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BOUNDARY BEHAVIOUR
OF HARMONIC FUNCTIONS
IN A HALF-SPACE
AND BROWNIAN MOTION (1)

by D. L. BURKHOLDER and R. F. GUNDY

The behaviour of harmonic functions in the half-space $\mathbb{R}^{n+1}_+$ has been discussed from two points of view: geometrical and probabilistic. In this paper, we compare these two view points with respect to (1) local convergence at the boundary and (2) the $H^p$-spaces. The results are as follows: (1) The existence of a nontangential limit for almost all points in a set $E$ of positive Lebesgue measure in $\mathbb{R}^n(=\partial\mathbb{R}^{n+1}_+)$ is more restrictive than the existence of a «fine» or probability limit almost everywhere in $E$ when $n \geq 2$. When $n = 1$, the existence of a nontangential limit almost everywhere in $E$ implies the existence of a «fine» limit almost everywhere in $E$ and conversely. (2) For all $n \geq 1$, the nontangential maximal function of $u$ belongs to $L^p(0 < p < \infty)$ if and only if the Brownian motion maximal function belongs to $L^p$. That is, in light of the results of Fefferman and Stein [10], we may say that the class $H^p$, defined probabilistically coincides with $H^p$ defined geometrically. This is proved in [3] for the half-plane $\mathbb{R}^2_+$. However, the arguments for $\mathbb{R}^2_+$ cannot be extended to

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basically, because of the potential-theoretic distinction between dimensions two and three. That this distinction is exhibited in the local statement (1) but not in the global statement (2) is something of a surprise.

From the geometrical viewpoint, the main results on local convergence are due to Marcinkiewicz and Zygmund [14], Spencer [18], and Privalov [17] for $n = 1$, and to Calderón [4], [5] and Stein [19] for $n > 1$. Theorem A below is a summary statement of these results. First, however, we need some notation. The cone in $\mathbb{R}^{n+1}$ with vertex at $x \in \mathbb{R}^n$, height $k$, and angle $a$, is denoted by

$$\Gamma(x; a, k) = \{(s, y) : |x - s| < ay, 0 < y < k\}.$$

The nontangential maximal function of a function $u$ defined on $\mathbb{R}^{n+1}$ is defined as

$$N(u; a, k)(x) = \sup_{(s, y) \in \Gamma(x; a, k)} |u(s, y)|$$

and the area function

$$A(u; a, k)(x) = \left( \int_{\Gamma(x; a, k)} |\nabla u(s, y)|^2 y^{1-n} \, dx \, dy \right)^{1/2}.$$

Notice that both $N(u; a, k)$ and $A(u; a, k)$ are monotone increasing in the parameters $a$ and $k$.

**Theorem A.** — Let $u$ be harmonic in $\mathbb{R}^{n+1}$. The following subsets of $\mathbb{R}^n = \partial \mathbb{R}^{n+1}$ are equal almost everywhere:

1. $\{x : N(u; a, k)(x) < \infty\}$;
2. $\{x : A(u; a, k)(x) < \infty\}$;
3. $\{x : \lim_{(s, y) \to x} u(s, y) \text{ exists and is finite}\}$.

A simplified proof of Theorem A, based on distribution function inequalities between the area function and the nontangential maximal function, is given in [2].

In order to state the probabilistic analogue of Theorem A, we recall the following facts: Let $u$ be an harmonic function defined in $\mathbb{R}^{n+1}$ and let $z_t = (x_t, y_t)$, $t \geq 0$ be $(n + 1)$-dimensional Brownian motion started from the point
(x_0, y_0) \in \mathbb{R}^{n+1}_+, stopped at time \tau = \inf \{t : y_t = 0\}. We refer to this process as Brownian motion in \( \mathbb{R}^{n+1}_+ \). It follows from Ito’s change of variables formula (see McKean [15]) that \( u(x_t, y_t) \) is a stochastic integral of the form

\[
u(x, y_t) = u(x_0, y_0) + \int_0^t \langle \nabla u(z_s), dz_s \rangle.
\]

We let \( \mathbb{P}_{x_0, y_0} \) denote the measure on the space of trajectories from \( (x_0, y_0) \) to \( \mathbb{R}^n \) corresponding to the process \( (x_t, y_t) \), \( t \geq 0 \). We may also define the conditional measure \( \mathbb{P}_{x_0, y_0}^{x, y_t} \) corresponding to a « Brownian » process that starts at \( (x_0, y_0) \) and terminates at the point \( x \in \mathbb{R}^n \). Explicit formulas for \( \mathbb{P}_{x_0, y_0} \) and \( \mathbb{P}_{x_0, y_0}^{x, y_t} \), as well as a discussion of these processes, is given by Doob [9].

Let the Brownian maximal function of \( u \) be defined as

\[ u^* = \sup_{t < \tau} |u(x_t, y_t)|. \]

The Brownian analogue of the area function \( A(u) \) is given by

\[
S(u) = \left[ u^2(x_0, y_0) + \int_0^t |\nabla u(x_t, y_t)|^2 \, dt \right]^{\frac{1}{2}}.
\]

With these definitions, we may state the following theorem.

