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On the lower order \((R)\) of an entire Dirichlet series


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ON THE LOWER ORDER (R)
OF AN ENTIRE DIRICHLET SERIES
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1. Introduction.

For an entire Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \ (s = \sigma + it, \ \lambda_1 \geq 0, \ \lambda_n \to \infty \text{ with } n) \quad (1.1)$$

the lower order (R) $\lambda$ is defined as :

$$\lim_{\sigma \to \infty} \inf \frac{\log \log M(\sigma)}{\sigma} = \lambda, \ (0 \leq \lambda \leq \infty), \quad (1.2)$$

where $M(\sigma) = \sup \{|f(\sigma + it)| : -\infty < t < \infty\}$.

Improving upon a result of Rahman [6], Juneja and Singh [4] have, very recently, proved the following theorem :

**Theorem A.** — Let $f(s)$ be an entire Dirichlet series given by (1.1) of lower order (R) $\lambda$ ($0 \leq \lambda \leq \infty$). Then

$$\lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_{n-1}^{-1}}{\log |a_n|^{-1}} \leq \lambda. \quad (1.3)$$

Further, if

$$\lim_{n \to \infty} \sup \frac{\log n}{\lambda_n} < \infty, \quad (1.4)$$

and

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forms a non-decreasing function of \( n \) for \( n > n_0 \), then

\[
\lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_{n-1}}{\log |a_n|^{-1}} = \lambda. \tag{1.6}
\]

In this paper, we obtain the estimations for the lower order \((R)\lambda\) in terms of the sequences \(\{\lambda_n\}\) and \(\{a_n\}\) which hold for every entire Dirichlet series, and one of our estimations includes (1.6) and the result of Rahman [6] as the special cases. In fact, we prove:

**Theorem.** — Let \( f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \) be an entire Dirichlet series given by (1.1) of lower order \((R)\lambda\) \((0 \leq \lambda \leq \infty)\) such that (1.4) is satisfied. Then

\[
\lambda = \max \lim_{p \to \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_p-1}}{\log |a_{n_p}|^{-1}} \tag{1.7}
\]

\[
\lambda = \max \lim_{p \to \infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_p-1}) \log \lambda_{n_p-1}}{\log |a_{n_p-1}/a_{n_p}|}. \tag{1.8}
\]

**2. Preliminary Discussions.**

Let \( \mu(\sigma) \) denote the maximum term of \( f(s) \) for \( \Re(s) = \sigma \) and \( \lambda_{\nu(\sigma)} = \max \{\lambda_n : \mu(\sigma) = |a_n| e^{\sigma \lambda_n}\} \). Let \( \{\rho_n\} \) be the sequence of jump points of \( \lambda_{\nu(\sigma)} \) (points of discontinuity of \( \lambda_{\nu(\sigma)} \)), every jump point is listed with multiplicity equal to size of the jump, such that \( \rho_1 \leq \rho_2 \leq \rho_3 \leq \cdots \rho_n \leq \cdots \). Since \( \lambda_{\nu(\sigma)} \to \infty \) as \( \sigma \to \infty \), \( \rho_n \to \infty \) as \( n \to \infty \). We denote by \( \{\lambda_{n_k}\} \) the range of \( \lambda_{\nu(\sigma)} \), so that \( \lambda_{\nu(\sigma)} = \lambda_{n_k} \) for \( \sigma = \rho_{n_k} \). Then

i) \( 0 < \rho_{n_k} < \rho_{n_k+1} = \rho_{n_k+2} = \cdots = \rho_{n_k+1}, k = 1, 2, 3, \ldots \)

ii) \( \lambda_{\nu(\sigma)} = \lambda_{n_k}, \) when \( \rho_{n_k} \leq \sigma < \rho_{n_k+1}, k = 1, 2, 3, \ldots \)
These arguments are analogue to those for entire power series, given and used by Gray and Shah in their works [1,2,3].

