

# ANNALES DE L'INSTITUT FOURIER

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*Annales de l'institut Fourier*, tome 23, n° 4 (1973), p. 65-74

[http://www.numdam.org/item?id=AIF\\_1973\\_\\_23\\_4\\_65\\_0](http://www.numdam.org/item?id=AIF_1973__23_4_65_0)

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## TOPICS ON KRONECKER SETS

by R. KAUFMAN

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In the first part of this note we consider relations between classes of differentiable functions and linear Kronecker sets. The problem in each of three theorems is to find a set  $E$  of some narrow type, and a differentiable map  $\varphi$ , so that  $\varphi(E)$  is a Kronecker set. The first theorem focusses on functions  $\varphi$  in a prescribed non-quasi-analytic class  $C(M_n)$ , as described in [2, V] and [6, Ch. 19]. The second deals with a qualitative study of  $\varphi'$ , and the third uses van der Corput's inequality to impose a very strong condition on  $E$ .

The second part illustrates the use of our method for constructing Kronecker sets; the lesson is that many special phenomena of exceptional sets are present in each set of multiplicity. Here we have in mind the work of Körner [5], and apply our method to strengthen a theorem on the union of two Kronecker sets.

### 1.

Let  $E$  be a compact subset of  $[0, 1]$ ; the exact condition on  $E$  is known, that there be a function  $\varphi$  of class  $C^1[0, 1]$  so that  $\varphi' > 0$  and  $\varphi(E)$  is a Kronecker set [3; 1, VII; 4]. There exist, however, sets  $E$  of this type such that  $\varphi(E)$  is an  $M_0$ -set whenever  $\varphi' > 0$  and  $\varphi \in C^2$ . Thus there is some interest in sets  $E$  for which  $\varphi \in C^\infty$  can be chosen as before; in the next theorem, we prove what is perhaps an extremal result on the possible smoothness of  $\varphi$ . Let  $(\lambda_n)$  be an increasing sequence of positive numbers and  $M_n = \lambda_1 \dots \lambda_n$ .

**THEOREM 1.** — *Let  $F$  be an  $M_0$ -set, and let  $\Sigma\lambda_n^{-1} < \infty$ . Then there is a  $M_0$ -set  $E \subseteq F$ , a function  $\varphi$  of class  $C^\infty(\mathbb{R})$  so that*

- (a)  $\varphi' \geq 1$  everywhere,  $\varphi^{(n)} = O(M_n)$  uniformly ( $n \geq 1$ ).
- (b)  $\varphi(E)$  is a Kronecker set.

To explain the significance of the condition  $\Sigma\lambda_n^{-1} < \infty$ , let  $\psi$  be  $C^\infty$  of compact support and  $P_n = \|\psi^{(n)}\|_2$  so that by the Plancherel theorem  $(P_n)$  is log-convex [2] and by the Denjoy-Carleman theorem  $\Sigma P_n P_{n+1}^{-1} < \infty$  [2, 6]. Thus the condition on  $(\lambda_n)$  is essential for the technique of partitions of unity, and Theorem 1 may be best-possible among results valid for all  $M_0$ -sets  $E$ .

In the proof of Theorem 1 we take a sequence  $(a_n)$  decreasing to 0 so that  $\Sigma(a_n\lambda_n)^{-1} < \infty$ . Then there is a function  $\psi$  of compact support,  $0 \leq \psi \leq 1$ ,  $\psi = 1$  on a neighbourhood of 0, and  $|\psi^{(n)}| = O(a_1\lambda_1 \dots a_n\lambda_n)$  [2, 6]. Next we can expand the interval on which  $\psi = 1$  at will, preserving the inequality  $0 \leq \psi \leq 1$  and the inequalities on  $\psi^{(n)}$ . Because  $\lim a_n = 0$ , each homothety of  $\psi$ , say  $\varphi(x) = \psi(Ax + B)$  also satisfies the inequalities  $|\psi_2^{(n)}| \leq C_A M_n$  ( $0 \leq n < \infty$ ), with  $C_A$  a function of  $A$  alone. In particular, if  $r < r_1 < s_1 < s$ , one of the homotheties equals 1 on  $(r_1, s_1)$  and 0 outside  $(r, s)$ .

