

ANNALES DE L'INSTITUT FOURIER

ROGER G. MCCANN

Another characterization of absolute stability

Annales de l'institut Fourier, tome 21, n° 4 (1971), p. 175-177

http://www.numdam.org/item?id=AIF_1971__21_4_175_0

© Annales de l'institut Fourier, 1971, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (<http://annalif.ujf-grenoble.fr/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

ANOTHER CHARACTERIZATION OF ABSOLUTE STABILITY

by Roger C. McCANN

It is well known that absolute stability of a compact subset M of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighborhoods, and also by the existence of a continuous Liapunov function φ defined on some neighborhood of $M = \varphi^{-1}(0)$, [1]. In a more general setting it has been shown that a set M is closed and absolutely stable if and only if $M = \bigcap \varphi_i^{-1}(0)$ for suitable Liapunov functions φ_i , [2]. This paper presents a more elementary description of absolute stability in terms of positively invariant neighborhoods only.

Throughout this paper \mathbb{R} and \mathbb{R}^+ will denote the reals and the non-negative reals respectively. A rational number r is called dyadic iff there are integers n and j such that $n \geq 0$, $1 \leq j < 2^n$, and $r = j/2^n$.

A dynamical system on a topological space X is a mapping π of $X \times \mathbb{R}$ into X satisfying the following axioms (where $x\pi t = \pi(x, t)$):

(1) $x\pi 0 = x$ for $x \in X$.

(2) $(x\pi t)\pi s = x\pi(t + s)$ for $x \in X$ and $t, s \in \mathbb{R}$.

(3) π is continuous in the product topology.

If $A \subset X$ and $B \subset \mathbb{R}$, then $A\pi B$ will denote the set $\{x\pi t : x \in A, t \in B\}$. A subset A of X is called positively invariant if and only if $A\pi \mathbb{R}^+ = A$.

A mapping $\varphi : X \rightarrow \mathbb{R}^+$ is called a Liapunov function (relative to π) if and only if φ is continuous and $\varphi(x\pi t) \leq \varphi(x)$ for all $x \in X$ and $t \in \mathbb{R}^+$.

Absolute stability is defined in terms of a prolongation ([1], [2]) and, in [1], is characterized in a special setting by the following theorem.

THEOREM A. — *Let M be a compact subset of a locally compact metric space. Then the following are equivalent:*

- (a) *There is a Liapunov function φ with $\varphi^{-1}(0) = M$.*
- (b) *M possesses a fundamental system of absolutely stable neighborhoods.*
- (c) *M is absolutely stable.*

In [2], absolutely stable sets, in a more general setting, are characterized by Liapunov functions.

THEOREM B. — *Let M be a subset of a space X which is Hausdorff paracompact, and locally compact. Then M is closed and absolutely stable if and only if $M = \bigcap \varphi_i^{-1}(0)$ for suitable Liapunov functions $\varphi_i: X \rightarrow [0, 1]$.*

In order to obtain our result we will need the following result [2, Corollary 18].

THEOREM C. — *In a locally compact metric space X , the closed absolutely stable sets are precisely the zero-sets of Liapunov functions mapping X into $[0, 1]$.*

THEOREM. — *Let M be a closed subset of a locally compact metric space X . Then M is absolutely stable if and only if M possesses a family \mathcal{F} of neighborhoods satisfying*

- (i) *If $U \in \mathcal{F}$, then U is open and positively invariant.*
- (ii) *$\bigcap \mathcal{F} = M$.*
- (iii) *If $U \in \mathcal{F}$, then there is a $V \in \mathcal{F}$ such that $\bar{V} \subset U$.*
- (iv) *If $U, V \in \mathcal{F}$ are such that $\bar{U} \subset V$, then there is a $W \in \mathcal{F}$ such that $\bar{U} \subset W \subset \bar{W} \subset V$.*

Proof. — *If.* Let $U \in \mathcal{F}$. For each dyadic rational r we construct a set $U(r) \subset U$ such that $U(r) \in \mathcal{F}$ and $\bar{U}(r) \subset U(s)$ if $r < s$. Then we construct a Liapunov function $\varphi_U: X \rightarrow [0, 1]$ and show that $M = \bigcap \{\varphi_U^{-1}(0) : U \in \mathcal{F}\}$. The result will then follow from Theorem B. First obtain from \mathcal{F} a system

of neighborhoods $U\left(\frac{1}{2^n}\right)$, n a non-negative integer, such that $U(1) = U$ and $U\left(\frac{1}{2^{n+1}}\right) \subset U\left(\frac{1}{2^n}\right)$. This is clearly possible by (iii). Using (iv) this system of neighborhoods can be extended to one with the desired properties. For example, we choose $U\left(\frac{3}{4}\right)$ to be any member W of \mathcal{F} such that $\bar{U}\left(\frac{1}{2}\right) \subset W \subset \bar{W} \subset U(1)$. Now define $\rho_U: X \rightarrow \mathbb{R}^+$ by $\rho_U(x) = 1$ if $x \in U = U(1)$ and $\rho_U(x) = \inf \{\rho : x \in U(\rho)\}$ if $x \in U$. If $x \in U(\rho)$ and $t \in \mathbb{R}^+$, then $x\rho t \in U(\rho)$ since $U(\rho)$ is positively invariant. Therefore

$$\rho_U(x) = \inf \{r : x \in U(r)\} \geq \inf \{r : x\rho t \in U(r)\} = \rho_U(x\rho t).$$

The continuity of ρ_U is proved as in the proof of Urysohn's lemma. Thus for each $U \in \mathcal{F}$ we have constructed a continuous Liapunov function ρ_U such that $M \subset \rho_U^{-1}(0) \subset U$. By (ii), $\cap \rho_U^{-1}(0) = M$.

Only if. — Let M be absolutely stable. Then by theorem C, $M = \rho_U^{-1}(0)$ for some Liapunov function ρ . Let \mathcal{F} consist of all sets of the form $\{x : \rho(x) < r\}$ where $r \in (0, 1)$. Evidently \mathcal{F} satisfies conditions (i)-(iv).

Remark. — In the « If » part of the proof we only need that X is Hausdorff, paracompact, and locally compact.

The author wishes to thank Professor Otomar Hájek for several helpful conversations during the preparation of this paper.

BIBLIOGRAPHY

- [1] J. AUSLANDER, P. SEIBERT, Prolongation and stability in dynamical systems, *Ann. Inst. Fourier*, Grenoble, 14 (1964), 237-268.
- [2] O. HÁJEK, Absolute stability of non-compact sets (to appear).

Manuscrit reçu le 20 décembre 1970.

Roger C. McCANN

Case Western Reserve University
Cleveland, Ohio 44 106.