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ON ABSOLUTE STABILITY
by Roger C. McCANN

It is well known that absolute stability of a compact subset \( M \) of a locally compact metric space can be characterized by the presence of a fundamental system of absolutely stable neighbourhoods, and also by the existence of a continuous Liapunov function \( V \) defined on some neighbourhood of \( M = V^{-1}(0) \), [1]. Here we characterize the absolute stability of \( M \) in terms of the cardinality of the set of positively invariant neighbourhoods of \( M \).

Throughout this paper \( \mathbb{R} \) and \( \mathbb{R}^+ \) will denote the reals and non-negative reals respectively. A rational number \( r \) is called dyadic if and only if there are integers \( n \) and \( j \) such that \( n > 0, 1 \leq j < 2^n \), and \( r = \frac{j}{2^n} \).

A dynamical system on a topological space \( X \) is a mapping \( \pi \) of \( X \times \mathbb{R} \) into \( X \) satisfying the following axioms (where \( x\pi t = \pi(x, t) \)):

1. \( x\pi 0 = x \) for \( x \in X \).
2. \((x\pi t)\pi s = x\pi(t + s)\) for \( x \in X \) and \( t, s \in \mathbb{R} \).
3. \( \pi \) is continuous in the product topology.

If \( M \subset X \) and \( N \subset \mathbb{R} \), then \( M\pi N \) will denote the set \( \{x\pi t : x \in M, t \in N\} \). A subset \( M \) of \( X \) is called positively invariant if and only if \( M\pi \mathbb{R}^+ = M \). A point \( x \in X \) is called a critical point if and only if \( x\pi \mathbb{R} = \{x\} \). A subset \( M \) of \( X \) is called stable if and only if every neighbourhood of \( M \) contains a positively invariant neighbourhood of \( M \).

A Liapunov function for a positively invariant compact subset \( M \) of \( X \) is a continuous mapping \( V \) of a neighbour-
hood $W$ of $M$ into $R^+$ such that $V^{-1}(0) = M$ and $V(x+it) \leq V(x)$ for $x \in W$ and $t \in R^+$.

Absolute stability is defined in terms of a prolongation and is characterized by the following theorem, [1].

**Theorem.** — Let $M$ be a compact subset of a locally compact metric space. Then the following are equivalent:

(i) There is a Liapunov function $V$ for $M$.

(ii) $M$ possesses a fundamental system of absolutely stable neighbourhoods.

(iii) $M$ is absolutely stable.

**Lemma 1.** — Let $A \subseteq R$ be uncountable. Then there exists an $x \in A$ such that every neighbourhood of $x$ contains uncountably many elements of $A$.

**Proof.** — [4, 6,23, III].

The following is a consequence of Lemma 1.

**Lemma 2.** — Let $A \subseteq R$ be uncountable. Then there exists an $x \in A$ such that the sets \{ $y \in A : y < x$ \} and \{ $y \in A : x < y$ \} are uncountable.

**Lemma 3.** — Let $S$ and $T$ be relatively compact sets of a locally compact connected metric space $X$ and $\mathcal{D}$ a family of open sets of $X$ such that

(i) for every $U \in \mathcal{D}$, $\bar{S} \subseteq U \subseteq \overline{U} \subseteq T$,

(ii) if $U, V \in \mathcal{D}$, then either $\overline{U} \subseteq V$ or $\overline{V} \subseteq U$.

Then there is a $W \in \mathcal{D}$ such that the sets \{ $U \in \mathcal{D} : U \subseteq W$ \} and \{ $U \in \mathcal{D} : W \subseteq U$ \} are uncountable.

**Proof.** — Since $X$ is connected, the boundary $\partial U$ of $U \in \mathcal{D}$ is nonempty. If $U \in \mathcal{D}$, then $\partial U$ is compact since $T$ is relatively compact. Let $d$ be a metric on $X$ and define $f : \mathcal{D} \to R^+$ by $f(U) = d(\bar{S}, \partial U)$. If $U, V \in \mathcal{D}$ with $\overline{U} \subseteq V$, then $f(U) < f(V)$. Let $A$ be the image of $\mathcal{D}$ under $f$.

Then $f$ is a one-to-one order preserving mapping of $\mathcal{D}$ onto $A$. $A$ is uncountable since $\mathcal{D}$ is such. By Lemma 2 there is an $x \in A$ such that the sets \{ $y \in A : x < y$ \} and
\{y \in A : y < x\} are uncountable. Set \(W = f^{-1}(x)\). It is easily verified that
\[
\{U \in D : U \subseteq W\} = \{f^{-1}(y) : y < x\},
\]
\[
\{U \in D : W \subseteq U\} = \{f^{-1}(y) : x < y\},
\]
and that both sets are uncountable.

**Theorem 4.** — A nontrivial compact subset \(M\) of a locally compact connected metric space is absolutely stable if and only if \(M\) possesses a fundamental system \(\mathcal{F}\) of open positively invariant neighbourhoods such that

(i) for each \(U \in \mathcal{F}\), the set \(\{V \in \mathcal{F} : V \subseteq U\}\) is uncountable,

(ii) if \(U, V \in \mathcal{F}\), then either \(U \subseteq V\) or \(V \subseteq U\).

