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ARUNAVA MUKHERJEA

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THE CONVOLUTION EQUATION $P = P^*Q$ OF CHOQUET AND DENY AND RELATIVELY INVARIANT MEASURES ON SEMIGROUPS

by Arunava MUKHERJEA

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1. Introduction. By a measure P on a Hausdorff space S, we mean a non-negative sigma-finite countably-additive set function on the Borel sets (generated by open sets) such that for any compact set K, $P(K) < \infty$ and for any Borel set B,

$$\begin{aligned} P(B) &= \sup \{P(C) : C \text{ compact and } C \subset B\} \\ &= \inf \{P(O) : O \text{ open and } O \supset B\}. \end{aligned}$$

From now on, S will always be a topological semigroup, i.e., S is algebraically a semigroup and the mapping $(x, y) \to xy$ from $S \times S$ into S is continuous. A measure P on S is called relatively right invariant if there is a homomorphism β from S into $R^+ = (0, \infty)$ such that for every Borel set B and x in S,

$$P(Bx^{-1}) = \beta(x)P(B), \qquad \text{where} \qquad Bx^{-1} = \{y \in S : yx \in B\}.$$

By $x^{-1}B$, $A^{-1}B$ and AB^{-1} we will mean respectively the sets

$$\{y \in S : \ xy \in B\}, \qquad \mathrm{U}\{x^{-1}B : \ x \in A\}, \qquad \mathrm{U}(\{Ax^{-1} : \ x \in B\}.$$

The support of a measure P, denoted by $S(P)$, is the set $\{x \in S :$ every open set containing x has positive P-measure$\}$. Then we know that $S(P)$ is closed and if $B \subset S - S(P)$, then $P(B) = 0$. [By $A - B$, we mean $A \cap B^c$ where B^c is the complement of B.]

It is known that when S is locally compact and P, Q are two finite measures, then

$$P^\star Q(B) = \int P(Bx^{-1})Q\,(dx) = \int Q(x^{-1}B)P\,(dx)$$

and that, in the case when P is infinite, Q is infinite, the above integrals are all well-defined when S is second countable and by Tonelli's theorem, they are equeal.

Choquet and Deny [C — D] considered on an abelian locally compact topological group the representation of a measure P as the convolution product of itself and a finite measure Q:

$$(1) \qquad P = P^\star Q$$

where mainly when P is infinite, they used Choquet's celebrated results on integral representations of points of compact convex sets on topological vector spaces. Later in [T], A. Tortrat considered the problem in the case of finite P when S is any topological group. Tortrat in [T], proved the following theorem:

Theorem [T]. — *When* S *is a topological group,* $P = P^\star Q$ *if and only if* $P(Bx^{-1}) = P(B)$ *for every Borel set* B *and each* x *in the smallest closed subgroup containing* S(Q).

To prove this theorem, Tortrat used very much the fact that the function $x \to P(Kx^{-1})$, whenever K is a compact set, attains its maximum on a topological group S. Since this need not be true in the case of a topological semigroup, his method does not extend to the latter case.

S will be called a left group if for each x in S, $Sx = S$ and S contains an idempotent. A left group is always right cancellative. If S is a left group and E is the set of all idempotents in S and if $G = eS$ for some fixed idempotent e in S, then eS is a group and S is topologically isomorphic to $E \times G$. If S is also locally compact, then G becomes a locally compact semigroup which is algebraically a group and so, by Ellis' theorem [1], G is a topological group.

Tortrat in [T′] made also an interesting study of the equations $P = P^\star Q$ and $P = Q^\star P$ when P and Q are

[1] R. Ellis: *Duke Math. J.*, 24 (1957), 119-126.

probability measures and S is completely regular and a left group.

In the case of infinite P, Choquet and Deny [C — D] showed that if P is an extremal point of the cone $\{P : P = P^{\star}Q\}$, then for every x in S(Q), there exists a positive number $\beta(x)$ such that the translate of P by x is $\beta(x).P$.