Since $|a_{n_k-1}|e^{\sigma \lambda n_k-1}$ and $|a_{n_k}|e^{\sigma \lambda n_k}$ are the two consecutive maximum terms, we have

$$\rho_{n_k} = \log |a_{n_k-1}|/|a_{n_k}|/(\lambda_{n_k} - \lambda_{n_k-1}) . \quad (2.1)$$

Also, we need the following:

**Lemma**

$$\lim_{k \to \infty} \inf \frac{\log \lambda_{n_k}}{\rho_{n_k+1}} = \lambda .$$

**Proof.** — Since ([5], Theorem B)

$$\lim_{\sigma \to \infty} \inf \frac{\log \lambda_{\nu(\sigma)}}{\sigma} = \lambda, \ (0 \leq \lambda \leq \infty) ,$$

there exists a sequence $(x_i)$, $x_i \to \infty$ with $i$ such that

$$\lim_{i \to \infty} \frac{\log \lambda_{\nu(x_i)}}{x_i} = \lambda .$$

It is always possible to find a subsequence $(\rho_{n_{ki}})$ of $(\rho_{n_k})$ which satisfies the inequalities:

$$\rho_{n_{ki}} \leq x_i < \rho_{n_{ki}+1}, \ i = 1, 2, 3, \ldots$$

In either of the cases, for $i \geq i_0 = i_0(\varepsilon), \ \varepsilon > 0$, we have

$$\frac{\log \lambda_{n_{ki}}}{\rho_{n_{ki}+1}} \leq \frac{\log \lambda_{\nu(x_i)}}{x_i} \leq \lambda + \varepsilon ,$$

which implies

$$\lim_{k \to \infty} \inf \frac{\log \lambda_{n_k}}{\rho_{n_k+1}} \leq \lambda .$$

The reverse inequality is obvious. Hence the lemma is proved.
3. Proof of the Theorem.

Using (2.1) in the lemma, we get

\[
\lambda = \lim_{k \to \infty} \inf \frac{(\lambda_{n_k} - \lambda_{n_k - 1}) \log \lambda_{n_k - 1}}{\log |a_{n_k - 1}/a_{n_k}|}.
\] (3.1)

We have proved (3.1) for a particular subsequence \(\{\lambda_{n_k}\}\) which is the range of the rank function \(\lambda_{\nu(o)}\). Thus the theorem will be proved completely if we establish, for any arbitrary subsequence (say) \(\{\lambda_{n_p}\}\) of \(\{\lambda_n\}\), the following inequalities:

\[
\lambda \geq \lim_{p \to \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_p - 1}}{\log |a_{n_p}|^{-1}} \geq \lim_{p \to \infty} \inf \times
\] (3.2)

Proof of the first inequality in (3.2): Let

\[
\lim_{p \to \infty} \inf \frac{\lambda_{n_p} \log \lambda_{n_p - 1}}{\log |a_{n_p}|^{-1}} = \alpha.
\]

Assume that \(\alpha > 0\), for otherwise the result is trivially true. Therefore, for any \(\epsilon > 0\), \(\exists a N = N(\epsilon)\) such that

\[
-\lambda_{n_p} \quad |a_{n_p}| > \lambda_{n_{p-1}}^{\alpha - \epsilon}, \quad (p \geq N).
\]

Let \(e^{\sigma_p} = 2 \lambda_{n_{p-1}}^{\frac{1}{\lambda_{n_{p-1}}}}\), \(p = 1, 2, 3, \ldots\). So if

\[
\sigma_p \leq \sigma \leq \sigma_{p+1}, \quad \text{we have}
\]

\[
\log M(\sigma) \geq \log |a_{n_p}| + \sigma_p \lambda_{n_p}
\]

\[
\geq \log |a_{n_p}| + \sigma_p \lambda_{n_p}
\]

\[
\geq \lambda_{n_p} \log 2
\]

\[
= e^{(\alpha - \epsilon)\sigma_p + 1} \log 2^{2^{\alpha - \epsilon}}.
\]
i.e.
\[
\log \log M(\sigma) \geq (\alpha - \epsilon) \sigma_{p+1} + \log \log 2 - (\alpha - \epsilon) \log 2.
\]

which gives
\[
\lambda = \lim_{\sigma \to \infty} \inf \frac{\log \log M(\sigma)}{\sigma} \geq \alpha.
\]