After a few reductions we can suppose that  $F$  is a closed, totally disconnected subset of  $[0, 1]$  and  $\mu$  is a probability measure in  $F$  with  $\hat{\mu}$  in  $C_0(\mathbb{R})$ . Let  $(f_m)_1^\infty$  be a dense sequence in the real Banach space  $C[0, 1]$ , and let  $(h_m)_1^\infty$  be a sequence of  $2\pi$ -periodic  $C^\infty$ -functions of mean 1, such that  $h_m \geq 0$  and  $h_m(t) = 0$  when  $m^{-1} \leq |t| \leq \pi - m^{-1}$ . Thus for any function  $g$  and real number  $y$ ,  $h_m(yg - f_m) = 0$  except on the set  $(|\exp iyg - \exp if_m| \leq m^{-1})$ .

We shall construct a sequence of measures  $\mu_m$ , beginning with  $\mu_0 = \mu$  and

$$(1) \quad \mu_m = h_m(y_m g_m - f_m) \cdot \mu_{m-1}$$

where  $1 < y_1 < \dots < y_m < \dots$  and  $g_m$  converges so rapidly to a limit  $\varphi$  that  $y_m |g_m - \varphi| \leq 2m^{-1}$ . Also,  $|\hat{\mu}_m - \hat{\mu}_{m-1}| < 3^{-m}$  so that the weak limit  $\sigma$  of the sequence

$(\mu_m)$  has norm in  $\left(\frac{1}{2}, \frac{3}{2}\right)$  and  $\hat{\sigma}$  is in  $C_0(\mathbb{R})$ . On the support of  $\sigma$ , which is contained in the support of  $\mu_m$ , we have  $|\exp iy_m\varphi - \exp if_m| \leq 2m^{-1}$ , and Theorem 1 will be proved by constructing the limit  $\varphi$  with the special properties listed.

Each  $h_m$  has an absolutely convergent Fourier expansion  $h_m(t) = \sum a_k \exp ikt, a_0 = 1$ . Thus

$$\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u) = \int \sum' a_k \exp ik(y_m g_m - f_m) \cdot \exp - iut \cdot d\mu_{m-1}.$$

Now each  $g_m$  shall be constructed so that  $g_m' = 0$  on a neighbourhood of  $F$ ; hence there are disjoint intervals  $I_j$ , covering  $F$ , on which  $g_m' = b_j$ , say. We express the integral above as a sum of integrals over  $I_j$ 's. The  $j$ -th term in the  $k$ -th integral has the same modulus as  $\hat{v}(u - ky_m b_j)$ , where  $v \equiv v(k, j)$  is absolutely continuous with respect to  $\mu_0$  and  $\sum_j \|v(k, j)\| = \|\mu_{m-1}\|$  for all  $k \neq 0$ . Each  $\hat{v} \in C_0(\mathbb{R})$ , whence to each  $\varepsilon > 0$  there is a  $T$  so that for all  $u$

$$|\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u)| \leq \varepsilon + \sum'' |a_k| \cdot \|v(k, j)\|,$$

where  $\sum''$  means summation over the set of indices

$$k \neq 0, \quad |u - ky_m b_j| < T.$$

Next, let us suppose that the values  $b_j$ , of  $g_m'$  on  $I_j$ , are distinct, differing among each other by at least  $r > 0$ . Each inequality  $|u - ky_m b_j| < T$  has at most one solution  $j$  for each  $k \neq 0$ , as soon as  $yr > 2T$ . When this condition is imposed, the sum  $\sum''$  does not exceed  $\max \mu_{m-1}(I_j) \cdot \sum |a_k|$ .

From this point is clear how to proceed. Let  $g_0 = x$  and suppose that  $g_{m-1}$  is a known function of class  $C^\infty$ , with  $g_{m-1}' = 0$  on a neighbourhood of  $F$ . Let  $(I_j)$  be a covering of  $F$  by disjoint intervals so that  $\mu_{m-1}(I_j) \sum |a_k| < 4^{-m}$ , say. Then there is a function  $p \geq 0$  of compact support, so that  $p = c_j > 0$  on  $I_j$ , where the numbers  $c_j$  are distinct; the indefinite integral  $q$  of  $p$ , with  $q(-\infty) = 0$ , has the property that  $|q| = 0(1)$  while

$$|q^{(n+1)}| = |p^{(n)}| = 0(M_n) = 0(M_{n+1}).$$

For all  $\delta > 0$  sufficiently small, the level sets of  $(g_{m-1} + \delta q)$ ,

form a covering of  $F$  finer than the covering  $(I_j)$  and we choose  $g_m = g_{m-1} + \delta q$ . Because  $\delta$  can be arbitrarily small it is clear that the sequence  $(g_m)$  can be made to converge to a function  $\varphi$  fulfilling the inequalities  $|\varphi^{(n)}| = O(M_n)$  for  $n \geq 1$ , and the inequality  $|g_m - \varphi| y_m < m^{-1}$ . This completes the proof of Theorem 1.