In this paper, we make an attempt to find, in the case of certain locally compact semigroups, those solutions P of equation (1) which are relatively invariant on S(Q). Our result in the infinite case is not very satisfactory. But it yields an interesting result in the finite case. We also present an interesting characterization of relatively invariant measures on certain locally compact semigroups. Here are our results :

In what follows, [S(Q)] will denote the semigroup generated by S(Q) and H^0 will denote the interior of H.

THEOREM 1. — *Let* S *be a second countable locally compact cancellative semigroup. Let* P *and* Q *be measures such that* $P = P^{\star}Q = Q^{\star}P$ *and for compact sets* A *and* B,

$$P(AB^{-1}) < \infty$$

and $P(A^{-1}B) < \infty$. *If there is a Borel set* H *which is a semigroup such that* (i) $S(Q) \subset H$ *and* $H^0 \cap S(P) \neq \emptyset$ (ii) *for every compact set* K,

$$\int_H P(Kx^{-1})P\,(dx) < \infty \qquad \textit{and} \qquad \int_H P(x^{-1}K)P\,(dx) < \infty,$$

then there exist homomorphisms β_1, β_2 *from* [S(Q)] *into* R^{+} *such that* $P(Bx^{-1}) = \beta_1(x)P(B)$ *and* $P(x^{-1}B) = \beta_2(x)P(B)$ *for every* x *in* [S(Q)].

THEOREM 2. — *Let* S *be a locally compact* (*not necessarily second countable*) *cancellative semigroup. Suppose* P *is finite. Then* $P = P^{\star}Q = Q^{\star}P \Longleftrightarrow P(Bx^{-1}) = P(x^{-1}B) = P(B)$ *for every Borel set* B *and each* x *in* $\overline{[S(Q)]}$. (*Here the bar on the head denotes closure.*)

COROLLARY 3. — *On a locally compact cancellative semigroup, a probability measure* P *is idempotent if and only if it is a* *normed Haar measure on a compact subgroup which is its support.*

THEOREM 4. — *Let* S *be a locally compact semigroup satisfying* (*) Kx^{-1} *is compact whenever* K *is compact and* x *is in* S. *If* P *is a relatively right invariant measure (not necessarily sigma-finite) on* S *with modulus* β, *then* S(P) *is a left group and* β *is continuous. If* E *is the set of all idempotents of the left group* S(P) *and* $G = e.S(P)$ *for some fixed idempotent* e *in* S(P), *then* S(P) *is topologically isomorphic to* $E \times G$ *and the measure* P *restricted to* S(P) *is the product of a measure* P_1 *on* E *and a relatively right invariant Haar measure* P_2 *on* G. *Conversely, every such product measure on a left group is relatively right invariant.*

Before we go into section 2, we note that it is easy to get trivial examples satisfying conditions of theorem 1, by considering a relatively invariant Haar measure on any totally disconnected locally compact abelian groups which necessarily contain many compact open subgroups.

2. In this section, S will always be a locally compact semigroup. We will assume that $P(Kx^{-1}) < \infty$, $P(x^{-1}K) < \infty$ for any compact K and x in S. We may note that Kx^{-1} and $x^{-1}K$ are closed for any closed K and may not be compact even for compact K. We need the following lemmas.

LEMMA 5. — *Let* $P_x(B) = P(Bx^{-1})$ *and* ${}_xP(B) = P(x^{-1}B)$. *Then* P_x *and* ${}_xP$ *are (regular) measures.*

Proof. — We consider only P_x. If $P_x(B) = \infty$, then given $n > 0$, we can find a compact set $K \subset Bx^{-1}$ such that $P(K) > n$ so that the compact set $Kx \subset B$ and

$$P_x(Kx) = P(Kxx^{-1}) \geqslant P(K) > n.$$

If $P_x(B) < \infty$, then given $\varepsilon > 0$, we can find compact $K \subset Bx^{-1}$ such that $P(Bx^{-1} - K) < \varepsilon$ so that $Kx \subset B$ and

$$P_x(B - Kx) = P(Bx^{-1} - Kxx^{-1}) \leqslant P(Bx^{-1} - K) < \varepsilon.$$

Hence P_x is regular relative to compact sets from inside.