**Theorem A’.** — Let \( u \) be harmonic in \( \mathbb{R}^{n+1}_+ \). The following subsets of \( \mathbb{R}^n = \partial \mathbb{R}^{n+1}_+ \) are equal almost everywhere (with respect to Lebesgue measure) for every \( (x_0, y_0) \in \mathbb{R}^{n+1}_+ \):

\[
\begin{align*}
(1') \quad & \{x : \mathbb{P}_{x_0, y_0}^{x, y_t} (u^* < \infty) > 0\} \\
(2') \quad & \{x : \mathbb{P}_{x_0, y_0}^{x, y_t} (S(u) < \infty) > 0\} \\
(3') \quad & \{x : \mathbb{P}_{x_0, y_0}^{x, y_t} (\lim_{t \to \tau} u(x_t, y_t) \text{ exists and is finite}) > 0\}.
\end{align*}
\]

We omit the details of the proof of Theorem A’; it follows from the fact that the sets \( \{u^* < \infty\} \) and \( \{S(u) < \infty\} \) are equal \( \mathbb{P}_{x_0, y_0}^{x, y_t} \)-almost everywhere. The set \( (3') \) can also be characterized as the set where \( u \) has a fine boundary limit in the sense of Lelong [13] and Naïm [16]. This fact is due to Doob [8].

One purpose of this paper is to compare the local behaviour
of \( u \) described in Theorems A and A'. We have the following:

**Theorem 1.** — a) For \( u \) harmonic in \( \mathbb{R}^2 \), the sets of Theorem A are equal almost everywhere with respect to Lebesgue measure on \( \mathbb{R}^2 \) to the sets of Theorem A'. b) For \( u \) harmonic in \( \mathbb{R}^{n+1} \), \( n \geq 2 \), the sets of Theorem A are contained in those of Theorem A', up to sets of measure zero. The converse is not true.

Part a) of Theorem 1 is due to Brelot and Doob [1] and Constantinescu and Cornea [7]. Part b) is due in part to Brelot and Doob [1]; our contribution is to show that, without additional hypotheses, the sets of Theorem A' can be strictly larger than those of Theorem A when \( n \geq 2 \). (If, however, one adds the hypothesis that \( u \) is positive, or even bounded below in each cone \( \Gamma(x; a, k) \) for \( x \in E \) of positive measure — the bound may depend on \( x \) — then \( u \) has a nontangential limit almost everywhere in \( E \) (Carleson [6]), as well as a fine limit almost everywhere in \( E \) (Brelot and Doob [1]).)

We now consider the geometric and probabilistic descriptions of the Hardy classes \( H^p \). For \( \mathbb{R}^2 \), it is shown in [3] that \( H^p, 0 < p < \infty \) may be described as the space of real harmonic functions \( u \) such that

\[
\text{sup}_{k \geq 0} \int_{\mathbb{R}^2} |N(u; a, k)|^p \, dx < \infty.
\]

Fefferman and Stein [10] extend this result to the \( H^p \) spaces introduced by Stein and Weiss [20] for harmonic functions in \( \mathbb{R}^{n+1} \), \( n \geq 2 \). Therefore, we take (4) as the definition of \( H^p \).

The probabilistic analogue of condition (4) is

\[
\text{sup}_{y > 0} \int_{\mathbb{R}^n} E_{z, y}(|u^*|^p) \, dx < \infty
\]

where \( E_{z, y} \) is the expectation corresponding to \( P_{z, y} \).

Fefferman and Stein show that the area function and non-tangential maximal function are related as follows:

**Theorem B.** — Let \( u \) be harmonic in \( \mathbb{R}^{n+1} \). Then for all \( p \) in the interval \( 0 < p < \infty \),

\[
\text{sup}_{k \geq 0} \int_{\mathbb{R}^n} A(u; a, k)(x)^p \, dx \leq c_{p, a} \text{sup}_{k \geq 0} \int_{\mathbb{R}^n} |N(u; a, k)(x)|^p \, dx.
\]
Furthermore, if the left-hand side of this inequality is finite, \( u \) may be normalized to vanish at infinity and with this normalization

\[
\sup_{k > 0} \int_{\mathbb{R}^n} |N(u; a, k)(x)|^p \, dx \leq C_{p,a} \sup_{k > 0} \int_{\mathbb{R}^n} |A(u; a, k)(x)|^p \, dx.
\]

The probabilistic version of Theorem B is stated in [3] for \( \mathbb{R}^2_+ \) ([3], Lemma 4). The proof, however, is valid in any number of dimensions. We restate it here as Theorem B'.