Theorem 2 is a variation on the idea of building jumps into the derivative, but using Lebesgue measure for the initial measure  $\mu$ , and differential calculus, we give a more precise conclusion. Let  $\varphi$  be of class  $C^1[0, 1]$ , let  $F$  be a closed subset of  $[0, 1]$ ,  $m(F) > 0$ , and let

$$H(y) = m(x \in F, \varphi'(x) < y), \quad -\infty < y < \infty.$$

be the relative distribution function of  $\varphi'$ .

**THEOREM 2.** —  $F$  contains an  $M_0$ -set  $E$  such that  $\varphi(E)$  is a Kronecker set, provided  $H$  is continuous.

An equivalent statement is that such a set  $E \subseteq F$  can be found, provided  $H$  is not a pure saltus-function. Theorem 2 is proved by the same inductive process as before, beginning now with the Lebesgue measure restricted to  $F$ , i.e.  $\mu_0 = \chi_F m$ . In place of arguments on the sequence of functions  $g_m$  we use

**LEMMA.** — Let  $H$ ,  $\varphi$ , and  $F$  be as in Theorem 2. Then  $\lim_T \sup_u \left| \int_F \exp - iut. \exp i T \varphi(t). dt \right| = 0$  as  $T \rightarrow +\infty$ .

*Proof.* — Let  $k$  be a positive integer,  $\delta > 0$ , and  $u$  real, and let  $S(k, \delta, u)$  be the union of all intervals

$$[pk^{-1}, (p+1)k^{-1}], \quad (p = 0, \dots, k-1)$$

containing a point  $x$  at which  $|\varphi' - u| < \delta$ . Using the uniform continuity of  $\varphi'$  on  $[0, 1]$  and of  $H$  on  $(-\infty, \infty)$ , we see that to each  $\varepsilon > 0$  there exist  $k, \delta$  so that  $m(F \cap S(k, \delta, u)) < \varepsilon$  for all real  $u$ . When  $u$  and  $T$  are specified let us denote by  $G$  any intersection  $F \cap (a, b)$  where  $|-u + T\varphi'| \geq \delta T$  throughout  $(a, b)$ . We shall give a uniform method of estimating  $\int_G \exp i - iut. \exp T\varphi(t). dt$  and this will prove the lemma.

For definiteness we suppose  $-u + T\varphi'(t) > 0$  on  $[a, b]$  and construct the sequence  $a = a_0, a_1, \dots$  such that

—  $ut + T\varphi$  increases by exactly  $2\pi T^{-1}$  between each pair  $a_n, a_{n+1}$ . Thus  $a_{n+1} - a_n \leq 2\pi (\delta T)^{-1}$  and  $(a, b)$  is covered by intervals  $(a_n, a_{n+1})$  and a remainder  $< 2\pi (\delta T)^{-1}$ . The polygonal interpolation  $\tilde{\varphi}$  of  $\varphi$ , with nodes  $a_0, a_1, \dots, b$  has the property  $|\tilde{\varphi} - \varphi| = o(T^{-1})$  by the mean-value theorem, and the estimation

$$\int_c \exp - iut. \exp i T\tilde{\varphi}(t). dt \rightarrow 0 \quad \text{as} \quad T \rightarrow + \infty$$

follows from the Lebesgue density theorem. This concludes the proof of the lemma.

**THEOREM 3.** — *Let  $\varphi$  have an absolutely continuous derivative on  $[0, 1]$ , and  $C_1 \leq \varphi'' \leq C_2$  almost everywhere ( $0 < C_1 < C_2 < \infty$ ). Let  $\omega(u)$  be positive on  $[0, \infty)$ , increasing to  $+\infty$ . Then there is a subset  $E$  so that  $\varphi(E)$  is a Kronecker set, and a measure  $\mu \geq 0$  in  $E$  such that  $\hat{\mu}(u) = 0 \left( |u|^{-\frac{1}{2}} \right) \omega(|u|)$ .*

As in the two previous proofs, all depends on a suitable estimate of an exponential integral. The sequence  $(f_m)$ , dense in  $C[0, 1]$ , is now supposed to contain functions of class  $C^2$ ; thus beginning with the Lebesgue measure  $m$  on  $[0, 1]$ , all measures constructed by the inductive process have the form  $\mu_m = p_m m$ , with  $p_m \in C^2$ . Thus there is a constant  $C_m$  so that

$$\left| \int f d\mu_m \right| \leq C_m \sup_x \left| \int_0^x f(t) dt \right| \quad (0 \leq t \leq 1).$$

Thus the following estimation enables us to complete the proof.