Now let $I(f) = \int f(s) P_x(ds)$ for every continuous function f with compact support. By Riesz-representation theorem (see [H], page 129), there is a (regular) measure Q such that $\int f(s)Q(ds) = I(f)$. We wish to show that $Q = P_x$.

If $Q(0) = \infty$, 0 being an open set, then there are compact sets $K_n \subset 0$ and continuous f_n with $f_n(s) = 1$, $s \in K_n$, $= 0$ if $s \notin 0$ such that f_n has compact support and $0 \leqslant f_n \leqslant 1$ and $n \leqslant Q(K_n) \leqslant I(f_n) \leqslant P_x(0)$ so that $P_x(0) = \infty$. If $P_x(0) = \infty$, then again it follows from the inner regularity of P_x that $Q(0) = \infty$. So if $Q(0) < \infty$, then $P_x((0) < \infty$. In this case, given $\varepsilon > 0$, we can find compact $K \subset 0$ with $P_x(0 - K) < \varepsilon$ and a continuous f with compact support with $f(s) = 1$ if $s \in K$, $= 0$ if $s \notin 0$ such that

$$P_x(0) \leqslant P_x(K) + \varepsilon \leqslant I(f) + \varepsilon \leqslant Q(0) + \varepsilon.$$

Similarly $Q(0) \leqslant P_x(0) + \varepsilon$ so that $Q(0) = P_x(0)$ for all open sets 0. Now if K is any compact set, then there is an open set $0 \supset K$ with $Q(0) < \infty$. Then

$$Q(K) = Q(0) - Q(0-K) = P_x(0) - P_x(0-K) = P_x(K).$$

Therefore by the inner regularity of both Q and P_x, it follows that $Q(B) = P_x(B)$ for all Borel sets B.

LEMMA 6. — *Suppose* $P(AB^{-1}) < \infty$ *for compact sets* A *and* B. *Then for any compact* (*resp. open*) *set* K, *the function* $x \to P(Kx^{-1})$ *is upper* (*resp. lower*)*-semicontinuous. The same is true for the function* $x \to P(x^{-1}K)$ *if* $P(\bar{A}^{1}B) < \infty$ *for compact sets* A *and* B.

We omit the proof since it follows easily from lemma 5 and standard arguments used for finite measures.

We are now ready to prove theorem 1.

Proof of theorem 1. — Let P and Q satisfy all the conditions stated in theorem 1. Let $x \in H^0 \cap S(P)$. Let K be any compact set. Let V be open with compact closure and $P(Vx^{-1}) > 0$. [Note that if $P(C) > 0$, then if V is open and $V \supset Cx$, then

$$P(Vx^{-1}) \geqslant P(Cxx^{-1}) \geqslant P(C) > 0.]$$

Let a be a real number such that $P(Kx^{-1}) = a.P(Vx^{-1})$.

Let $\varepsilon > 0$. Let $A = \{s : P(Ks^{-1}) < (a + \varepsilon) . P(Vs^{-1})\}$. Then A is open by lemma 6 and A contains x in S(P) so that $P(A) > 0$. Since the measure P_x is regular by lemma 5, we can find open sets W_1 and M_1 and compact sets W_2 and M_2 such that $K \subset W_1 \subset W_2$ and $V \supset M_2 \supset M_1$ and

$$P(W_2 x^{-1}) < (a + \varepsilon) . P(M_1 x^{-1}).$$

Then the set $B = \{s : P(W_2 s^{-1}) < (a + \varepsilon) . P(M_1 s^{-1})\}$ is open and contains x so that $P(B) > 0$. Let