**Theorem B'.** — Let \( u \) be harmonic in \( \mathbb{R}^{n+1}_+ \). For all \( p \) in the interval \( 0 < p < \infty \),

\[
c_p \sup_{\gamma > 0} \int_{\mathbb{R}^n} E_{x, \gamma}(|S(u)|^p) \, dx \leq \sup_{\gamma > 0} \int_{\mathbb{R}^n} E_{x, \gamma}(|u^*|^p) \, dx \leq C_p \sup_{\gamma > 0} \int_{\mathbb{R}^n} E_{x, \gamma}(|S(u)|^p) \, dx.
\]

The second purpose of this paper is to compare Theorems B and B'.

**Theorem 2.** — Let \( u \) be harmonic in \( \mathbb{R}^{n+1}_+ \), \( n \geq 2 \). Then

\[
c_{p,a} \sup_{\gamma > 0} \int_{\mathbb{R}^n} E_{x, \gamma}(|u^*|^p) \, dx \leq \sup_{k > 0} \int_{\mathbb{R}^n} |N(u; a, k)|^p \, dx \leq C_{p,a} \sup_{\gamma > 0} \int_{\mathbb{R}^n} E_{x, \gamma}(|u^*|^p) \, dx.
\]

Thus, while the probabilistic and nontangential local convergence criteria are different in \( \mathbb{R}^{n+1}_+ \) for \( n \geq 2 \), the \( H^p \) spaces, defined probabilistically or geometrically, coincide in all dimensions. It then follows from Theorems B and B' that the Brownian and nontangential area functions have equivalent \( L^p \)-norms for \( 0 < p < \infty \).

**Proof of Theorem 1.** — Since the first two statements of Theorem 1 may be found in Brelot and Doob [1], we prove only the last by constructing an example: There is a function \( u \) that is harmonic in \( \mathbb{R}^{n+1}_+ \) such that \( a) \lim_{t \to 0} u(x_t, y_t) \) exists and is finite with \( P_{x_0, y_0} \)-probability one for almost all \( x \in \mathbb{R}^n \); \( b) \) nontangential convergence of \( u \) holds for no \( x \in \mathbb{Q} \), the unit cube in \( \mathbb{R}^n \). That is, the set \( (1') \) is strictly larger than the set \( (1) \).
For simplicity, we carry out the details for $\mathbb{R}^3_+$. Roughly speaking, we construct a bed with an infinite number of vertical spines of varying height on the unit square. The function $u$ defined on $\mathbb{R}^3_+$ is to be large and of varying sign at the end of each spine, but small nearly everywhere else. The set where $u$ is largest — the tips of the spines — has small capacity, so the Brownian paths from $(x_0, y_0)$ miss these points with high probability. On the other hand, any cone $\Gamma(x), x \in Q$ is punctured by infinitely many of the spines, so that the oscillation of $u$ over $\Gamma(x)$ is infinite for every $x \in Q$.

Let

$$D_n = \left\{ \left( \frac{2j - 1}{2^n}, \frac{2k - 1}{2^n}, \frac{a^{-1}}{2^{n-1}} \right) : j = 1, \ldots, 2^{n-1}, k = 1, \ldots, 2^{n-1} \right\}$$

so that $\Gamma(x; a, k)$ contains at least one point of $D_n$ for each $n \geq n(a, k)$. The function $u$ to be constructed satisfies

$$u(x, y) \geq n, \quad (x, y) \in D_n$$

for $n$ odd, and

$$u(x, y) \leq -n, \quad (x, y) \in D_n$$

for $n$ even. Therefore, the oscillation of $u$ over the cone $\Gamma(x; a, k), x \in Q$ is infinite, so that $u$ has a nontangential limit nowhere in the set $Q$. For simplicity, we may assume that $a = 1, k = 2$, and denote the corresponding cone by $\Gamma(x)$.

The function $u$ to be constructed is of the form

$$u = \sum_{j=1}^{\infty} u_j$$

where each $u_j$ is harmonic in all of $\mathbb{R}^3$ and the series is uniformly convergent on compact subsets of $\mathbb{R}^3_+$. Therefore,

$$\lim_{t \to \infty} u_j(x_t, y_t) = u_j(x, 0)$$

almost everywhere with respect to $P_{x_0, y_0}$. Also, we show that with $P_{x_0, y_0}$-probability one,

$$\sum_{j=1}^{\infty} u_j^* < \infty$$
so that by the Lebesgue dominated convergence theorem,

$$\lim_{t \to \tau} u(x_t, y_t) = \sum_{j=1}^{\infty} \lim_{t \to \tau} u_j(x_t, y_t) = \sum_{j=1}^{\infty} u_j(x_t, 0)$$

almost everywhere \(P_{x_0, y_0}\). By definition of the conditional measures \(P_{x_0, y_0}\), we have

$$P_{x_0, y_0} \left( \lim_{t \to \tau} u(x_t, y_t) \text{ exists and is finite} \right) = 1$$

for almost every \(x \in Q\), with respect to Lebesgue measure. In other words, \(u\) has a fine limit for almost every \(x \in Q\), but a nontangential limit nowhere in \(Q\).

