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ON NONBORNOLOGICAL BARRELLED SPACES (1)

by Manuel VALDIVIA

L. Nachbin [5] and T. Shirota [6], give an answer to a problem proposed by N. Bourbaki [1] and J. Dieudonné [2], giving an example of a barrelled space, which is not bornological. Posteriorly some examples of nonbornological barrelled spaces have been given, e.g. Y. Kômura, [4], has constructed a Montel space which is not bornological. In this paper we prove that if \( E \) is the topological product of an uncountable family of barrelled spaces, of nonzero dimension, there exists an infinite number of barrelled subspaces of \( E \), which are not bornological. We obtain also an analogous result replacing « barrelled » by « quasi-barrelled ».

We use here nonzero vector spaces on the field \( K \) of real or complex number. The topologies on these spaces are separated.

If \( E \) is a separated locally convex space, we denote, as usual, by \( E' \), \( \sigma(E', E) \) and \( \beta(E', E) \), the topological dual of \( E \), the weak topology on \( E' \), and the strong topology on \( E' \), respectively. If \( A \) is a bounded, closed and absolutely convex set of \( E \), we denote by \( E_A \) the linear hull of \( A \) equipped with the norm associated to \( A \).

We shall need the following result of J. Dieudonné [3]:

a) Let \( E \) be a bornological space. If \( F \) is a subspace of \( E \), of finite codimension, then \( F \) is bornological.

Theorem 1. — If \( E \) is the topological product of the barrelled spaces \( E_i, i \in I \), where \( I \) is an uncountable set, there exists

(1) Supported in part by the « Patronato para el Fomento de la Investigación en la Universidad ». 
an infinite family $\mathcal{F}$ of barrelled dense subspaces of $E$, which are not bornological.

Proof. — Let $G$ be the subspace of $E$, whose points have all components zero except a countable set. Let $\mathcal{F}$ be the family of all the subspaces of $E$, such that $F \in \mathcal{F}$ if and only if $G \subset F$ and the codimension of $G$ in $F$ is finite and different from zero. Obviously $\mathcal{F}$ is infinite. If $F \in \mathcal{F}$, then $F$ is barrelled since $G$ is barrelled. (It can be proved that $G$ is barrelled taking any $\sigma(G', G)$-bounded set $A$ of $G' = E'$, and noticing that there exists a finite set $\{i_1, i_2, \ldots, i_n\} \subset I$, such that $A \subset \prod_{p=1}^{n} E'_p$, hence $A$ is $\sigma(E', E)$-bounded and, therefore, equicontinuous respect to $E$ and respect to $G$ also.) Now we suppose that $F$ is bornological. According to $a)$ we can find a bornological space $L$, such that $G \subset L \subset F$, being $G$ an hyperplane of $L$. In $L$ let $\mathcal{B}$ be the family of all the absolutely convex, closed and bounded sets. Since $L$ is the inductive limit of $\{E_B : B \in \mathcal{B}\}$ and $G$ is a dense hyperplane of $L$, there exists a $M \in \mathcal{B}$, such that $G \cap E_M$ is dense in $E_M$ and $G \not\supset E_M$. Therefore we can find in $E_M$ a sequence $\{x_n\}_{n=1}^{\infty} \subset G$, which converges to $x \notin G$. That is in contradiction with being $G$ sequentially closed in $E$ and also in $L$. Q.E.D.

Theorem 2. — If $E$ is the topological product of the bornological barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, there exists a family $\mathcal{F}$ of barrelled dense subspaces of $E$, which are not bornological, so that if $F \in \mathcal{F}$, there exists a subspace $H$ of $F$, of finite codimension, such that $H$ is bornological.

Proof. — It is enough to prove that the space $G$ defined in the proof of Theorem 1 is bornological. Let $\mathcal{A}$ be the family of the parts of $I$, which have a countable infinity of elements. For each $M \in \mathcal{F}$, we denote by $E(M)$ the subspace of $E$, whose points have all the components zero except at most those with indices in $M$. It is immediate that $E(M)$ is bornological. Since $G$ is the inductive limit of the family of spaces $\{E(M) : M \in \mathcal{A}\}$ then $G$ is bornological. (We can prove that $G$ is the inductive limit of $\{E(M) : M \in \mathcal{A}\}$ of the following
way: let $u$ be any linear form on $G$, such that its restriction $u_M$ to $E(M)$ is continuous, $M \in \mathcal{M}$. Let $\nu_M$ be the continuous extension of $u_M$ to $G$, such that if $x \in G$ and $x(M)$ is the projection of $x$ on $E(M)$, then $\nu_M(x) = u_M(x(M))$. Obviously the net $\{\nu_M: M \in \mathcal{M}, < \}$ converges weakly to $u$. Furthermore, if $x \in G$, it is easy to prove that $\{\nu_M(x): M \in \mathcal{M}\}$ is a bounded set in $K$, as since $G$ is barrelled, it results that $\{\nu_M: M \in \mathcal{M}\}$ is equicontinuous set, hence $u$ is continuous in $G$. Therefore, the space $G$ and the inductive limit of $\{E(M): M \in \mathcal{M}\}$ have the same topological dual and since, the topology of $G$ is the Mackey one, both spaces are the same.) Q.E.D.

Note. — From the anterior proof it can be deduced that if there exists the strongly inaccessible cardinal $\beta$, then there exists a bornological space $G$, whose completion $\hat{G}$ is not bornological. It is enough to carry out the topological product $E$ of nonzero Frechet spaces, in number equal to $\beta$, and to take the subspace $G$ formed by all points of $E$, whose components are nulle, except a countable set. Then $G$ is bornological and its completion $\hat{G} = E$ is not it.

Theorem 3. — If $E$ is the topological product of the quasi-barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, there exists an infinite family of quasi-barrelled dense subspaces, which are not bornological.

Proof. — The proof is analogous to that of Theorem 1, replacing barrelled by quasi-barrelled. (The proof of being $G$ quasi-barrelled can be done, taking any set $A$ of $G' = E'$, $\beta(G', G)$-bounded, and taking into account that there exists a finite set $\{i_1, i_2, \ldots, i_n\} \subseteq I$ so that $A \subseteq \prod_{\rho=1}^{n} E_{i_{\rho}'}$, hence it is easy to deduce that $A$ is bounded for the topology $\beta(E', E)$, and since $E$ is quasi-barrelled it results that $A$ is equicontinuous respect to $E$, and also respect to $G$.) Q.E.D.

Theorem 4. — If $E$ is the topological product of the quasi-barrelled spaces $E_i$, $i \in I$, where $I$ is an uncountable set, and there exists a $i_0 \in I$, such that $E_{i_0}$ is not barrelled, then there
exists a infinite family $\mathcal{F}$ of quasi-barrelled dense subspaces of $E$, which are not bornological nor barrelled.

Proof. — It is enough to prove if in the Theorem 3, $F \in \mathcal{F}$, then $F$ is not barrelled. Indeed, if $F$ is barrelled, then its closure in $E$, which is equal to $E$, is a barrelled space. In contradiction with the fact that $E_4$ is not barrelled. Q.E.D.

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Manuscrit reçu le 22 juin 1971.
accepté par J. Dieudonné.

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