$$f(s) = \max \{P(W_2 s^{-1}) - (a + \varepsilon) . P(M_1 s^{-1}), 0\}.$$

Then by hypothesis, $\int_H f(s) P(ds) < \infty$ and since

$$P = P^* Q = Q^* P,$$

we have, $\int f(s) I_H(s) P(ds) = \int \int f(st) I_H(st) Q(ds) P(dt)$, where $I_H(s)$ is the characteristic function of H. It can also be easily checked that $f(t) \leqslant \int f(st) Q(ds)$. Since $S(Q) \subset Ht^{-1}$ for all t in H (a semigroup), we have

$$f(t) I_H(t) \leqslant \int f(st) I_H(st) Q(ds) \quad \text{for all } t \text{ in } S.$$

Therefore we have,

$$f(t) I_H(t) = \int f(st) I_H(st) Q(ds) = \int f(st) Q(ds)$$

for all t in $H - E$ where $P(E) = 0$. Let $B' = B \cap H^0$ which is an open set containing x in S(P) so that $P(B') > 0$. We observe that x is in the closure of $(B' - E) \cap S(P)$; for, if G is any open set containing x, then $G \cap B'$ is relatively open in S(P) and hence it has positive P-measure and therefore it must intersect $(B' - E) \cap S(P)$ since E has P-measure zero. Now if t is in $(B' - E) \cap S(P)$, then $0 = f(t) = \int f(st) Q(ds)$ which implies that $P(W_2 t^{-1} s^{-1}) \leqslant (a + \varepsilon) . P(M_1 t^{-1} s^{-1})$ for Q — almost all S. This implies that this inequality is valid for all s in S(Q), when we replace W_2 by W_1 and M_1 by M_2 since then the set

$$\{s : P(W_1 t^{-1} s^{-1}) > (a + \varepsilon) . P(M_2 t^{-1} s^{-1})\}$$

is open.

[Note that since W_1 is open and M_2 is compact, as in lemma 6, it follows that for fixed t, the functions

$$s \to P(M_2 t^{-1} s^{-1}) \qquad \text{and} \qquad s \to P(W_1 t^{-1} s^{-1})$$

are respectively upper and lower semicontinuous.] So now if s is in $S(Q)$, then for t in $(B' - E) \cap S(P)$,

$$\begin{aligned} P_s(W_1 t^{-1}) = P(W_1 t^{-1} s^{-1}) &\leqslant (a + \varepsilon).P(M_2 t^{-1} s^{-1}) \\ &= (a + \varepsilon).P_s(M_2 t^{-1}). \end{aligned}$$

Therefore, using lemma 6 for the measure P_s, we get for all s in $S(Q)$,

$$P(W_1 x^{-1} s^{-1}) \leqslant (a + \varepsilon).P(M_2 x^{-1} s^{-1})$$

so that $P(Kx^{-1}s^{-1}) \leqslant a.P(Vx^{-1}s^{-1})$. Since $P = P^\star Q$, the choice of a implies that

$$\int [P(Kx^{-1}s^{-1}) - a.P(Vx^{-1}s^{-1})]Q\,(ds) = 0.$$

Hence $P(Kx^{-1}s^{-1}) = a.P(Vx^{-1}s^{-1})$ for Q — almost all s and therefore for all s in $S(Q)$. This means that for any given compact K and open V with compact closure with $P(Vx^{-1}) > 0$, we have

$$P(Kx^{-1}s^{-1})P(Vx^{-1}) = P(Kx^{-1})P(Vx^{-1}s^{-1})$$

for all s in $S(Q)$. This equality is also true when $P(Vx^{-1}) = 0$.

Now given s in $S(Q)$, we can always find an open set V with compact closure such that

$$P(Vx^{-1}) > 0, \qquad P(Vx^{-1}s^{-1}) > 0.$$

Therefore for every s in $S(Q)$, there is a positive real number $\beta_1(s)$ such that for every compact K, we have

$$P(Kx^{-1}s^{-1}) = \beta_1(s)P(Kx^{-1}).$$

Since cancellation by x is permitted, for any compact C, $Cxx^{-1} = C$ so that $P(Cs^{-1}) = \beta_1(s)P(C)$. [This is the first time we are using concellation.] Hence by the regularity of P_s and P, for any Borel set B, $P(Bs^{-1}) = \beta_1(s)P(B)$ for all s in $S(Q)$ and therefore for all s in $[S(Q)]$ and

$$\beta_1(st) = \beta_1(s)\beta_1(t)$$

for all s and t in $S(Q)$.