The basic device in the construction is Runge's theorem for harmonic functions in \(\mathbb{R}^n\). (Walsh [21]; also see Lelong's review [12], for other references.)

**Runge's Theorem for \(\mathbb{R}^{n+1}\).** Let \(K\) be a compact set in \(\mathbb{R}^{n+1}\) such that \(\mathbb{R}^{n+1} - K\) is connected. Suppose that \(u\) is harmonic on an open set containing \(K\). Then \(u\) can be uniformly approximated by harmonic polynomials on \(K\).

We now proceed with the construction. For convenience, assume that the initial point \((x_0, y_0)\) for the Brownian motion satisfies \(y_0 \geq 2\). Let \(0 < \varepsilon_n < \frac{1}{2^{n-1}}, b_n > y_0 + n\) be chosen so that

$$(5) \quad P_{x_0, y_0}((x_t, y_t) \in Q_n - T_n \quad \text{for all} \quad 0 \leq t \leq \tau) \geq 1 - \frac{1}{2^n}$$

where

$$Q_n = [-b_n, b_n] \times [-b_n, b_n] \times [0, 2b_n]$$

and

$$T_n = \left\{(s, y) : |x - s| < \varepsilon_n, \quad 0 \leq y < \frac{1}{2^{n-1}} + \varepsilon_n, \right\}$$

for some point \(\left(x, \frac{4}{2^{n-1}}\right) \in D_n\).

Notice that \(T_n\) is the union of \(2^{n-2}\) disjoint cylinders or «spines» each of which contains a point of \(D_n\) in its interior. Notice also that, because of the transience of Brownian motion in \(\mathbb{R}^3\), the choice of \(\varepsilon_n, b_n\) in (5) is possible in \(\mathbb{R}^3\) but not in \(\mathbb{R}^2\). The set \(K_n = (Q_n - T_n) \cup D_n\) is compact and
\( \mathbb{R}^3 - K_n \) is connected, so that the hypotheses of Runge’s theorem apply. Let \( U \) and \( V \) be disjoint open sets such that

\[ Q_n - T_n \subset U \]

and

\[ D_n \subset V. \]

Let \( \omega(x, y) \) be defined on \( U \cup V \), equal to zero on \( U \), \( \lambda_n \) on \( V \), where \( \lambda_n \) is a constant to be chosen later. Then \( \omega \) is harmonic on \( U \cup V \) and by Runge’s theorem, there is a harmonic polynomial \( u_n \) such that

\[ |u_n(x, y) - \omega(x, y)| < \frac{1}{2^n} \quad \text{on} \quad K_n. \]

Therefore,

(6) \[ |u_n(x, y)| < \frac{1}{2^n} \quad \text{for} \quad (x, y) \in Q_n - T_n \]

and

\[ |u_n(x, y) - \lambda_n| < \frac{1}{2^n} \quad \text{for} \quad (x, y) \in D_n. \]

The first claim is that the series \( \sum_{n=1}^{\infty} |u_n(x, y)| \) converges uniformly on compact subsets of \( \mathbb{R}^3_+ \). Any compact subset of \( \mathbb{R}^3_+ \) is a subset of \( Q_n - T_n \) for all large \( n \), so uniform convergence follows from (6). It follows that

\[ u = \sum_{j=1}^{\infty} u_j \]

is harmonic in \( \mathbb{R}^3_+ \).

Finally, we must choose the constants \( \lambda_n \). Let \( \lambda_1 = 2 \) and note that the point \( \left( \frac{1}{2}, \frac{1}{2}, 1 \right) \in D_1 \) but

\[ \left( \frac{1}{2}, \frac{1}{2}, 1 \right) \in Q_n - T_n \]

for all \( n > 1 \). Therefore

\[ u\left( \frac{1}{2}, \frac{1}{2}, 1 \right) = \sum_{j=1}^{\infty} u_j \left( \frac{1}{2}, \frac{1}{2}, 1 \right) > 2 - \sum_{j=1}^{\infty} \left( \frac{1}{2} \right)^j = 1. \]
Suppose $\lambda_1, \ldots, \lambda_{n-1}$ have been chosen so that $u(x, y) \geq k$ for $(x, y) \in D_k$, $k$ odd, and $u(x, y) \leq -k$ for $(x, y) \in D_k$, $k$ even. Simply choose $\lambda_n$ so that

$$\inf_{(x, y) \in D_n} \sum_{k=1}^{n-1} u_k(x, y) + \lambda_n > n + 1$$

if $n$ is odd. Then

$$u(x, y) > n + 1 - \frac{1}{2^n} - \frac{1}{2^{n+1}} - \cdots \geq n$$

for $(x, y) \in D_n$ since this also implies $(x, y) \in Q_m - T_m$ for $m > n$. If $n$ is even, then choose $\lambda_n$ so that

$$\sup_{(x, y) \in D_n} \sum_{k=1}^{n-1} u_k(x, y) + \lambda_n < -(n + 1);$$

then $u(x, y) \leq -n$ for $(x, y) \in D_n$ in the same way. Finally, by (5) and (6),

$$P_{x_0, y_0} \left( u_n^* > \frac{1}{2^n} \right) \leq \frac{1}{2^n}$$

so that $\sum_{n=1}^{\infty} u_n^* < \infty$ almost everywhere $(P_{x_0, y_0})$. This completes the construction.