For all  $y > y_0, k \geq 1$  or  $k \leq -1$ , and real  $u$

$$\left| \int_0^x \exp - iut. \exp ik(y\varphi - f_m). dt \right| \leq C'_m |u|^{-\frac{1}{2}},$$

and moreover, the integrals are uniformly  $o(1)$  as  $y \rightarrow \infty$ .

To prove this we use the inequality  $\varphi'' \geq C_1 > 0$  and  $f_m \in C^2$  to choose  $y$  so large that  $y\varphi'' - f_m'' \geq \frac{1}{2} C_1 y$ . By van der Corput's inequality [7, p. 197] the integrals are uniformly  $0 \left( |ky|^{-\frac{1}{2}} \right)$ , so the second part is disposed of. More-

over, the first inequalities are valid on domains of the type  $|ky| \geq \varepsilon|u|$ , for any fixed  $\varepsilon > 0$ . For the complementary domain  $|u|\varepsilon > |ky|$ ,

$$|k(y\varphi - f_m)'| \leq \varepsilon|u| \cdot |\varphi'| + y^{-1} \varepsilon |u| |f_m'|.$$

Thus for small  $\varepsilon$  and large  $y$ ,  $-ut + k(y\varphi - f_m)$  has derivative  $\geq \frac{1}{2}|u|$  or  $\leq -\frac{1}{2}|u|$ . We can then write

$$-ut + k(y\varphi - f_m) = up(t) \text{ where } \frac{1}{2} \leq |p'(t)| \text{ and } |p''(t)| \leq C_m''.$$

The integral then takes the form  $\int_a^b \exp -ius q(s) ds$  where  $|q(s)| \leq 2$  and  $q(s)$  has total variation  $\leq 2C_m''$ . The integral then has modulus  $< 8C_m''|u|^{-1} \leq C_m'|u|^{-\frac{1}{2}}$ , because  $|u| > |y| \rightarrow +\infty$ . This proves the required estimation.

The last theorem about differentiable functions is a complement to the first and second; its proof involves a lemma on interpolation of differentiable functions. The set  $E$  of Theorem 1 can be mapped by a diffeomorphism  $\varphi$  of class  $C(M_n)$  onto a Kronecker set, and in fact  $\varphi'' = 0$  on  $E$ ; by [1]  $E$  can also be mapped by a  $C^1$ -diffeomorphism  $\psi$  onto a Kronecker set, and here  $\psi' = 1$  on  $E$ . Quite possibly  $E$  could be constructed so that the diffeomorphism  $\varphi$  is smooth and has derivative 1 on  $E$ , but the method of Theorem 1 plainly fails to accomplish this. The theorem to be proved shows that the existence of diffeomorphisms  $\varphi$  and  $\psi$  by no means implies that their characteristics can be attained simultaneously. A similar property of stability of  $M_0$ -sets is obtained in [4] by an entirely different technique.

**THEOREM 2'.** — *Let  $F$  and  $\varphi$  be as in Theorem 2, and let  $\omega(u)$  be positive and increasing on  $(0, \infty)$ ,  $\omega(0+) = 0$ . Then the set  $E \subseteq F$  can be so chosen that  $\varphi(E)$  is a Kronecker set, but  $\psi(E)$  is an  $M_0$ -set whenever  $\psi \in C^1[0, 1]$ ,  $\psi' = 1$  on  $E$ , and  $|\psi'(s) - \psi'(t)| \leq \omega(|s - t|)$  for  $0 \leq s < t \leq 1$ .*

When  $F$  is totally disconnected,  $\varphi$  can be constructed in any non-quasi-analytic class  $C(M_n)$  so that  $\varphi'$  is strictly increasing and  $\varphi'' = 0$  on  $F$ ; the simple example  $\varphi(x) = x^2$  illustrates that analyticity has no obvious consequences about  $E$ . The necessity of the lemma needs to be explained.