Since $P = P{}^\star Q = Q{}^\star P$, by Tonelli's theorem, we have

$$P(B) = \int P(x^{-1}B)Q\,(dx) = \int P(Bx^{-1})Q\,(dx).$$

Therefore, in the above arguments, if we start with $P(x^{-1}K)$ and $P(x^{-1}V)$ in place of $P(Kx^{-1})$ and $P(Vx^{-1})$ respectively, then it follows as before that there is a homomorphism β_2 from $[S(Q)]$ into R^+ such that $P(s^{-1}B) = \beta_2(s)P(B)$ for all s in $[S(Q)]$.

Remark 7. — If in the statement of theorem 1, the existence of a semigroup H satisfying the conditions stated is replaced by the existence of a compact semigroup containing $S(Q)$ and requiring no other condition, the theorem is still valid.

Proof of theorem 2. — In theorem 1, we needed second countability only to be able to write

$$\int P(Bx^{-1})Q\,(dx) = \int Q(x^{-1}B)P\,(dx).$$

When P and Q are finite, this equality is true on any locally compact semigroup since then for every bounded continuous function f, we have

$$\iint f(st)P\,(ds)Q\,(dt) = \iint f\,(st)\;\; Q\,(dt)P\,(ds).$$

[See the paper by Glicksberg, *Pacific J. of Math.*, 11, 1961, 205-14.] Then it follows from theorem 1 that

$$P = P{}^\star Q = Q{}^\star P \Longleftrightarrow P(Bx^{-1}) = P(x^{-1}B) = P(B)$$

for all x in $[S(Q)]$. Now if $x \in \overline{[S(Q)]}$, then there is a net x_p in $[S(Q)]$ converging to x; then, given $\varepsilon > 0$, we can find compact K and open V such that (i) $V \supset B \supset K$ (ii) $P(V - K) < \varepsilon$ so that

$$P(Bx^{-1}) \geqslant P(Kx^{-1}) \geqslant \limsup P(Kx_p^{-1}) = P(K) \geqslant P(B) - \varepsilon$$

and

$$P(Bx^{-1}) \leqslant P(Vx^{-1}) \leqslant \liminf P(Vx_p^{-1}(= P(V) \leqslant P(B) + \varepsilon.$$

Therefore $P(Bx^{-1}) = P(B)$ for all x in $\overline{[S(Q)]}$. Similarly, the same is true for $P(x^{-1}B)$ by dual arguments.

Proof of corollary 3. — When $P = P^\star P$, the support of P is a locally compact semigroup. By theorem 2,

$$P(Bx^{-1}) = P(x^{-1}B) = P(B)$$

for every x in $S(P)$ and every Borel set B. This means that P is a $r^\star$-invariant and $l^\star$-invariant probability measure, and by a result in [M— T], $S(P)$ is a compact group. The rest follows easily.

3. In this section, we present a few results on relatively invariant measures on locally compact semigroups. Let β be a homomorphism from S into R^+ such that

$$P(Bs^{-1}) = \beta(s)P(B)$$

for all Borel sets B. If for compact sets A and B, $P(AB^{-1}) < \infty$ then β is continuous on S; because, if x_p is a net converging to x and K (compact), V (open) are sets with finite positive P-measure, then by lemma 6, we have

$$\beta(x)P(K) = P(Kx^{-1}) \geqslant \limsup P(Kx_p^{-1}) = P(K).\limsup \beta(x_p)$$

and

$$\beta(x)P(V) = P(Vx^{-1}) \leqslant \liminf P(Vx_p^{-1}) = P(V).\liminf \beta(x_p)$$

so that β is continuous on S. Since in a locally compact semigroup which is a left group, AB^{-1} is compact for compact sets A and B, β is continuous on a left group.