**Proof of Theorem 2.** — We begin with a series of lemmas.

**Lemma 1.** — For $b > a > 0$, and $\lambda > 0$,

$$m(N(u; b, k) > \lambda) \leq Cm(N(u; a, k) > \lambda)$$

The choice of $C$ depends only on the dimension $n$ and the ratio $a/b$. In particular,

$$\|N(u; b, k)\|^p \leq C\|N(u; a, k)\|^p.$$

This lemma corresponds to Lemma 2 of [2], stated for $N(u; a)$ and $N(u; b)$. The proof, however, is valid for any measurable function $u$ defined on $\mathbb{R}^{n+1}_+$. Therefore, we may simply apply that argument to

$$u_k(x, y) = u(x, y) \text{ if } y \leq k$$

$$= 0 \text{ otherwise.}$$
The second assertion of the lemma follows from the integration formula
\[ \|N(u; a, k)\|_p = p \int_0^\infty \lambda^{p-1} m(N(u; a, k) > \lambda) \, d\lambda. \]

The next lemma is due to Hardy and Littlewood [11] for the case \( p < 1 \). They state it without proof; a full proof is given by Fefferman and Stein (Lemma 2 in [10]).

**Lemma 2.** Let \( B_R \) be a ball in \( \mathbb{R}^{n+1} \) with center at \((x_0, y_0)\), radius \( R > 0 \), and \( B_r \subset B_R \) be another ball with the same center but with radius \( r < R \). Then for \( 0 < p < \infty \),
\[
\sup_{(s, t) \in B_r} |u(s, t)|^p \leq C_{p, r/R} \frac{1}{m(B_R)} \int_{B_R} |u(x, y)|^p \, dx \, dy.
\]

**Lemma 3.** Let
\[ D(u; a, k) = \sup_{(s, y) \in \Gamma(x; a, k)} y|\nabla u(s, y)|; \]
then
\[ D(u; a, k) \leq C N(u; b, 2k) \]
for \( b > a \), with \( C \) depending only on the dimension \( n \) and the ratio \( a/b \).

This lemma is taken from Stein [19] (see Lemma 4). We omit the proof.

**Lemma 4.** Let \( u \) be harmonic in \( \mathbb{R}^{n+1}_+ \) and satisfy the condition
\[
\sup_{y > 0} \int_{\mathbb{R}^n} |u(x, y)|^p \, dx < \infty
\]
for some \( p \) in the interval \( 0 < p < \infty \). Then
\[
\|N(u_{a}; a, k)\|_p < \infty
\]
for all \( a > 0, k > 0 \), where \( u_{a}(x, y) = u(x, y + a) \) for \( a > 0 \). Furthermore, there exists a \( k_0 > 0 \) such that for all \( k \geq k_0 \) we have
\[
\|N(u_{a}; 2a, 2k)\|_p \leq C\|N(u_{a}; a, k)\|_p.
\]
The constant $k_0$ depends on $u$, but $C$ depends only on $p$ and the dimension $n$.

Proof. — If $\lim_{k \to \infty} \|N(u_\alpha; a, k)\|_p < \infty$ for some $a > 0$, then the same is true for $2a$ by Lemma 1. Also, (8) holds for $k > 0$ sufficiently large.

We now assume that $\lim_{k \to \infty} \|N(u_\alpha; a, k)\|_p = \infty$ for $a \leq \frac{1}{2}$.

Consider the ball $B(x, \frac{a}{2})$ with center at $(x, \frac{a}{2})$, radius $\frac{3a}{2}$. Then $B(x, \frac{a}{2})$ contains the cone $\Gamma(x; a, \alpha)$ and all points of $\Gamma(x; a, \alpha)$ lie at a distance of more than $(\frac{3}{2} - 1/\sqrt{2})a$ from the boundary of the ball $B(x, \frac{a}{2})$. Therefore, by Lemma 2,

$$|N(u_\alpha; a, \alpha)(x)|^p \leq C_p \frac{1}{m(B(x, \frac{a}{2}))} \int_{B(x, \frac{a}{2})} |u_\alpha(s, s)|^p \, ds \, dy.$$  

If we integrate both sides of the above inequality with respect to $x$, and use Fubini's theorem, we obtain

$$\int_{\mathbb{R}^n} |N(u_\alpha; a, \alpha)|^p \leq C_p \sup_{0 < y < \frac{3a}{2}} \int_{\mathbb{R}^n} |u(x, y)|^p \, dx \leq C_p \sup_{\gamma > 0} \int_{\mathbb{R}^n} |u(x, y)|^p \, dx < \infty.$$  