Beginning from the set  $F$ , the subset  $E \subseteq F$  is to be defined, and thereby a certain subset of  $C^1$ , say  $S(E)$ . But  $S_1(E)$  is then known only in principle and is obviously much larger than  $S(F)$ . Thus the construction seems to be circular, because it requires some knowledge of  $S(E)$  to proceed. To circumvent this obstacle we consider all sets  $S(E)$  simultaneously, attempting to replace each function  $\psi_1$  in  $S(E)$  by a function  $\psi_1$  in  $C^1[0, 1]$ , such that  $\psi = \psi_1$  in  $E$  and  $\psi'_1 = 1$  in  $E \cup F$ . This, however, is possible only if  $[0, 1] \sim F$  meets each interval  $(a, b) \subseteq (0, 1)$  in a subset whose measure is not much smaller than  $(b - a) \cdot \omega(b - a)$ , because  $\psi'_1 = 1$  on  $F$ . To solve this problem of interpolation we must therefore replace  $F_1$  by a subset whose complementary intervals are specially constructed.

**LEMMA.** — *Corresponding to the function  $\omega$  there is a closed set  $F_1$  with this property: whenever  $E \subseteq [0, 1]$  and  $\psi \in S_1(E)$ , then  $\psi$  coincides with a function  $\psi_1 \in C^1[0, 1]$  whose derivative is 1 on  $E \cup F_1$ . Moreover all the derivatives  $\psi'_1$  so constructed are equicontinuous on  $[0, 1]$ . Finally,  $m(F \cap F_1) > 0$ .*

*Proof.* — Let  $T_n$  be an increasing sequence of positive numbers and  $R \sim F_1$  the set defined by  $x \notin F_1$  if  $|T_n x - q| \leq n^{-2}$  for some integer  $q$  and  $n \geq 1$ . When  $(T_n)$  increases rapidly,  $m(F \cap F_1) > 0$ ; we specify that  $T_1 > 8$  and  $\omega(8T_n^{-1}) \leq (n + 1)^{-3}$ . Thus, if  $b - a > 8T_n^{-1}$ ,  $(R \sim F_1) \cap (a, b)$  contains intervals of length  $2n^{-2}T_n^{-1}$ , whose total length exceeds  $n^{-2}(b - a)$ . Let now  $E$  be a closed subset of  $[0, 1]$  and  $\psi$  a function in the class  $S_1(E)$ , and write  $\psi_2(x) = x - \psi(x)$ . Now  $\psi_2$  is considered as a function defined only on  $E$ ; then

$$|\psi_2(s) - \psi_2(t)| \leq |s - t| \omega(|s - t|)$$

by the mean-value theorem. To each interval  $[a, b]$  meeting  $E$  only in its end-points  $a$  and  $b$ , there is a least integer  $n$  with  $b - a > 8T_n^{-1}$ . Then  $(a, b)$  meets  $R \sim F_1$  in a certain set of intervals, of total length  $> n^{-2}(b - a)$ . The derivative of  $\psi_2$  will have a triangular graph over these intervals, of height  $h$ , and will vanish elsewhere in  $(a, b)$ .



The common height  $h$  of these triangles fulfills an inequality  $n^{-2}(b-a) \cdot |h| \leq 2|\psi_2(b) - \psi_2(a)| \leq 2(b-a)\varpi(b-a)$ . In case  $1 \geq b-a > 8T_1^{-1}$ , we obtain  $|h| \leq 2\varpi(1)$ ; when  $8T_{n-1}^{-1} \geq b-a > 8T_n^{-1}$ , we have  $\varpi(b-a) \leq n^{-3}$  and then  $|h| \leq 2n^{-1}$ . To complete the extension of  $\psi_2$  onto  $[0, 1]$ , we extend to be constant to the left and right of  $[0, 1]$ . The equicontinuity of the aggregate  $\{\psi'_1\}$  follows from the triangular shape and the fact that  $|\psi'_1| \leq 2n^{-1}$  on the interval  $(a, b)$  provided  $b-a < 2n^{-2}T_n^{-1}$ . Finally, set

$$\psi_1(x) = x - \psi_2(x)$$

and the lemma is complete.

To prove the main result, we can assume that  $F_1 = F$ , and construct  $E$  as in Theorem 2 so that  $\psi_1(E)$  is an  $M_0$ -set for each function  $\psi_1$  constructed in the lemma. This can be accomplished with the aid of uniform estimates for integrals of the form

$$\int_F \exp -iu\psi_1(t) \cdot \exp iy\varphi(t) \cdot dt.$$

To each  $\delta > 0$  there is a neighbourhood  $V_\delta \supseteq F$  on which  $|\psi'_1 - 1| < \delta$  for each function  $\psi_1$ , and from this point the argument of Theorem 2 is valid, so the integrals tend to 0 uniformly as  $y \rightarrow \infty$ .