A measure P is called $r^\star$-invariant if $P(Bx^{-1}) = P(B)$ for every Borel set B and x in S. These measures were studied in [A]. It is conjectured in [A] that the support of such a measure on a locally compact semigroup must be a left group. The problem is solved in [M — T] for finite $r^\star$-invariant measures. The general problem seems to be still open. We think that the conjecture is true even for relatively right invariant measures. However, we are unable to solve it. The importance of the fact that the support of a $r^\star$-invariant measure is a left group, lies in the fact shown in [A] that such a measure is the product of a (regular) measure and a right invariant Haar measure.

Proof of theorem 4. — Let $P(Bx^{-1}) = \beta(x)P(B)$ for all Borel B and x in S, where $\beta(x) > 0$ for all x in S.

The preceding discussion shows that β is a continuous homomorphism on $S(P)$, if $S(P)$ is a left group. Now for any y in $S(P)$ and x in S, Vx^{-1} has positive P-measure for every open neighbourhood V of y so that $V \cap \overline{(S(P).x)}$ is non-empty. This means that $S(P) \subset \overline{S(P).x}$ for every x in S. Now let y be in $S(P)$ and W be its compact neighbourhood such that $P(W) > 0$. Then for any x in S, $P(Wx) = 0$ implies $P(Wxx^{-1}) = 0$ which is false since $Wxx^{-1} \supset W$. Therefore every open neighborhood of yx has positive P-measure so that $yx \in S(P)$. Hence for every x in S, $\overline{S(P).x} = S(P)$. Since S satisfies condition (*), stated in the theorem, it is easy to check that the right translations by x in S are closed. This means that $S(P).x = S(P)$ for every x in S. Then for z in $S(P)$, there is a w in $S(P)$ such that $wz = z$; then $zz^{-1} \cap S(P)$ is non-empty and therefore, by condition (*), it is compact and since it is also a semigroup, it contains an idempotent by a result of Numakura. [See [N].] Hence $S(P)$ is a left group. The proof of the rest of the theorem can be easily given following theorem 3 in [A].

Remark 8. — Under condition (*), introduced in theorem 4, $K(S)$, the continuous functions on S with compact support is closed under right translations. Let $f_a(s) = f(sa)$. Let us say that a positive integral I on S (i.e. a positive linear functional on $K(S)$) is relatively right invariant if for any a in S, there is a number $\beta(a) > 0$ such that $I(f_a) = \beta(a)I(f)$ for all f in $K(S)$. Then from theorem 1 in [A], results easily the equivalence of the following:

(*i*) S admits a relatively right invariant integral,

(*ii*) S admits a relatively right invariant measure,

(*iii*) S contains a unique minimal left ideal which is closed.

Remark 9. — Let us remark that the support of a relatively invariant (both sided) measure on a arbitrary locally compact semigroup is a topological group. The proof follows from that of a similar result for invariant measures given in a recent paper of Tserpes and the author in Semigroup Forum.

Remark 10. — In theorem 4, we do not need local compactness to prove that $S(P)$ is a left group. But unless S is locally compact, the factor G of S(P) (see the statement of theorem 4) may not be a topological group so that for compact sets A and B in S(P), the set $AB^{-1} \cap S(P)$ may not be compact and consequently we cannot use lemma 6 to prove the continuity of β on S(P). Also, we need local compactness to characterize P as a product measure as stated in the theorem.

Remark 11. — In theorem 2, local compactness can be replaced by metric topology since for finite measures P and Q, every Borel set is completion measurable on $S \times S$ so that we can write $\int P(Bx^{-1})\,Q(dx) = \int Q(x^{-1}B)P\,(dx)$, by Fubini's theorem.

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A. MUKHERJEA
University of S. Florida
Tampa, Florida 33620.