Proof of the Second Inequality in (3.2): Let
\[
\lim_{p \to \infty} \inf \frac{(\lambda_{n_p} - \lambda_{n_p-1}) \log \lambda_{n_p-1}}{\log |a_{n_p-1}/a_{n_p}|} = \beta.
\]

Again, without any loss of generality, assume \(\beta > 0\), so that
\[
|a_{n_p-1}/a_{n_p}| < \frac{\lambda_{n_p} - \lambda_{n_p-1}}{\beta - \epsilon},
\]
for \(p \geq p_0 = p_0(\epsilon), \epsilon > 0\). This implies
\[
\left| \frac{a_{n_p}}{a_{n_p}} \right| = \left| \frac{a_{n_p}}{a_{n_p}} \right| \cdot \left| \frac{a_{n_p+1}}{a_{n_p+2}} \right| \cdots \left| \frac{a_{n_p-1}}{a_{n_p}} \right| < \prod_{m=p_0+1}^{p} \frac{\lambda_{n_m} - \lambda_{n_{m-1}}}{\beta - \epsilon},
\]
\[
\Rightarrow \log |a_{n_p}|^{-1} < o(1) + \frac{1}{\beta - \epsilon} \sum_{m=p_0+1}^{p} (\lambda_{n_m} - \lambda_{n_{m-1}}) \log \lambda_{n_{m-1}}
\]
\[
\Rightarrow \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} < o(1) + \frac{1}{\beta - \epsilon} - \frac{1}{\beta - \epsilon} \frac{\lambda_{n_0}}{\lambda_{n_p} \log \lambda_{n_{p-1}}}
\]
\[
(\beta - \epsilon)^{-1} \sum_{m=p_0+1}^{p-1} \lambda_{n_m} (\log \lambda_{n_m} - \log \lambda_{n_{m-1}})
\]
\[
\frac{\lambda_{n_p} \log \lambda_{n_{p-1}}}{\lambda_{n_p} \log \lambda_{n_{p-1}}}
\]
\[
\Rightarrow \lim_{p \to \infty} \sup \frac{\log |a_{n_p}|^{-1}}{\lambda_{n_p} \log \lambda_{n_{p-1}}} \leq \frac{1}{\beta}.
\]
Hence

\[
\lim_{p \to \infty} \inf \frac{\lambda_n \log \lambda_n}{\log |a_n|} \geq \lim_{p \to \infty} \inf \frac{(\lambda_n - \lambda_{n-1}) \log \lambda_n}{\log |a_{n-1}|/|a_n|}.
\]

**Remark.** – If, in addition to the hypothesis of our theorem, (1.5) is satisfied, then our result (1.7) reduces to (1.6). Further, if \( \log \lambda_{n+1} \sim \log \lambda_n \), as \( n \to \infty \), the result of Rahman [6] is also obtained.

**Justification.** – Since (1.5) is satisfied, each term of \( f(s) \) is a maximum term and so \( \lambda_{n_k} = \lambda_k \), for \( k = 1, 2, \ldots \). Therefore, the result (3.1) reduces to

\[
\lambda = \lim_{k \to \infty} \inf \frac{(\lambda_k - \lambda_{k-1}) \log \lambda_{k-1}}{\log |a_{k-1}|/a_k}.
\]

Further, as the result (3.2) is true for every subsequence \( \{\lambda_n\} \) of \( \{\lambda_n\} \), it is also true for the sequence \( \{\lambda_n\} \), since \( \{\lambda_n\} \) may be regarded as a subsequence of \( \{\lambda_n\} \). Hence

\[
\lambda \geq \lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_n}{\log |a_n|} \geq \lim_{n \to \infty} \inf \frac{(\lambda_n - \lambda_{n-1}) \log \lambda_{n-1}}{\log |a_{n-1}|/a_n}.
\]

Thus, the result (1.6) follows from (3.3) and (3.4).

Furthermore, if \( \log \lambda_{n+1} \sim \log \lambda_n \), as \( n \to \infty \), then (1.6) implies that

\[
\lambda = \lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_n}{\log |a_n|} = \lim_{n \to \infty} \inf \frac{\lambda_n \log \lambda_n}{\log |a_n|}.
\]

which is a result of Rahman [6].

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