Theorem 2' is valid for the weaker inequality

$$|\psi'(s) - \psi'(t)| = O(\varpi|t-s|),$$

since there is a function  $g$ , locally constant on  $E$ , so that  $g' + \psi'$  has modulus of continuity at most  $\varpi^{\frac{1}{2}}$ .

## 2.

**THEOREM 4.** — *Let  $\lambda$  be a continuous, finite measure on  $R$ , and  $F$  an  $M_0$ -set. Then there is a Kronecker set  $E$  and a positive measure  $\mu \neq 0$  in  $E$  such that each set  $\{|\hat{\mu}(u)| \geq \delta, |\hat{\lambda}(u)| \geq \delta\}$  is compact. Moreover to each  $\delta > 0$  there is a  $u_0$  so that the set  $\{|\hat{\mu}(u+u_0)| \geq \delta, |\hat{\lambda}(u)| \geq \delta\}$  is empty.*

Here we set  $\varphi(x) = x$  so that the support of the limit measure  $\mu$  is a Kronecker set. Of course we cannot obtain

uniform convergence of  $\hat{\mu}_m$ , but only pointwise convergence, sufficient to ensure that  $\frac{3}{2} > \mu(E) > \frac{1}{2}$ . However, we can obtain uniform convergence on each of the subsets  $R_\delta = \{|\hat{\lambda}(u)| \geq \delta\}$ . A classical theorem of Wiener shows that  $|\hat{\lambda}|^2$  has mean-value 0, and for each  $\delta$  there is an  $\eta > 0$  such that  $R_\delta + (0, \eta) \subseteq R_{\frac{1}{2}\delta}$ . Thus  $R_\delta + (0, \eta)$  meets  $[-a, a]$  in a set of measure  $o(a)$ . An elementary covering argument shows that this remains valid for  $R_\delta + I$ ,  $I$  being a fixed, finite interval, and plainly that property is preserved by dilations and translations of  $R_\delta$ . Thus we obtain the important property of  $R_\delta$ : there is a sequence  $y_m$  such that  $\lim d(ky_m, R_\delta) = +\infty$  for each  $\delta > 0$  and each integer  $k \neq 0$ . Examination of the formula for  $\hat{\mu}_m(u) - \hat{\mu}_{m-1}(u)$  shows that it is possible to force the sequence  $\hat{\mu}_m$  to converge uniformly on each  $R_\delta$ . Because each  $\hat{\mu}_m \in C_0$ , the limit  $\hat{\mu} \in C_0(R_\delta)$  and this expresses the first property claimed for  $\hat{\mu}$ .

To obtain the second we choose a sequence  $(u_m)$  along with  $(\mu_m)$ . Now  $\hat{\mu}_m \in C_0(\mathbb{R})$  so there is a number  $-u_m$  so far from  $R_m - 1$  that  $|\hat{\mu}_m(u - u_m)| < m^{-1}$  whenever  $|\hat{\lambda}(u)| \geq m^{-1}$ ; or  $|\hat{\mu}_m| < m^{-1}$  on  $R_m - 1 + u_m$ . From here the argument is almost as before, except that  $\hat{\mu}_m - \hat{\mu}_{m-1}$  must be controlled on a set of the type finite  $+ R_\delta$  and this is easily attained. In the limit we have, for example,  $|\hat{\mu}| < 2m^{-1}$  on  $R_m - 1 + u_m$  so that the inequalities  $|\hat{\lambda}| > m^{-1}$  and  $|\hat{\mu}(u - u_m)| > 2m^{-1}$  exclude each other.

Here is a simple consequence of Theorem 4. To each uncountable closed set  $E$  there are a Kronecker set  $E_1$ , disjoint from  $E$  and probability measures  $\mu$  in  $E_1$ ,  $\lambda$  in  $E$ , so that  $\limsup |\hat{\mu}| + |\hat{\lambda}| \leq 1$ , and a sequence of characters  $\chi_m$  so that  $\widehat{|\chi_m \mu|} + |\hat{\lambda}| < +m^{-1} + 1$ . Thus  $E \cup E_1$  is at most  $H_{\frac{1}{2}}$ , in a sense somewhat stronger than in [5]; of course the most interesting case occurs when  $E$  is itself a Kronecker set.

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Manuscrit reçu le 4 décembre 1972.

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