That is, we have shown that $\|N(u_\alpha; a, k)\|_p < \infty$ for $k = \alpha$ provided $a \leq 1/2$. The same kind of argument shows that if $\|N(u_\alpha; a, k)\|_p < \infty$ for some $k < \infty$, then

$$\|N(u_\alpha; a, 2k)\|_p \leq \|N(u_\alpha; a, k)\|_p + C_p \sup_{\gamma > 0} \int_{\mathbb{R}^n} |u(x, y)|^p \, dx.$$  

In fact, if

$$M(u_\alpha; a, 2k)(x) = \sup \{|u_\alpha(s, y)| : (s, y) \in \Gamma(x; a, 2k) - \Gamma(x; a, k)\}$$  

then

$$|N(u_\alpha; a, 2k)|^p \leq |N(u_\alpha; a, k)|^p + |M(u_\alpha; a, 2k)|^p.$$  

If $B(x, \frac{3k}{2})$ is the ball centered at $(x, \frac{3k}{2})$, then the «top half» of the cone $\Gamma(x; a, 2k)$, that is, the set $\Gamma(x; a, 2k) - \Gamma(x; a, k)$, is contained in the ball $B(x, \frac{3k}{2})$,.
and lies at a distance of \( \left( \frac{3 - \sqrt{5}}{2} \right) k \) from the boundary of the ball. Therefore, again by Lemma 1 and the argument leading to (9), we find that

\[
\|M(u_2; a, 2k)\|_p^p \leq C_p \sup_{y \geq 0} \int_{\mathbb{R}^2} |u(x, y)|^p \, dx.
\]

Therefore, (10) follows from this and inequality (12).

The argument to this point shows that \( \|N(u_2; a, k)\|_p < \infty \) for all \( k > 0 \) since this statement is true for \( k = a, 2a, \ldots \). A slight amplification of the argument shows that \( \|N(u_2; a, k)\|_p \) is a continuous, increasing function of \( k \) with range equal to the interval \([0, \infty)\). Therefore, for some \( k_0 > 0 \), we have

\[
\|N(u_2; a, k_0)\|_p = \sup_{y \geq 0} \int_{\mathbb{R}^2} |u(x, y)|^p \, dx.
\]

For any \( k \geq k_0 \), from (10) we have

\[
\|N(u_2; a, 2k)\|_p^p \leq \|N(u_2; a, k)\|_p^p + C_p \sup_{y \geq 0} \int_{\mathbb{R}^2} |u(x, y)|^p \, dx \\
\leq (1 + C_p) \|N(u_2; a, k)\|_p^p.
\]

Finally, by Lemma 1, we may replace \( a \) by \( 2a \) and obtain

\[
\|N(u_2; 2a, 2k)\|_p^p \leq C_p \|N(u_2; a, k)\|_p^p.
\]

The lemma is proved.

**Lemma 5.** — Given \( D > 0 \) and \( 0 < p < \infty \), let \( f_i, i = 1, 2 \), be a pair of functions that satisfy the inequality

\[
\int |f_i|^p \leq D \int |f_1|^p < \infty.
\]

Then

\[
\int |f_1|^p \geq 2 \int_{\|f_i\|_p > a} |f_i|^p
\]

**Proof.** Since \( \|f_i\|_p < \infty \), either the conclusion of the lemma holds, or, with strict inequality, we have

\[
\int |f_1|^p < 2 \int_{\|f_i\|_p < a} |f_i|^p \leq \frac{1}{D} \int |f_2|^p \leq \int |f_1|^p,
\]

which is a contradiction.
Given any point \((s, y) \in \mathbb{R}^{n+1}_+\), recall that \(P_{s,y}^x\) is the measure associated with conditional Brownian motion with initial point \((s, y)\) and terminal point \(x \in \mathbb{R}^n\).

**Lemma 6.** — Let \(B(x', y')\) be the ball in \(\mathbb{R}^{n+1}_+\) with center at \((x', y')\), radius \(\theta y'\), \(0 < \theta < 1\), and with \(|x' - x| \leq ay'\). If \(|s - x| \leq ay, y \geq 2y'\), then

\[
P_{s,y}^x((x, y) \text{ hits } B(x', y')) \geq C > 0
\]

where \(C\) depends only on \(\theta\) and \(a\).

