM_\Delta(R) = \{f \in M(R) \mid \Delta f \in M(R)\}$ and

$$M_\delta M_{\delta\Delta}(R) = \{g \in M_\delta(R) \mid \Delta g \in M_\delta(R)\} \ .$$

As in Theorem 5 we have :

THEOREM. – $MM_\Delta(R) = WCC_\Delta(R) + M_\delta M_{\delta\Delta}(R)$.

We remark that the integral representation of WCC_Δ-functions along Δ_M is also valid, and that a characterization of WCC_Δ-minimal functions, similar to that in Theorem 7, can be given in terms of the Royden harmonic boundary.

BIBLIOGRAPHY

[1] S. Bergman and M. Schiffer, Kernel functions and elliptic differential equations in mathematical physics, Academic Press, New York, (1953), 432 p.

[2] C. Constaninescu and A. Cornea, Ideale Ränder Riemannscher Flächen, Springer, (1963), 244 p.

[3] P.R. Garabedian, Partial differential equations, Wiley, New York, (1964), 672 p.

[4] M. Nakai and L. Sario, Biharmonic classification of Riemannian manifolds, (to appear).

[5] M. Nakai and L. Sario, Quasiharmonic classification of Riemannian manifolds, (to appear).

[6] G. De Rham, Variétés différentiables, Hermann, Paris, (1960), 196 p.

[7] L. Sario – M. Nakai, Classification theory of Riemann surfaces, Springer, (1970), 446 p.

[8] L. Sario – M. Schiffer – M. Glasner, The span and principal functions in Riemannian spaces, *J. Analyse Math.* 15 (1965), 115-134.

[9] I.N. Vekua, New methods for solving elliptic equations, North-Holland, Amsterdam, (1967), 358 p.

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Y.K. Kwon
Department of Mathematics
University of Texas
Austin, Texas 78 712

L. Sario
Department of Mathematics
University of California
Los Angeles, California 90 024

B. Walsh
Department of Mathematics
Rutgers University
New Brunswick N.J. 08 903 (USA)