**Proof.** — Since \(X\) is connected, no nontrivial subset of \(X\) is both open and closed. If \(M\) is absolutely stable, then there is a continuous Liapunov function \(V\) for \(M\). Set \(\mathcal{F} = \{V^{-1}([0, r]) : r \text{ in the range of } V\}\). It is easily verified that \(\mathcal{F}\) possesses the desired properties. Now assume that \(\mathcal{F}\) is a fundamental system of open positively invariant neighbourhoods of \(M\) with properties (i) and (ii). For each dyadic rational we will construct a set \(U(r) \in \mathcal{F}\) such that \(U(r) \subseteq U(s)\) whenever \(r < s\). We first obtain from \(\mathcal{F}\) a fundamental system of neighbourhoods \(\left\{U \left(\frac{1}{2^n}\right) : n \text{ a non-negative integer}\right\}\) such that \(U \left(\frac{1}{2^{n+1}}\right) \subseteq U \left(\frac{1}{2^n}\right)\) and the set
\[
\left\{A \in \mathcal{F} : U \left(\frac{1}{2^{n+1}}\right) \subseteq A \subseteq U \left(\frac{1}{2^n}\right)\right\}
\]
is uncountable. This is done by induction in the following manner. Let \(N_1\) be a countable fundamental system of neighbourhoods of \(M\). Let \(U(1) \in N_1\) be an element of \(\mathcal{F}\) which is relatively compact. Suppose that \(U \left(\frac{1}{2^n}\right)\) has been defined. By Lemma 3 and property (ii), there is a \(B \in \{W \in \mathcal{F} : W \subseteq U \left(\frac{1}{2^n}\right)\}\) such that \(B \subseteq N_{n+1}\) and both \(\{W \in \mathcal{F} : V \subseteq B\}\) and
\[
\{W \in \mathcal{F} : B \subseteq V \subseteq U \left(\frac{1}{2^n}\right)\}
\]
are uncountable. Set $\cup \left( \frac{1}{2^{n+1}} \right) = B$. Now extend this system to one with the desired properties. For example, we chose $\cup \left( \frac{3}{4} \right)$ to be any element $C$ of $\mathcal{F}$ such that the sets 
\[
\left\{ W \in \mathcal{F} : \cup \left( \frac{1}{2} \right) \subset V \subset C \right\} \quad \text{and} \quad \left\{ W \in \mathcal{F} : C \subset V \subset \cup (1) \right\}
\]
are uncountable. This is possible by the properties of the sets $\cup \left( \frac{1}{2^n} \right)$ and Lemma 3. Now define $V : \cup (1) \to \mathbb{R}^+$ by

\[
V(x) = \inf \{ r : x \in \cup (r) \}.
\]
Evidently $V(x) = 0$ if and only if $x \in M$. If $x \in \cup (r)$ and $t \in \mathbb{R}^+$, then $x \pi t \in \cup (r)$ since $\cup (r)$ is positively invariant. Therefore,

\[
V(x) = \inf \{ r : x \in \cup (r) \} \geq \inf \{ r : x \pi t \in \cup (r) \} = V(x \pi t).
\]

The continuity of $V$ is proved as in the proof of Urysohn’s lemma. Thus we have constructed a Lyapunov function for $M$. $M$ is absolutely stable.

**Example.** — Let $X = [-1, 1]$, $M = \{0\}$, and $\pi$ be the dynamical system indicated by the following diagram where the points $\pm 2^{-n}$, $n$ a non-negative integer, are critical points.

![Diagram](attachment:image)

Clearly $M$ is stable. The only open positively invariant neighbourhoods of $M$ are $X$ and intervals of the form $(-2^{-m}, 2^{-n})$ where $m$ and $n$ are non-integers. There are only countably many such neighbourhoods. Hence, $M$ is not absolutely stable.

**Proposition 5.** — Let $X$ be the plane and $p$ an isolated critical point. If each neighbourhood of $p$ contains uncountably many periodic trajectories (cycles), then $p$ is absolutely stable.

**Proof.** — Let $W$ be a disc neighbourhood of $p$ which contains no critical points other than $p$. A cycle $C$ is a Jordan curve and, hence, decomposes the plane into two components, one bounded (denoted by $\text{int } C$) and the other unbounded. If $C$ is a cycle, then $\text{int } C$ contains a critical point, $[3, \text{ VII},
4.8]. Hence, if $C$ is a cycle in $W$, then $C$ is the boundary of a neighbourhood (necessarily invariant) of $p$. It can be shown (the proof is almost identical with that of Proposition 1.10 of [6]) that if $C_1$ and $C_2$ are distinct cycles in $W$, then either $\text{int } C_1 \subseteq \text{int } C_2$ or $\text{int } C_2 \subseteq \text{int } C_1$. Theorem 4 may now be applied to obtain the desired result.

Another characterization of absolute stability of compact sets is found in [5]. Non-compact absolutely stable sets are characterized in [3].

BIBLIOGRAPHY


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