**Proof.** — Let \(\tau = \inf \{t : y_t = y'\}\). The conditional measure associated with the random vector \((x_\tau, y_\tau)\) is given by

\[
h_{x}(s, y)P_{s,y}^x((x_\tau, y_\tau) \in A) = \int h_{x}(x_\tau, y_\tau)P_{x_\tau,y_\tau}^x(dx_\tau, dy_\tau) \quad \{(x_\tau, y_\tau) \in A\}
\]

where \(h_{x}\) is the Poisson kernel for \(\mathbb{R}^{n+1}_+\) with pole at \(x \in \mathbb{R}^n\). This formula may be obtained by a standard stopping time argument. The probability \(P_{s,y}^x((x_\tau, y_\tau) \in A)\) has a density with respect to Lebesgue measure on the hyperplane \(y = y'\) in \(\mathbb{R}^{n+1}_+\) given by

\[
f^x(s, y, y') = C_n \frac{y - y'}{(|s - |^2 + |y - y'|^2)^{(n+1)/2}} \frac{h_{x}(s, y')}{h_{x}(s, y)}.
\]

It follows that

\[
\int_{S(x', y')} f^x(\omega; (s, y), y') \, d\omega \geq C > 0
\]

where \(S(x', y')\) is the projection of \(B(x', y')\) on the hyperplane \(y = y'\). (The constant \(C\) depends only on \(\theta\) and \(a\).) The integral represents the probability that the \(n\)-dimensional sphere \(S(x', y')\) is hit by a conditional path from \((s, y)\) to \(x\) before the path hits the complement of \(S(x', y')\) on the hyperplane \(y = y'\). Since this probability is smaller than the one we wish to estimate, the lemma is proved.

**Lemma 7.** — Let \(u\) be a continuous function in \(\mathbb{R}^{n+1}_+\). Then

\[
\sup_{\lambda > 0} \int_{\mathbb{R}^n} P_{x,y}(u^* > \lambda) \, dx \leq C \sup_{k > 0} m(N(u; a, k) > \lambda).
\]
Remarks. — This inequality is stated as part of Theorem 3 of [3] for the case \( n = 1 \). The proof may be extended without difficulty for \( n > 1 \), so we omit the details. It should be noted, however, that in Theorem 3 of [3], we assume that \( u \) is harmonic. As is clear from the proof, this assumption is used to obtain the converse inequality only.

We are now in a position to prove Theorem 2. First, we note that

\[
\sup_{y > 0} \int_{\mathbb{R}^n} E_{x,y}(|u^*|^p) \, dx \leq C \sup_{k > 0} \int_{\mathbb{R}^n} |N(u; a, k)|^p \, dx
\]

where \( C \) depends only on \( a \). This follows immediately by integrating the stronger estimate given in Lemma 7.

Now, we prove that

\[
\sup_{k > 0} \int_{\mathbb{R}^n} |N(u; a, k)|^p \, dx \leq C \sup_{y > 0} \int_{\mathbb{R}^n} E_{x,y}(|u^*|^p) \, dx
\]

where \( C \) depends only on \( a \) and \( p \). We may assume that the right hand side is finite, so that, in particular,

\[
\sup_{y > 0} \int_{\mathbb{R}^n} |u(x, y)|^p \, dx < \infty.
\]

The hypothesis of Lemma 4 is satisfied, and therefore, we have

\[
(14) \quad \|N(u_\alpha; 2a, 2k)\|^p_p \leq C\|N(u_\alpha; a, k)\|^p_p
\]

with the right hand side finite for \( \alpha > 0 \). We now apply Lemma 5 with \( f_1 = N(u_\alpha; a, k) \) and \( f_2 = N(u_\alpha; 2a, 2k) \). The hypothesis of Lemma 5 is satisfied with the constant \( D = C \) where \( C \) is given in Lemma 1, independent of \( a, \alpha, m, \) and \( k \). Let

\[
G = \{ x : N(u_\alpha; a, k)(x) \geq (2C)^{-1/p} N(u_\alpha; 2a, 2k)(x) \}.
\]

From this and Lemma 3, we may conclude that

\[
(15) \quad G \subseteq \left\{ x : D \left( u_\alpha; \frac{3a}{2}, \frac{3}{2} k \right)(x) \leq CN(u_\alpha; a, k)(x) \right\}
\]

for another constant \( C \) independent of \( a, \alpha, m, \) or \( k \). Fix a point \( x \in G \) and consider the cone \( \Gamma(x; a, k) = \Gamma \). We may
select a point \((x', y') \in \Gamma\) and a ball \(B(x', y')\) with center \((x', y')\), radius \(\theta y'\) such that \(|u_\alpha(s, t)| \geq \frac{1}{4} N(u_\alpha; a, k)(x)\) for every point \((s, t) \in B(x', y')\). This may be done as follows: 

Choose \((x', y') \in \Gamma\) so that \(|u_\alpha^*(x', y')| > \frac{1}{2} N(u_\alpha; a, k)(x)\). Since \(x \in G\), we know (15) holds so that

\[
t|\nabla u_\alpha(s, t)| \leq CN(u_\alpha; a, k)(x)
\]

for all points \((s, t)\) in a ball of radius \(\theta y'\), centered at \((x', y')\), and contained in the cone \(\Gamma \left( x; \frac{3a}{2}, \frac{3}{2} k(x) \right)\). The constant \(\theta\) depends only on the angle \(a\), at this point; we may assume \(\theta < \frac{1}{2}\). By the mean value theorem,

\[
|u_\alpha(s, t) - u_\alpha(x', y')| \leq 2CN(u_\alpha; a, k)(x) \frac{(x' - s)^2 + (y' - t)^2}{y'}
\]

for all points \((s', t') \in B(x', y')\). Now choose \(\theta\) so that \(\theta C < 1\); it follows from the above inequalities that

\[
|u_\alpha(s, t)| \geq |u_\alpha(x', y')| - |u_\alpha(s, t) - u_\alpha(x', y')| \geq \frac{1}{4} N(u_\alpha; a, k)(x)
\]

for all \((s', t') \in B(x', y')\) with radius \(\theta y'\).

If we now apply Lemma 6 to \(B(x', y')\), we obtain

\[
(16) \quad P_{x,y}^x \left( u_\alpha^* \geq \frac{1}{4} N(u_\alpha; a, k)(x) \right) \geq P_{x,y}^x((x, y) \text{ hits } B(x', y')) \geq C > 0
\]

for \(x \in G, |s - x| \leq a y, y \geq 2y'\). Notice that \(y' \leq k\), so points \((s, y)\) such that \(|s - x| \leq a y, y \geq 2k\) satisfy the above requirements. The last restriction, \(y \geq 2k\), prevents us from making a direct estimation of integrals from the probabilities (16). To overcome this obstacle, we cut into \(G\) in the following way: Let \(R\) be chosen large enough so that \(G_R = G \cap \{|x| \leq R\}\) satisfies

\[
(17) \quad \int_{R^a} |N(u_\alpha; a, k)(x)|^p \, dx \leq 2C^p \int_{c_R} |N(u_\alpha; a, k)(x)|^p \, dx
\]
where \( C = C(14) \). With this choice of \( R \), let \( y_0 \) be chosen so that \( y_0 \geq 2k \) and such that each point \((s, y_0)\) in the \( n\)-dimensional ball \(|s| \leq \frac{a}{2} y_0\) is contained in the cone \( \Gamma(x; a, y_0) \) for every \( x \) such that \(|x| \leq R\). In particular, since \( y_0 \geq 2k \), the points in the ball \( \{(s, y_0) : |s| \leq \frac{a}{2} y_0\} \) satisfy the requirements of inequality (16) for every \( x \in G_R \). Let \( E_{x,y_0}^\infty \) be the conditional expectation corresponding to \( P_{x,y_0}^\infty \) and \( \chi_{G_R}(x) \) the indicator function of the set \( G_R \); we have

\[
E_{x,y_0}^\infty(|u_\alpha^*|^p) = E_{x,y_0}^\infty(E_{x,y_0}^\infty(|u_\alpha^*|^p)) \\
\geq E_{x,y_0}^\infty(E_{x,y_0}^\infty(|u_\alpha^*|^p)\chi_{G_R}(x)) \\
\geq CE_{x,y_0}^\infty(|N(u_\alpha; a, k)(x)|^p\chi_{G_R}(x)) \\
= C \int_{R^n} \chi_{G_R}(x)|N(u_\alpha; a, k)(x)|^p \frac{y_0}{(|s-x|^2 + y_0^{(a+1)/2})} dx.
\]

Therefore,

\[
\sup_{y > 0} \int_{R^n} E_{x,y}(|u_\alpha^*|^p) \, ds \geq \int_{|s| \leq \frac{a}{2} y_0} E_{x,y_0}(|u_\alpha^*|^p) \, ds \\
\geq C \int_{|s| \leq \frac{a}{2} y_0} \int_{R^n} \chi_{G_R}(x)|N(u_\alpha; a, k)(x)|^p \frac{y_0}{(|s-x|^2 + y_0^{(a+1)/2})^2} \, dx \, ds \\
\geq C \int_{R^n} \chi_{G_R}(x)|N(u_\alpha; a, k)(x)|^p \, dx \\
\geq C \int_{R^n} |N(u_\alpha; a, k)(x)|^p \, dx.
\]

Here we have used Fubini's theorem, inequality (17), and the fact that \( y_0 \geq 2R \) and \(|s| \leq \frac{a}{2} y_0\) implies

\[
\frac{y_0}{(|s-x|^2 + y_0^{(a+1)/2})^2} \simeq \frac{1}{y_0^2}.
\]

In summary, we have shown that

\[
\sup_{y > 0} \int_{R^n} E_{x,y}(|u_\alpha^*|^p) \, ds \geq C \int_{R^n} |N(u_\alpha; a, k)(x)|^p \, dx
\]

for \( k \geq k_0 \) and all \( \alpha > 0 \). By successive applications of the
monotone convergence theorem, we finally conclude that
\[
\sup_{y>0} \int_{\mathbb{R}^n} E_{z,y}(|u^*|^p) \, dx \geq C \sup_{k>0} \int_{\mathbb{R}^n} |N(u; a, k)(x)|^p \, dx.
\]

Theorem 2 is proved.

BIBLIOGRAPHIE


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